Combined degree and connectivity conditions for H-linked graphs

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Abstract

For a given multigraph H, a graph G is H-linked, if $|G| \geq |H|$ and for every injective map $\tau : V(H) \to V(G)$, we can find internally disjoint paths in G, such that every edge from uv in H corresponds to a $\tau(u) - \tau(v)$ path.

To guarantee that a G is H-linked, you need a minimum degree larger than $\frac{|G|}{2}$. This situation changes, if you know that G has a certain connectivity k. Depending on k, even a minimum degree independent of $|G|$ may suffice. Let $\delta(k, H, N)$ be the minimum number, such that every k-connected graph G with $|G| = N$ and $\delta(G) \geq \delta(k, H, N)$ is H-linked. We study bounds for this quantity. In particular, we find bounds for all multigraphs H with at most three edges, which are optimal up to small additive or multiplicative constants.

1 Introduction and notation

All graphs and multigraphs considered here are loopless. For concepts and notation not defined here we refer the reader to Diestel's book ([1]).

A *separation* of a graph G consists of two sets $A, B \subseteq V(G)$ with $A \cup B = V$ and no edges between $A \setminus B$ and $B \setminus A$. If $|A \cap B| = k$ then the separation is called a k*-separation*.

Now let H be a multigraph. A graph G is H*-linked*, if $|G| \geq |H|$ and for every injective map $\tau : V(H) \to V(G)$, we can find internally disjoint paths in G, such that every edge from uv in H corresponds to a $\tau(u) - \tau(v)$ path. This concept generalizes several concepts of connectivity studied before. If H is a star with k edges (or a k-multi-edge), then H-linked graphs are exactly the k-connected graphs. If H is a cycle with k edges, then H-linked graphs are exactly the k-ordered graphs. Finally, if H is a matching with k edges, then H-linked graphs are exactly the k-linked graphs.

The following are easy facts about H-linked graphs. Detailed proofs for Facts 1.1 and 1.2 can be found in [5].

Fact 1.1. *Let H_1 and H_2 be multigraphs and suppose that H_2 is a submultigraph of H_1. Then every H_1-linked graph is H_2-linked.*

Fact 1.2. *Let H_1 and H_2 be multigraphs and suppose that one gets H_2 from H_1 through the identification of two non-adjacent vertices, one of which has degree 1. Then every H_1-linked graph is H_2-linked.*

Corollary 1.3. *Let H be a multigraph without isolated vertices. Then every $|E(H)|$-linked graph is H-linked.*

Fact 1.4. *Let H be a multigraph with a k-multi-edge. Then, every H-linked graph is $(|H| - 2 + k)$-connected.*

The minimum degree required for a graph to be k-linked is well understood. Kawabarayashi, Kostochka and Yu prove the following sharp bounds.

Lemma 1.5 ([4]). *Let G be a graph on $N \geq 2k$ vertices with minimum degree*

$$\delta(G) \geq \begin{cases} \frac{N+2k-3}{2}, & \text{if } N \geq 4k-1 \\ \frac{N+5k-5}{3}, & \text{if } 3k \leq N \leq 4k-2 \\ N-1, & \text{if } 2k \leq N \leq 3k-1 \end{cases}.$$

Then G is k-linked.

Note that the degree bounds above imply that the graph G is $(2k-1)$-connected. Further, if G has a $(2k-1)$-separation (A, B), the bounds allow missing edges in $G[A]$ and $G[B]$ only inside $A \cap B$. On the other hand, if G is $2k$-connected, then an average (and thus a minimum) degree constant in N is sufficient, the best known bound was found by Thomas and Wollan.

Theorem 1.6 ([6]). *If G is $2k$-connected and G has at least $5k|V(G)|$ edges, then G is k-linked.*

For $k = 3$, Thomas and Wollan strengthen this bound to a sharp bound. Given a graph G and a set $X \subset V(G)$, the pair (G, X) is called *linked*, if for every set $\{x_1, \ldots, x_k, y_1, \ldots, y_k\} \subseteq X$ of $2k \leq |X|$ disjoint vertices, there are k disjoint $x_i - y_i$ paths with no internal vertices in X.

