# Linear Forests and Ordered Cycles


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**Abstract**

A collection $L = P^1 \cup P^2 \cup \cdots \cup P^t$ $(1 \leq t \leq k)$ of $t$ disjoint paths, $s$ of them being singletons with $|V(L)| = k$ is called a $(k,t,s)$-*linear forest*. A graph $G$ is $(k,t,s)$-*ordered* if for every $(k,t,s)$-linear forest $L$ in $G$ there exists a cycle $C$ in $G$ that contains the paths of $L$ in the designated order as subpaths. If the cycle is also a hamiltonian cycle, then $G$ is said to be $(k,t,s)$-*ordered hamiltonian*. We give sharp sum of degree conditions for nonadjacent vertices that imply a graph is $(k,t,s)$-ordered hamiltonian.


## 1 Introduction

Over the years hamiltonian graphs have been widely studied. A variety of related properties have also been considered. Some of the properties are weaker, for example traceability in graphs, while others are stronger, for example hamiltonian connectedness. Recently a new strong hamiltonian property was introduced in [7] and further studied in [5], [2], and [3].

We say a graph $G$ on $n$ vertices, $n \geq 3$ is *k-ordered* for an integer $k$, $1 \leq k \leq n$, if for every sequence $S = (x_1, x_2, ..., x_k)$ of $k$ distinct vertices in $G$, there exists a cycle that contains all the vertices of $S$ in the designated order. A graph is *k-ordered hamiltonian* if for every sequence $S$ of $k$ vertices there exists a hamiltonian cycle which encounters $S$ in its designated order.

Hu, Tian and Wei [4] considered a different question; when is it possible to find a long cycle passing through a collection of paths?

In this paper we combine these two ideas. In order to treat this in generality, we say $L$ is a $(k,t,s)$-*linear forest* if $L$ is a collection $L = P^1 \cup P^2 \cup \cdots \cup P^t$ $(1 \leq t \leq k)$ of $t$ disjoint paths, $s$ of them being singletons such that $|V(L)| = k$. A graph $G$ is $(k,t,s)$-*ordered* if for every $(k,t,s)$-linear forest $L$ in $G$ there exists a cycle $C$ in $G$ that contains the paths of $L$ in the designated order as subpaths. Further, if the paths of $L$ are each oriented and $C$ can be chosen to encounter the paths of $L$ in the designated order and according to the designated orientation on each path, then we say $G$ is *strongly* $(k,t,s)$-*ordered*. If $C$ is a hamiltonian cycle then we say $G$ is $(k,t,s)$-*ordered hamiltonian* and *strongly* $(k,t,s)$-*ordered hamiltonian*, respectively. Note that saying $G$ is $(s,s,s)$-ordered is the same as saying $G$ is $s$-ordered.

We will think of all cycles being directed. For a cycle $C$ and vertices $x, y \in V(C)$, we denote the $x-y$ path on $C$ following the direction of $C$ by $xCy$.

As usual, we will denote the minimum degree of a graph $G$ by $\delta(G)$, and the minimum degree sum of two non adjacent vertices in a graph $G$ by $\sigma_2(G)$.

We will say that a graph $G$ on at least $2k$ vertices is *$k$-linked*, if for every vertex set $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ of $2k$ vertices, there are $k$ disjoint $x_i - y_i$ paths. The property remains the same if we allow repetition in $T$, and ask for $k$ internally disjoint $x_i - y_i$ paths. Thus, as an easy consequence, every $k$-linked graph is $k$-ordered and $(2k-s, k, s)$-ordered.

An important theorem about $k$-linked graphs is the following theorem of Bollobás and Thomason [1]:

**Theorem 1** *Every $22k$-connected graph is $k$-linked.*

The following lemmas will be used later.