Theorem 1.7 ([7]). *Let G be a graph, an let $X \subset V(G)$ with $|X| = 6$. If G has no 5-separation (A, B) with $X \subseteq A$ and G has at least $5|V(G)| - 26$ edges outside of $G[X]$, then (G, X) is linked.*

Corollary 1.8 ([7]). *If G is 6-connected and G has at least $5|V(G)| - 14$ edges, then G is 3-linked.*

Corollary 1.9 ([7]). *If G is 6-connected and $\delta(G) \geq 10$, then G is 3-linked.*

Similarly, bounds have been known for a long time for the case $k = 2$.

Theorem 1.10 ([3]). *Let G be a 4-connected graph, which is either non-planar or triangulated. Then G is 2-linked.*

Corollary 1.11. *If G is 4-connected and $\|G\| \geq 3|G| - 6$, then G is 2-linked.*

Corollary 1.12. *If G is 4-connected and $\delta(G) \geq 6$, then G is 2-linked.*

In a sense, there is a rather sharp threshold for k-linked graphs. If $\kappa(G) = 2k - 2$, then G is not k-linked. If $\kappa(G) = 2k - 1$, we need to give very strong (linear) degree conditions to guarantee that G is k-linked. If $\kappa(G) = 2k$, then weak (constant) degree conditions suffice. Our program is to study similar behavior in the more general setting of H-linked graphs. In particular, we want to study the following quantity.

Definition 1.13. *Let H be a multigraph, and let $k \geq 0$. Choose $N \geq k + 1$ large enough, so that K^N is H-linked, and define*

$$\delta(k, H, N) := \min\{\delta \in \mathbb{N}_{\geq k} : \text{ every } k\text{-connected graph on } N \text{ vertices with } \delta(G) \geq \delta \text{ is } H\text{-linked}\}.$$

Due to the following simple fact, we will restrict our attention to multigraphs H without isolated vertices for the rest of the paper.

Fact 1.14. *Let H be a multigraph. Then $\delta(k + 1, H \cup v, N + 1) = \delta(k, H, N) + 1$.*

We can state some of the above Theorems and facts along the lines of our program.

Theorem 1.15. *Let H be a connected bipartite multigraph with ℓ edges, where one of the two parts of the bipartition contains only one vertex. Then*

$$\delta(k,H,N)=\begin{cases}\left\lceil\frac{N+\ell-2}{2}\right\rceil, & \text{for } k<\ell,\\ k, & \text{for } k\geq \ell.\end{cases}$$

Theorem 1.16.

$$\begin{array}{rcccll} & & \delta(k,\ell\, K^2,N) & = & \left\lceil\frac{N+2\ell-3}{2}\right\rceil+o(1), & \text{if } k<2\ell,\\ \max\{2\ell+2-o(1),k\} & \leq & \delta(k,\ell\, K^2,N) & \leq & \max\{10\ell,k\}, & \text{if } k\geq 2\ell.\end{array}$$

Theorem 1.17.

$$\begin{array}{rcccll} & & \delta(k,3\, K^2,N) & = & \left\lceil\frac{N+3}{2}\right\rceil+o(1), & \text{if } k<6,\\ \max\{8-o(1),k\} & \leq & \delta(k,3\, K^2,N) & \leq & \max\{10,k\}, & \text{if } k\geq 6.\end{array}$$

Theorem 1.18.

$$\begin{array}{rcccll} & & \delta(k,2\, K^2,N) & = & \left\lceil\frac{N+1}{2}\right\rceil+o(1), & \text{if } k<4,\\ \max\{6-o(1),k\} & \leq & \delta(k,2\, K^2,N) & \leq & \max\{6,k\}, & \text{if } k\geq 4.\end{array}$$

To get the lower bounds in Theorems 1.16, 1.17 and 1.18, construct a not ℓ-linked but $(2\ell+1)$-connected graph from a planar, not triangulated 5-connected graph by adding $2\ell-4$ *universal* vertices, which are connected to all other vertices of the graph. We will finish this section with a small new result.

Theorem 1.19.

$$\delta(k,K^3,N)=\begin{cases}\left\lceil\frac{N}{2}\right\rceil, & \text{if } k<2,\\ \left\lceil\frac{N+2}{3}\right\rceil, & \text{if } k=2,\\ k, & \text{if } k\geq 3.\end{cases}$$

Proof. Let $\{x,y,z\}\subseteq V(G)$. For $k<2$, the statement is trivial, as only 2-connected graphs can be K^3-linked, and we need $\delta(G)\geq\frac{N}{2}$ to guarantee $\kappa(G)\geq 2$, in which case the lower bound for $k=2$ gives the result. For $k=2$, let C be a longest cycle in G containing $\{x,y\}$. Then, with a standard Dirac type argument ([2]), $|C|\geq 2\delta(G)\geq\frac{2N+4}{3}$. If z is on C, we are done. Otherwise, $|N(z)\cap C|\geq 3$, and z has at least two neighbors on at least one of xCy and yCx, so we can find a cycle through x, y and z. On the other hand, the graph consisting of three complete graphs on $\frac{N-2}{3}$ (rounded up or down appropriately) vertices, each of them completely connected to two independent vertices, shows the sharpness of the bound.

For $k\geq 3$, the statement is very easy again, as every 3-connected graph admits a cycle through any three given vertices, and is thus K^3-linked. □

Note that the only multigraphs with three edges we have not considered yet are P^4, $K^2\cup P^3$ and $K^2\cup C^2$, where C^2 denotes two vertices connected by a double edge. We will find good bounds for these graphs in the following three sections.

2 $H=P^4$

Let us start this section with a definition.

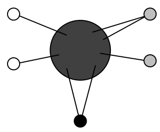
Definition 2.1. *Let G be a graph, and let $\{a,a',b,b',c,c'\}\subseteq V(G)$. Then $(G,\{b,b'\},\{c,c'\},(a,a'))$ is an* obstruction *if for any three vertex disjoint paths from $\{a,b,b'\}$ to $\{a',c,c'\}$, one path is from a to a'.*

Note that if G is a graph which does not contain a path through $a, c, b, a' \in V(G)$ in this order, we can construct an obstruction $(G_{b,c}, \{b, b'\}, \{c, c'\}, (a, a'))$ from G through addition of two vertices $\{b', c'\}$ with $N(b') = N(b)$ and $N(c') = N(c)$. Thus, if we want to find bounds on $\delta(k, P^4, n)$, it will be helpful to know the structure of obstructions.

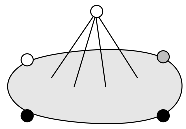
Yu has characterized obstructions in [8]. We will be concerned mostly with connectivity $k \geq 4$, so we will state his results here only for 4-connected graphs. In particular, we omit case (4) in the following definition.

Definition 2.2 ([8]). *Let G be a graph, and $\{a, b, b'\}, \{a', c, c'\} \subseteq V(G)$. Suppose $\{a, b, b'\} \neq \{a', c, c'\}$, and assume that G has no proper 3-separation (G_1, G_2) such that $\{a, b, b'\} \subseteq G_1$ and $\{a, c, c'\} \subseteq G_2$. Then we call $(G, (a, b, b'), (a', c, c'))$ a* rung *if one of the following is satisfied, up to permutation of $\{b, b'\}$ and $\{c, c'\}$.*

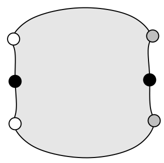
(1) $a = a'$ *or* $\{b, b'\} = \{c, c'\}$;



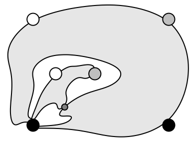
(2) $b = c$ *and* $(G - b, b', c', a', a)$ *is plane;*



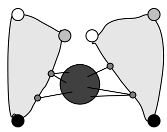
(3) $\{a, b, b'\} \cap \{a', c, c'\} = \emptyset$ *and* (G, b, a, b', c', a', c) *is plane;*



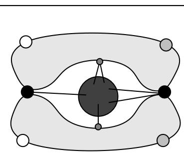
(5) $\{a, b, b'\} \cap \{a', c, c'\} = \emptyset$, (G, b, c, a', a) *is plane, and G has a separation* (G_1, G_2)*, such that* $V(G_1 \cap G_2) = \{z, a\}$ *(or* $V(G_1 \cap G_2) = \{z, a'\}$*),* $\{b, c, a, a'\} \subseteq G_1$*,* $\{b', c'\} \subseteq G_2$*, and* (G_2, b', c', z, a) *(or* (G_2, b', c', a', z)*) is plane;*