**Lemma 1** *If a $2k$-connected graph $G$ has a $k$-linked subgraph $H$, then $G$ is $k$-linked.*

**Proof:** Let $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a set of $2k$ vertices in $V(G)$. Since $G$ is $2k$-connected, there are $2k$ disjoint paths from $T$ to $V(H)$. Choose the paths from $T$ to $V(H)$ such that each path contains exactly one element of $V(H)$ (if $x_i \in T \cap V(H)$ then the corresponding path consists only of this one vertex). Now we can connect these paths in the desired way inside $H$, since $H$ is $k$-linked. $\square$

**Lemma 2** *If $G$ is a graph, $v \in V(G)$ with $d(v) \geq 2k-1$, and if $G - v$ is $k$-linked, then $G$ is $k$-linked.*

**Proof:** Let $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a set of $2k$ vertices in $V(G)$. If $v \notin T$, we can find disjoint $x_i - y_i$ paths inside $G - v$. Thus we may assume that $v = x_1$. If $y_1 \in N(v)$, we can find disjoint $x_i - y_i$ paths for all $i \geq 2$ in $G - v - y_1$, since $G - v - y_1$ is $(k-1)$-linked. Adding the path $vy_1$ completes the desired set of paths in $G$. If $y_1 \notin N(v)$, then there exists a vertex $x_1' \in N(v) - T$, since $d(v) \geq 2k-1$. We can find disjoint $x_i - y_i$ paths for $i \geq 2$ and a $x_1' - y_1$ path in $G - v$, which we can then extend to an $x_1 - y_1$ path in $G$. $\square$

Further, we will use a Theorem of Mader [6] about dense graphs:

**Theorem 2** *Every graph $G$ with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq (2k-3)(n-k+1)+1$ has a $k$-connected subgraph.*

**Corollary 3** *Every graph $G$ with $|V(G)| = n \geq 2k-1$, and $|E(G)| \geq 2kn$ has a $k$-connected subgraph.*

## 2 Degree Conditions

In this section we examine minimum degree conditions sufficient to insure a graph is either $(k, t, s)$-ordered hamiltonian or strongly $(k, t, s)$-ordered hamiltonian. Sharp results for $s = t = k$ were shown in [5], [2] and [3]:

**Theorem 4** [5] *Let $k \geq 2$ be a positive integer and let $G$ be a graph of order $n$, where $n \geq 11k-3$. Then $G$ is $k$-ordered hamiltonian if $\delta(G) \geq \left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

**Theorem 5** [3] *Let $k \geq 3$ be a positive integer and let $G$ be a graph of order $n \geq 2k$. If $\sigma_2(G) \geq n + \frac{3k-9}{2}$, then $G$ is $k$-ordered hamiltonian.*

As a first step, we prove the following theorem:

**Theorem 6** *Let $s, t, k$ be integers with $0 \leq s < t < k$ or $s = t = k \geq 3$. If $G$ is a (strongly) $(k, t, s)$-ordered graph on $n \geq k$ vertices with*

$$\sigma_2(G) \geq \begin{cases} n + k - t & \text{if } s = 0 \\ n + k - t + s - 1 & \text{if } s > 0 \end{cases},$$

*then $G$ is (strongly) $(k, t, s)$-ordered hamiltonian.*

As a corollary, we obtain the following theorem.

**Theorem 7** *For $k \geq 1$ and $1 \leq t \leq k$, if $G$ is a (strongly) $(k, t, s)$-ordered graph on $n \geq k$ vertices with $\delta(G) \geq \frac{n+k-t+s}{2}$, then $G$ is (strongly) $(k, t, s)$-ordered hamiltonian.*

In the same spirit, we will prove another theorem, which is not needed for our main result, Theorem 10.

**Theorem 8** *Let $s, t, k$ be integers with $1 < t/2 < s \leq t \leq k$. If $G$ is a (strongly) $(k, t, s)$-ordered graph on $n \geq 11k$ vertices with*

$$\sigma_2(G) \geq n + k - \frac{t+3}{2},$$

*then $G$ is (strongly) $(k, t, s)$-ordered hamiltonian.*

**Proof of Theorem 6 and Theorem 8:** Since $G$ is (strongly) $(k, t, s)$-ordered, we may choose a longest cycle $C$ containing the paths of a given $(k, t, s)$-linear forest $L$ in the designated order and with the designated orientations (if there are any) on each path. We need to show that $C$ is hamiltonian.