(6) $\{a, b, b'\} \cap \{a', c, c'\} = \emptyset$*, and there are pairwise edge disjoint subgraphs* G_1*,* G_2 *and* M *of* G *such that* $G = G_1 \cup G_2 \cup M$*,* $V(G_1 \cap M) = \{u, w\}$*,* $V(G_2 \cap M) = \{p, q\}$*,* $G_1 \cap G_2 = \emptyset$*,* $\{a, b, c\} \subseteq G_1$*,* $\{a', b', c'\} \subseteq G_2$*, and* (G_1, a, b, c, w, u) *and* (G_2, a', c', b', p, q) *are plane;*



(7) $\{a, b, b'\} \cap \{a', c, c'\} = \emptyset$*, and there are pairwise edge disjoint subgraphs* G_1*,* G_2 *and* M *of* G *such that* $G = G_1 \cup G_2 \cup M$*,* $V(G_1 \cap M) = \{a, a', w\}$*,* $V(G_2 \cap M) = \{a, a', p\}$*,* $V(G_1 \cap G_2) = \{a, a'\}$*,* $\{b, c\} \subseteq G_1$*,* $\{b', c'\} \subseteq G_2$*, and* (G_1, b, c, a', w, a) *and* (G_2, c', b', a, p, a) *are plane;*



Definition 2.3 ([8]). *Let L be a graph and let $R_1, \ldots, R_m$ be edge disjoint subgraphs of L such that*

1. $(R_i, (v_{i-1}, x_{i-1}, y_{i-1}), (v_i, x_i, y_i))$ *is a rung for* $1 \leq i \leq m$*,*
2. $V(R_i \cap R_j) = \{v_i, x_i, y_i\} \cap \{v_{j-1}, x_{j-1}, y_{j-1}\}$ *for* $1 \leq i < j \leq m$*,*
3. *for* $0 \leq i \leq k \leq j \leq m$*, we have* $(v_i = v_j \implies v_i = v_k)$*,* $(x_i = x_j \implies x_i = x_k)$ *and* $(y_i = y_j \implies y_i = y_k)$*,*
4. $L = (\bigcup_{i=1}^m R_i) + S$*, where* $S \subseteq \bigcup_{i=0}^m \{v_i x_i, v_i y_i, x_i y_i\}$*.*

Then we call L a ladder along $v_0 \dots v_m$.

Due to condition 4 in the last definition, we may assume that there are no edges in $R_i[v_{i-1}, x_{i-1}, y_{i-1}]$ and $R_i[v_i, x_i, y_i]$ for $1 \le i \le m$.

For a sequence S, the *reduced sequence of* S is the sequence obtained from S by removing a minimal number of elements such that consecutive elements differ. After these definitions, we are ready to formulate the version of the characterization theorem for obstructions, restricted to 4-connected graphs.

Theorem 2.4 ([8]). *Let G be a 4-connected graph, $\{a, b, b'\}, \{a', c, c'\} \subseteq V(G)$. Then the following are equivalent.*

1. $(G, \{b, c\}, \{b', c'\}, (a, a'))$ *is an obstruction,*
2. *G has a separation (J, L) such that $V(J \cap L) = \{w_0, \dots, w_n\}$, $(J, w_0, \dots, w_n)$ is plane, and $(L, (a, b, b'), (a', c, c'))$ is a ladder along $v_0 \dots v_m$, where $v_0 = a$, $v_m = a'$ and $w_0 \dots w_n$ is the reduced sequence of $v_0 \dots v_m$.*

With the help of Theorem 2.4, we can determine fairly sharp bounds for $\delta(k, P^4, N)$.