Let $L = P^1 \cup P^2 \cup \ldots \cup P^t$, and $x_1, \ldots, x_t, y_1, \ldots, y_t \in V(C)$, such that $P^i = x_i C y_i$ for all $1 \leq i \leq t$. Note that $x_i = y_i$ if $P_i$ is a singleton. Let $R^i = y_i C x_{i+1}$ for $1 \leq i \leq t-1$, and $R^t = y_t C x_1$. Let $R = \bigcup_i R^i$.

Suppose $C$ is not hamiltonian and let $H$ be a component of $G - C$.

**Claim 1** *No $R^i$ contains more than one vertex adjacent to $H$.*

Suppose there exists an interval $R^i$ with at least two vertices adjacent to $H$. Without loss of generality we may assume that $R^1$ is such an interval. Pick two of these vertices $v_1, v_2$ such that there are no other adjacencies of $H$ in $v_1 C v_2 \subset R^1$. Note that $r = |v_1 C v_2| - 2 \geq 1$, otherwise $C$ can be extended by at least one vertex.

Let $u_1 \in N(v_1) \cap H$, let $u_2 \in N(v_2) \cap H$. Note that we allow $u_1 = u_2$. Consider now $X = (N(u_1) \cup N(u_2)) \cap C$. There cannot be two vertices consecutive on $R$ in $X$, otherwise $C$ can be extended by at least one vertex. Further, $X$ does not contain any vertices of

$v_1^+ C v_2^-$ by our choice of $v_1, v_2$. Note that $R \setminus v_1^+ C v_2^-$ consists of $t-s+1$ paths, and $|C \setminus R| = k - 2t + s$, thus

$$d(u_1) + d(u_2) \leq 2|X| + d_H(u_1) + d_H(u_2) \leq 2\left(|H| - 1 + \frac{|R| - r + t - s + 1}{2} + k - 2t + s\right).$$

Now concentrate on $v_1^+$ and $v_2^-$. There cannot be two consecutive vertices in $R \setminus v_1^+ C v_2^-$, such that one is adjacent to $v_1^+$ and the other adjacent to $v_2^-$, otherwise the whole segment $v_1^+ C v_2^-$ could be inserted between those two vertices, and a longer cycle through $u_1$ could be found. Thus,

$$d(v_1^+) + d(v_2^-) \leq 2\left(r - 1 + \frac{|R| - r + 1 + t - s}{2} + k - 2t + s + n - |C| - |H|\right).$$

But now,

$$\begin{aligned} 2(n+k-t) &\leq d(v_1^+) + d(u_1) + d(v_2^-) + d(u_2) \\ &\leq 2(n + k - t - 1 + |R| + k - 2t + s - |C|) = 2(n+k-t-1), \end{aligned}$$

a contradiction. Therefore, there can be at most one vertex adjacent to $H$ in each $R^i$.

To prove Theorem 6, observe that the degree condition forces $G$ to be complete or $(k-t+s+1)$-connected. If $G$ is complete we are done. So we may assume that $G$ is $(k-t+s+1)$-connected. Since $|C - R| = k - 2t + s$, there are at least $t+1$ vertices adjacent to $H$ in $R$. Thus, there exists an $R^i$ with two such vertices, a contradiction proving Theorem 6.

To prove Theorem 8, we first prove the following claim.

**Claim 2** *$H$ is the only component of $G - C$.*

Otherwise, let $H'$ be a different component, let $v_1 \in H, v_2 \in H'$. For $i = 1, 2$, let

$$\begin{aligned} a_i &= |\{v \in N(v_i) \cap (C \setminus L)\}|, \\ b_i &= |\{v \in N(v_i) : v = x_j \text{ or } v = y_j \text{ for some } j \text{ with } x_j \neq y_j\}|, \\ c_i &= |\{v \in N(v_i) : v = x_j = y_j \text{ for some } j\}|. \end{aligned}$$

We know that $a_i + b_i + 2c_i \leq t$, since by Claim 1, $v_i$ can have at most one neighbor in each $R_j$. Further, $b_i \leq 2(t-s)$. Thus,

$$\begin{aligned} 2d(v_1) &\leq 2(|H| - 1 + k - 2t + s + a_1 + b_1 + c_1) \\ &= 2|H| + k + a_1 + k - t - 2 + (b_1 - 2(t-s)) + (a_1 + b_1 + 2c_1 - t) \\ &\leq 2|H| + k + a_1 + k - t - 2. \end{aligned}$$

Similarly,

$$2d(v_2) \leq 2|H'| + k + a_2 + k - t - 2.$$

Therefore,

$$n + k - \frac{t+3}{2} \leq d(v_1) + d(v_2) \leq |H| + |H'| + k + \frac{a_1 + a_2}{2} + k - t - 2 \leq n + k - t - 2,$$

a contradiction, proving the claim.

The degree condition forces $G$ to be complete or $(k - \frac{t-1}{2})$-connected. If $G$ is complete we are done. So we may assume that $G$ is $(k - \frac{t-1}{2})$-connected. Since $|C - R| = k - 2t + s$, there are at least $\frac{3t+1}{2} - s$ neighbors of $H$ in $R$.

**Claim 3** *For some $i$, $1 \le i \le t$, the following is true: $x_i = y_i$ and $H$ has two neighbors in $y_{i-1}Cx_{i+1}^- \setminus x_i$.*

Let $h_i$ count the number of neighbors of $H$ in $y_{i-1}Cx_i \cup y_iCx_{i+1}^-$. We know that $h_i \in \{0,1,2\}$ for all $1 \le i \le t$. Further, $\sum_i h_i \ge 3t+1-2s-(t-s)$, since the sum counts every neighbor of $H$ in $\{x_i : x_i \ne y_i\}$ once and all other neighbors of $H$ in $R$ twice. Thus, at least $(t-s)+1$ of the $h_i$ are equal to 2. Therefore, $h_i = 2$ for some $i$ with $x_i = y_i$. The vertex $x_i$ cannot be one of the two neighbors of $H$ by Claim 1, establishing the claim.

Let $i$ be as in Claim 3, let $y \in y_{i-1}Cx_i^-$ and $z \in y_i^+Cx_{i+1}^-$ be the two neighbors of $H$. If $y^+z^+ \in E$, then $yHzC^-y^+z^+Cy$ is a longer cycle. Thus, $y^+z^+ \notin E$ and, since $y^+$ and $z^+$ are not in $N(H)$,

$$|C| \ge 2 + \frac{d(y^+)+d(z^+)}{2} > \frac{n+k}{2} - \frac{t}{4} + 1.$$

This implies that

$$|R| = |C| - k + 2t - s > \frac{n-k}{2} > 5k.$$

Now let $u \in H$, $v \in C - N(H)$. Then

$$d(v) \ge n+k-\frac{t+3}{2} - d(u) \ge n+k-\frac{t+3}{2} - (k-2t+s) - t - |H| \ge |C| - 1 - s + \frac{t-1}{2}.$$

Therefore, $v$ is adjacent to all but at most $\frac{s}{2}$ vertices on $C$.

For the final contradiction we differentiate two cases.

**Case 1** *Suppose $y^+ \ne x_i$ or $z^+ \ne x_{i+1}$.*

Let $w \in \{y^+, z^+\} - \{x_i, x_{i+1}\}$. Let $N = N(x_i) \cap N(x_{i+1}) \cap N(w)$. Since none of the vertices $x_i, x_{i+1}, w$ is adjacent to $H$, each is adjacent to all but at most $\frac{s}{2}$ vertices of the cycle. Thus, $|N| \ge |C| - \frac{3s}{2}$.