Theorem 2.5.

$$\begin{array}{rcccl} & & \delta(k, P^4, N) & = & \left\lceil \frac{N+1}{2} \right\rceil, \text{ if } k \le 3 \text{ and } N \ge 14, \\ \sqrt{N+1} & \le & \delta(4, P^4, N) & \le & \sqrt{N} + 5, \\ \sqrt[3]{N} + 2.7 & \le & \delta(5, P^4, N) & \le & \sqrt[3]{N} + 4.2 + o(1) \le \sqrt[3]{N} + 6, \\ 6 & \le & \delta(6, P^4, N) & \le & 8, \\ & & \delta(6, P^4, N) & = & 8, \text{ if } N \ge 418 \\ & & \delta(k, P^4, N) & = & k, \text{ if } k \ge 7. \end{array}$$

Proof.

Case 2.5.1. $k \le 3$

If G has minimum degree $\delta(G) \ge \frac{N+1}{2}$ and a 3-separation (A, B), then $G[A]$ and $G[B]$ can have missing edges only inside $A \cap B$. In this case, it is easy to check that G is P^4-linked for $N \ge 6$. If G has no 3-separation, then G is 4-connected and the result follows from the case $k = 4$ for $N \ge 14$.

To show that $\delta(k, P^4, N) > \lfloor \frac{N}{2} \rfloor$ consider a graph G consisting of two complete graphs G_1 and G_2 with $|G_1| = \lceil \frac{N+2}{2} \rceil$, $|G_2| = \lfloor \frac{N+2}{2} \rfloor$, and $|G_1 \cap G_2| = 2$, and an additional edge p_1p_4 with $p_1 \in V(G_1 \setminus G_2)$ and $p_4 \in V(G_2 \setminus G_1)$. If we choose $p_3 \in V(G_1 \setminus G_2)$ and $p_2 \in V(G_2 \setminus G_1)$, then G contains no path passing through p_1, p_2, p_3, p_4 in order.

Case 2.5.2. $k = 4$

First, we will construct a graph G demonstrating that $\delta(4, P^4, N) \ge \sqrt{N+1}$. Let $\delta \ge 4$. Let Z_i, $1 \le i \le \delta - 1$ be complete graphs with $|V(Z_i)| = \delta + 1$. Let $\{a_i, b_i, x_i, y_i\} \subset V(Z_i)$, where $a_1 = b_2$, and otherwise the $V(Z_i)$ are disjoint. Let $V(G) = \{p_1, p_4\} \cup \bigcup V(Z_i)$, and add all edges a_ib_{i+1}, x_ix_{i+1}, y_iy_{i+1} for $1 \le i \le \delta - 2$. Further, add the edge p_1p_4 and edges from p_1 to the first $\delta - 1$ vertices of the path $P = b_1b_2a_2b_3 \dots b_{\delta-1}a_{\delta-1}$, and from p_4 to the last $\delta - 1$ vertices of P (see Figure 2). Then $\delta(G) = \delta$, G is 4-connected, and $N = |V(G)| = \delta^2$. Further, there is no path containing p_1, $a_{\delta-1}$, b_1, p_4 in order.

To show that $\delta(4, P^4, N) < \sqrt{N} + 5$, assume that G is a 4-connected graph on n vertices with minimum degree $\delta \ge 7$ (the statement is trivial for $\delta \le 6$), and assume that G contains vertices a, c, b, a', but no path contains the vertices in the given order. Then $(G_{b,c}, \{b, b'\}, \{c, c'\}, (a, a'))$ is an obstruction and has the structure described in Theorem 2.4. Let us focus on the structure of the first rung $R_1 \subseteq L \subset G_{b,c}$. Since $N(b) = N(b')$, types (3), (5), (5'), (6), (6'), and (7) are not possible. If R_1 is of type (2) or (2'),

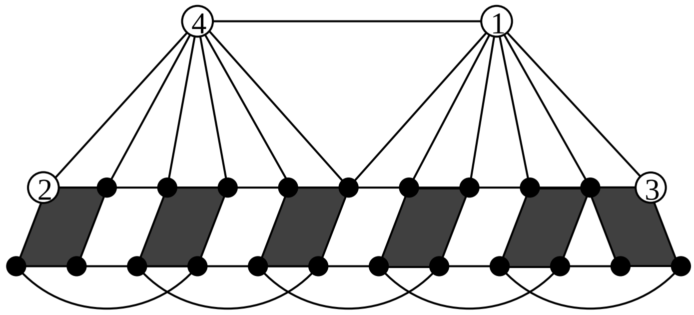


Figure 1: The graph G in Case 2.5.2

then it is in fact also of type (1) by the same reasoning. Since G is 4-connected (and G is obtained from $G_{b,c}$ by contracting $\{b,b'\}$ and $\{c,c'\}$), we can conclude that R_1 is of type (1) with $\{b,b'\} = \{x_1,y_1\}$ and $V(R_1) = \{a,b,b',v_1\}$.