**Claim 4** *For some $j$, $|N \cap y_jCx_{j+1}| \ge 4$.*

Otherwise,

$$5k < |R| \le 3t + |R| - |N| \le 3t + \frac{3s}{2},$$

a contradiction.

Let $j$ be as in the last claim, and let $v_1, v_2, v_3, v_4 \in N \cap y_jCx_{j+1}$ be the first four of these vertices in that order.

If $v_4 \in y^+Cx_i$, define a new cycle as follows: $C' = zC^-v_4x_{i+1}CyHz$.

If $v_4 \in z^+Cx_{i+1}$, let $C' = zC^-x_iv_4CyHz$.

Otherwise observe that there is at most one neighbor $x$ of $H$ in $v_1Cv_4$.

For $j \ne i$, define the new cycle $C'$ as follows:

If $x \in v_1Cv_2$, let $C' = zC^-x_iv_3x_{i+1}Cv_2wv_4CyHz$.

If $x \in v_3Cv_4$, let $C' = zC^-x_iv_2x_{i+1}Cv_1wv_3CyHz$.

Otherwise, let $C' = zC^-x_iv_2Cv_3x_{i+1}Cv_1wv_4CyHz$.

For $i = j$, a very similar construction works:

let $C' = zC^-v_4wv_1C^-x_iv_2Cv_3x_{i+1}CyHz$.

In any case, no vertex in $C - C'$ is adjacent to $H$, so all of them have high degree to $C$ and thus high degree to $R \cap C'$. Therefore, we can insert them one by one into $C'$ creating a longer cycle, a contradiction, completing Case 1.

**Case 2** *Suppose* $y^+ = x_i, z^+ = x_{i+1}$.

Let $N' = N(x_i) \cap N(x_{i+1})$. Then $|N'| \geq |C| - s$.

**Claim 5** *For some* $l$, $|N' \cap y_l C x_{l+1}| \geq 5$.

Otherwise,

$$5k < |R| \leq 4t + |R| - |N'| \leq 4t + s,$$

a contradiction.

Let $l$ be as in the last claim, and let $z_1, z_2, z_3, z_4, z_5 \in N' \cap y_l C x_{l+1}$ be the first five of these vertices in that order. At most one of them is adjacent to $H$, say $z_2$. Now a very similar argument as in the last case gives the desired contradiction, just replace $x_i$ by $z_1$, $x_{i+1}$ by $z_5$, and $w$ by $z_4$. One possible cycle would then be (for $l < j < i$): $C' = zC^- x_i z_2 C z_3 x_{i+1} C z_1 v_2 C v_3 z_5 C v_1 z_4 v_4 C y H z$. □

**Theorem 9** *If* $s = t = k \geq 3$ *or* $0 \leq s < t < k$, *and* $G$ *is a graph of order* $n \geq \max\{178t + k, 8t^2 + k\}$ *with*

$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0 \\ n + k + s - 4 & \text{if } 0 < 2s \leq t \\ n + k + \frac{t-9}{2} & \text{if } 2s > t \end{cases},$$

*then* $G$ *is strongly* $(k, t, s)$*-ordered.*

**Proof of Theorem 9.** To simplify the proof, we will first use an induction argument on $k$: The statement is obviously true for the base cases $(s = 0, t = 1, k = 2)$ and $(s = t = k = 3)$, since $G$ then is 2-connected. Suppose the statement is true for all $k \leq k_0$. We need to show the statement for $k = k_0 + 1$. So, let $G$ be a graph of order $n \geq \max\{178t + k, 8t^2 + k\}$ satisfying the degree condition for some triple $(k, t, s)$. We need to show that for any $(k, t, s)$-linear forest $L$ in $G$, we can find a cycle passing through it in the designated order and direction. Let $L$ be such a forest. Delete all inner vertices of the paths from $V(G)$, and replace the paths by edges to create a new graph $G'$ and a new linear forest $L'$. If there are any paths of three or more vertices in $G$, this will reduce the order of $G$ and the order of $L$. Finding a cycle in $G'$ through $L'$ yields a cycle in $G$ through $L$. Since $k' = 2t - s, n' = n - (k - k') \geq \max\{178t + k', 8t^2 + k'\}$, and