Similarly, R_m is of type (1) with $V(R_m) = \{a',c,c',v_{m-1}\}$. Thus, a and a' each have no neighbors outside of $J \cup \{b,b',c,c'\}$ (and thus, each of them has at least $\delta-1$ neighbors in J). We may assume that $aa' \in E(J)$, otherwise we can add it. Due to Euler's formula, all triangles in J are facial, so

$$|V(J)| \geq |N(a) \cup N(a')| \geq 2(\delta-1) - |N(a) \cap N(a')| \geq 2\delta - 3.$$

Therefore, again with Euler's formula, J has a lot of outgoing edges.

$$|E(J, G \setminus J)| \geq \delta|V(J)| - (6|V(J)| - 12). \tag{1}$$

For every $1 \leq j \leq m-2$, there are at most two vertices in $V(L) \cap N(v_j) \cap N(v_{j+1}) \cap N(v_{j+2})$ due to the ladder structure of L. Every other vertex in $V(L)$ has at most two neighbors in $V(J)$. Noting that $|V(J)| > n$ and $|V(L \cap J)| = n+1$, we have

$$2|V(L)| \geq |E(J, G \setminus J)| - 2n + 2 \geq \delta|V(J)| - 8|V(J)| + 14 \geq 2\delta^2 - 19\delta + 38,$$

and thus $N > (\delta-5)^2$. This shows the claim for $k=4$.

Case 2.5.3. $k = 5$

Construct the graph G as follows (see Figure 2). Let

$$\begin{aligned}
P^1 &= p_1p_2p_3p_4,\\
P^2 &= r_1^1 \ldots (r_1^{\delta-3} = r_2^1) r_2^2 \ldots (r_2^{\delta-2} = r_3^1) r_3^2 \ldots (r_3^{\delta-2} = r_4^1) r_4^2 \ldots r_4^{\delta-3},\\
P^3 &= v_{1,1}^1 \ldots (v_{1,1}^{\delta-4} = v_{1,2}^1) v_{1,2}^2 \ldots (v_{1,2}^{\delta-3} = v_{1,3}^1) \ldots v_{4,\delta-3}^{\delta-4}
\end{aligned}$$

be paths, let $Z_{1,1}^1, \ldots, (Z_{1,1}^{\delta-3} = Z_{1,2}^1), \ldots, Z_{4,\delta-2}^{\delta-3}$ be complete graphs on $\delta+1$ vertices each with $v_{i,j}^k, w_{i,j}^k, x_{i,j}^k, y_{i,j}^k, z_{i,j}^k \in V(Z_{i,j}^k)$ and $\{w_{i,j}^k, x_{i,j}^k\} = \{y_{i,j}^{k+1}, z_{i,j}^{k+1}\}$ for $1 \leq i \leq 4$, $1 \leq j \leq \delta-2$, and $1 \leq k \leq \delta-3$. Add edges $p_ir_i^j$, $r_i^jv_{i,j}^k$, $p_1y_{1,1}^1$, $p_1^1z_{1,1}^1$, $r_1^1y_{1,1}^1$, $r_1^1z_{1,1}^1$, $p_4w_{4,\delta-3}^{\delta-4}$, $p_4x_{4,\delta-3}^{\delta-4}$, $r_4^{\delta-3}w_{4,\delta-3}^{\delta-4}$, and $r_4^{\delta-3}x_{4,\delta-3}^{\delta-4}$.

Then $\delta(G) = \delta$, G is 5-connected, and $N = |V(G)| = 4\delta^3 - 33\delta^2 + 84\delta - 58 < 4(\delta-2.7)^3$. Further, there is no path containing $p_3,\ p_1,\ p_4,\ p_2$ in order. Therefore, $\delta(5, P^4, N) > \sqrt[3]{\frac{N}{4}} + 2.7$.

To show the upper bound for $\delta(5, P^4, N)$, assume that G is a 5-connected graph on N vertices with minimum degree $\delta \geq 7$ (the statement is trivial for $\delta \leq 6$), and assume that G contains vertices

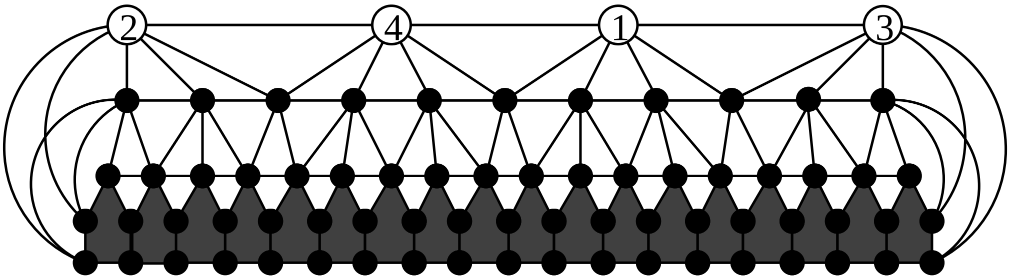


Figure 2: The graph G in Case 2.5.3

a, c, b, a', but no path contains the vertices in the given order. Then $(G_{b,c}, \{b, b'\}, \{c, c'\}, (a, a'))$ is an obstruction and has the structure described in Theorem 2.4. By the same argument as in Case 2.5.2, we can conclude that the first rung $R_1 \subseteq L \subset G_{b,c}$ is of type (1) with $\{b, b'\} = \{x_1, y_1\}$ and $V(R_1) = \{a, b, b', v_1\}$.

But since $d(b) \geq \delta$ and G (and thus $G_{b,c}$) is 5-connected, we have in fact that for $1 \leq i \leq \delta - 3$, $V(R_i) = \{v_{i-1}, v_i, b, b'\}$. Similarly, $V(R_i) = \{v_{i-1}, v_i, c, c'\}$ for $m - \delta + 4 \leq i \leq m$. Further, for $1 \leq i \leq m$, R_i is either of type (1) with $v_{i-1} = v_i$ or $V(R_i) = \{v_{i-1}, x_{i-1}, y_{i-1}, v_i, x_i, y_i\}$. Otherwise, we could find a 4-cut, or there would be a contradiction to Euler's formula in one of the planar subgraphs inside R_i.

Next, we will look at $N(a) \cap L$. If $av_i \in E(G_{b,c})$ for some $2 \leq i \leq m - 1$, then $\{a, v_i, x_i, y_i\}$ is a 4-cut of $G_{b,c}$, so $N(a) \cap L \subseteq \{v_1, v_m, b, b'\}$. Similarly, $N(a') \cap L \subseteq \{v_0, v_{m-1}, c, c'\}$. Therefore,

$$|(N(a) \cup N(a')) \setminus L| \geq 2(\delta - 3) - 1 = 2\delta - 7.$$

Observe that $V(J) \cup \{b, c\}$ induce a planar graph. The vertices in

$$(N(a) \cup N(a')) \setminus L \cup \{v_1, \dots v_{\delta-3}, v_{m-\delta+4}, \dots v_{m-1}\}$$

induce a subgraph of a path, as otherwise $G[V(J) \cup \{b, c\}]$ would contain a separating cycle of length at most 6, which would lead to a contradiction with Euler's formula. This implies that

$$|V(J)| \geq (4\delta - 10)(\delta - 3) + 4 = 4\delta^2 - 22\delta + 34.$$

For every $1 \leq j \leq m - 1$, if $v_j \neq v_{j+1}$, then $V(L) \cap N(v_j) \cap N(v_{j+1}) \subseteq \{x_j, y_j\}$ due to the ladder structure of L. Every other vertex in $V(L)$ has at most one neighbor in $V(J)$. Noting that $|V(J)| \geq m + 2\delta - 7$ and $|V(L \cap J)| = n + 1$, we have (using (1))

$$|V(G)| \geq |E(J, G \setminus J)| - 2(m - 1) + |V(J)| \geq \delta|V(J)| - 7|V(J)| + 4\delta > 4(\delta - 6)^3,$$

and thus $\delta(5, P^4, N) < \sqrt[3]{\frac{N}{4}} + 6$. The last inequality also gives us $|V(L)| > 4(\delta - 4.2)^3$ for $\delta \geq 50$, so $\delta(5, P^4, N) < \sqrt[3]{\frac{N}{4}} + 4.2 + o(1)$.