$$\sigma_2(G') \geq \sigma_2(G) - 2(k - k') \geq \begin{cases} n' + k' - 3 & \text{if } s = 0 \\ n' + k' + s - 4 & \text{if } 0 < 2s \leq t \\ n' + k' + \frac{t-9}{2} & \text{if } 2s > t \end{cases},$$

there is such a cycle in $G'$ if $k' < k$, by the induction hypothesis. Thus, we may assume that $k' = k$, and so $L = L'$, meaning that $L$ consists only of paths with one or two vertices.

**Claim 1** $G$ *has a* $t$*-linked subgraph* $H$.

All vertices of $G$ with $d(v) < \frac{n}{2}$ have to be adjacent. If there are at least $2t$ of them, this clique is $H$. Otherwise $|E(G)| \geq (n-2t)\frac{n}{4} \geq 44tn$, which implies by Corollary 3 that $G$ contains a $22t$-connected subgraph $H$. By Theorem 1, $H$ is $t$-linked.

**Claim 2** *$G$ is $t$-linked (and thus $(2t-s,t,s)$-ordered) or $V(G) = V(A) \cup V(B)$, where $|A| \leq |B| + 2t - 1$, $B$ is $t$-linked, and $A$ is either $t$-linked or complete.*

If $G$ is $2t$-connected, then $G$ is $t$-linked by Lemma 1. So assume there is a cut set $K$ with $|K| < 2t$. Let $A'$ and $B'$ be two components of $G - K$ with $|A'| \leq |B'|$. Let $v \in A', w \in B'$. Then

$$n + 2t - s - 3 \leq d(v) + d(w) \leq |A'| + |B'| + 2|K| - 2 \leq n + 2t - 3,$$

so $u$ and $v$ can miss a total of at most $s$ possible adjacencies. Since $|B'| > \frac{n}{2} - t$, this ensures $B'$ to be $22t$-connected and thus $t$-linked. If $A'$ is complete, we are done. Otherwise, the degree sum condition insures $|A'| \geq \frac{n-2t-s+1}{2}$, so $A'$ is $22t$-connected and thus $t$-linked. To find $A$ and $B$, we now partition the vertices of $K$ as follows one-by-one: Add any vertex $u \in K$ with degree $d_{B'}(u) \geq 2t-1$ to $B'$, and add the remaining vertices to $A'$. The result will be as desired, as can be seen step by step: If $u$ has high ($\geq 2t-1$) degree to $B'$, adding it to $B'$ will leave $B'$ $t$-linked by Lemma 2. If $u$ has low degree to $B'$, it must be either adjacent to all of $A'$ or have high degree to $A'$ by the degree sum condition. In both cases, $A'$ stays complete (if $|A'| < 2t$), or $A'$ stays $t$-linked (note that a complete graph on $2t$ vertices is $t$-linked), again by Lemma 2. This proves the claim.

**Case 1** *Suppose $t < 2s$.*

First, we may assume that $t \geq 3$. Otherwise, $t = s \leq 2$, and there is nothing to prove. We will use $A'$ and $B'$ as defined in the proof of Claim 2 above. There is a vertex $v \in B'$ with $d_A(v) = 0$: For every vertex $w \in A'$ we have $d_{B'}(w) = 0$, and for every $w \in A \cap K$ we have $d_{B'}(w) \leq 2t-2$. Since there are at most $2t-1$ vertices in $A \cap K$, at most $(2t-2)(2t-1) < |B'|$ vertices can have $d_A(v) > 0$.

Therefore, by the degree sum condition, we have $d_B(w) \geq 2t - s + \frac{t-5}{2}$ for every $w \in A$. Let $L = \{x_1y_1, x_2y_2, \ldots, x_ty_t\}$, where $x_i = y_i$ if the path is a singleton, and all paths are directed from $x_i$ to $y_i$ (remember: all paths are either edges or singletons by the induction hypothesis). We need to find paths from $y_i$ to $x_{i+1}$. Let

$$\begin{aligned}
L_A &= L \cap A, \\
L_B &= L \cap B, \\
L'_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cup \{y_i \in L_A | x_{i+1} \in L_B\}, \\
L'_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cup \{y_i \in L_B | x_{i+1} \in L_A\}, \\
S_A &= \{x_i \in L_A | y_{i-1} \in L_B\} \cap \{y_i \in L_A | x_{i+1} \in L_B\}, \\
S_B &= \{x_i \in L_B | y_{i-1} \in L_A\} \cap \{y_i \in L_B | x_{i+1} \in L_A\}.
\end{aligned}$$

By these definitions we get

$$|L'_A| + |S_A| = |L'_B| + |S_B|.$$

For $x_i \in L'_A$, let $N'(x_i) = (N(x_i) \cap B) - (L - \{y_{i-1}\})$.
For $y_i \in L'_A$, let $N'(y_i) = (N(y_i) \cap B) - (L - \{x_{i+1}\})$.

For $X \subset L'_A$, let
$$N'(X) = \bigcup_{x_i \in X} N'(x_i) \cup \bigcup_{y_i \in X} N'(y_i).$$
For $t = s = 3$, there is nothing to prove. For $t = 3, s = 2$, we get for every nonempty $X \subset L'_A$,
$$|N'(X)| \geq 3 - |L_B| + |X| + |X \cap S_A| \geq |X| + |X \cap S_A|.$$
For $t \geq 4$ we get for every nonempty $X \subset L'_A$,
$$\begin{aligned} |N'(X)| &\geq 2t - s + \frac{t-5}{2} - |L_B| + |X| + |X \cap S_A| - |S_B| \\ &= |X| + |X \cap S_A| + |L_A| - |S_B| + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + |L'_A| - |S_B| + \frac{t-5}{2} \\ &= |X| + |X \cap S_A| + \frac{|L'_A| - |S_B| + |L'_B| - |S_A|}{2} + \frac{t-5}{2} \\ &\geq |X| + |X \cap S_A| + \frac{t-5}{2}. \end{aligned}$$
Thus, $|N'(X)| \geq |X| + |X \cap S_A|$, and thus by Hall's Theorem, we can find disjoint neighbors for all $x_i, y_i \in L'_A$ in $N'(x_i)$ or $N'(y_i)$, respectively. Using that $B$ is $t$-linked and that $A$ is $t$-linked or complete, we can now find the desired cycle.

**Case 2** *Suppose* $s = 0$.

The degree condition forces $G$ to be $(2t-1)$-connected. If $G$ is $2t$-connected, then it is $t$-linked and we are done. If $G$ has a cut set $K$ of size $2t-1$, the degree condition forces $G - K$ to consist of two complete components $A'$ and $B'$, both of which are adjacent to all ertices in $K$. It is easy to see that such a graph is $t$-linked.

**Case 3** *Suppose* $0 < s \leq t/2$.

The degree condition forces $G$ to be $(2t-2)$-connected. If $G$ is $2t$-connected, then it is $t$-linked and we are done. If $G$ has a cut set $K$ of size $2t-2$, the degree condition forces $G - K$ to consist of two complete components $A'$ and $B'$, both of which are adjacent to all vertices in $K$. It is easy to see that such a graph is $(2t-s, t, s)$-ordered. If $K$ has size $2t-1$, $G$ has a very similar structure. Again, it is straightforward to verify the claim. □

**Theorem 10** *If* $0 \leq s \leq t \leq k$, *and* $G$ *is a graph of order* $n \geq \max\{178t + k, 8t^2 + k\}$ *with*
$$\sigma_2(G) \geq \begin{cases} n + k - 3 & \text{if } s = 0, t \geq 3 \\ n + k + s - 4 & \text{if } 0 < 2s \leq t, t \geq 3 \\ n + k + \frac{t-9}{2} & \text{if } 2s > t \geq 3 \\ n + k - 2 & \text{if } s \leq 1, t = 2 \\ n + k - 1 & \text{if } s = 0, t = 1 \\ n & \text{if } s = t \leq 2 \end{cases},$$
*then* $G$ *is strongly* $(k, t, s)$*-ordered hamiltonian.*

**Proof:** Apply Theorem 6 and Theorem 9. □

## 3 Sharpness

Theorem 6 is sharp for $s = 0$, illustrated by the following graph: Let $A = K_{\frac{n+k-t-1}{2}}$, and $B$ be a set of $\frac{n-k+t+1}{2}$ isolated vertices. Add all edges between $A$ and $B$. For $n$ sufficiently large, $G$ is strongly $(k,t,s)$-ordered, and $\sigma_2(G) = n+k-t-1$. But $G$ is not strongly $(k,t,s)$-ordered hamiltonian, since no hamiltonian cycle can contain a $(k,t,s)$-linear forest $L$ which completely lies inside $A$: Every hamiltonian cycle has exactly $k-t-1$ edges in $A$, one edge less than $L$.

The following graph shows sharpness of Theorem 9, $s = 0$. Let $G$ consist of three complete graphs: $A = K_{\frac{n-k+2}{2}}, K = K_{k-2}, B = K_{\frac{n-k+2}{2}}$. Add all edges between $A$ and $K$ and all edges between $K$ and $B$. The degree sum condition is just missed, but $G$ is not $(k,t,0)$-ordered: Let $x_1 \in A$, $y_t \in B$, $\langle L - \{x_1, y_t\}\rangle = K$.

The following graph shows sharpness of Theorem 9, $t \geq 2s \geq 2$. Let $G$ consist of four complete graphs: $S = K_s, T = K_{k-s}, A = K_{2s-1}, B = K_{n-k-2s+1}$. Add all edges from $A$, all edges between $T$ and $B$. For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$. Add all edges between $S$ and $T$ but the edges $s_iu_i, s_iv_i$. We have $\sigma_2(G) = n+k+s-5$, but if we pick $V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq s$, there is no cycle passing through $L$ in the designated order and direction.

The following graph shows sharpness of Theorem 9, $2s > t$. Let $G$ consist of four complete graphs: $S = K_{\lceil \frac{t}{2} \rceil}, T = K_{k-\lceil \frac{t}{2} \rceil}, A = K_{t-1}, B = K_{n-k-2s+1}$. Add all edges from $A$, all edges between $T$ and $B$. For every vertex $s_i \in S$, pick two vertices $u_i, v_i \in T$, with the exception that $v_{i+1} = u_i$ for $1 \leq i \leq s - \lceil \frac{t}{2} \rceil$. Add all edges between $S$ and $T$ but the edges $s_iu_i, s_iv_i$. We have $\sigma_2(G) = n+k+\lfloor \frac{t}{2} \rfloor - 5$, but if we pick $V(L) = V(S) \cup V(T)$, such that $x_{2i} = y_{2i} = s_i, x_{2i+1} = u_i, y_{2i-1} = v_i$ for all $i \leq \lceil \frac{t}{2} \rceil$, there is no cycle passing through $L$ in the designated order and direction.

## 4 Note added in proofs

Very recently, Thomas and Wollan [8] have improved the bound in Theorem 1 to the following.

**Theorem 11** *If a graph $G$ is $2k$-connected and has at least $5k|V(G)|$ edges, then $G$ is $k$-linked.*

**Corollary 12** *Every $10k$-connected graph is $k$-linked.*

Using these results in place of Theorem 1 will improve some of the bounds on $n$.

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