Case 2.5.4. $k = 6$

It follows from Theorem 1.6 that $\delta(6, P^4, N) \leq 10$, but we can do a little better. First, we will construct a 6-connected graph G with $\delta(G) = 7$, which is not P^4-linked. This is a graph very similar to a graph constructed by Yu in [8], although there he falsely claims that this graph is 7-connected.

Choose n large enough to be able to construct a 3-connected plane graph $(J, w_0, \dots, w_n)$ along the lines of the construction in Case 2.5.3. Add a ladder L along $J \cap L = w_0 w_1 \dots w_n$ as follows. Add vertices x, y, and x_i, y_i for $6 \le i \le n-5$, and edges xw_j, yw_{n-j} for $0 \le j \le 4$, xx_6, xy_6, w_4x_6, w_4y_6, w_5x_6, w_5y_6, yx_{n-5}, yy_{n-5}, $w_{n-4}x_{n-5}$, $w_{n-4}y_{n-5}$, $w_{n-5}x_{n-5}$, $w_{n-5}y_{n-5}$, and all possible edges in $\{w_i, x_i, y_i, x_{i+1}, y_{i+1}\}$ for $6 \le i \le n-6$ (see Figure 3).

If we construct J carefully, then G is 6-connected and has $\delta(G) = 7$. But there is no path through $p_1 = w_0$, $p_2 = y$, $p_3 = x$, $p_4 = w_n$ in order. This construction works for $N = 394$, and with slight adjustments for all $N \ge 418$. Note that $\{x_i, y_i, w_i, w_{i+1}, x_{i+2}, y_{i+2}\}$ is a 6-cut for $6 \le i \le n-7$, so G is not 7-connected.

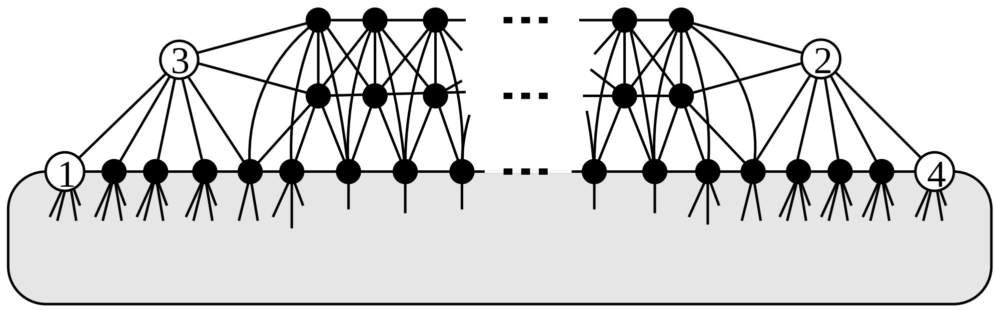


Figure 3: The graph G in Case 2.5.4

To conclude that $\delta(6, P^4, N) \le 8$, assume that there is a 6-connected graph G with $\delta(G) \ge 8$, which is not P^4-linked, i.e., $(G_{b,c}, \{b, b'\}, \{c, c'\}, (a, a'))$ is an obstruction for some $a, a', b, c \in V(G)$, and has the structure given by Theorem 2.4.

Following the same arguments as in Case 2.5.3, we can see that $V(R_i) = \{v_{i-1}, x_{i-1}, y_{i-1}, v_i, x_i, y_i\}$ for $1 \le i \le m$, while $\{x_i, y_i\} = \{b, b'\}$ and $\{x_{m-i}, y_{m-i}\} = \{c, c'\}$ for $0 \le i \le 5$. We now introduce two new types (8) and (8') of rungs $(R, (a, b, b'), (a', c, c'))$. Note that these rungs have proper 3-separations, as opposed to all other types.

(8) $|\{a, b, b', a', c, c'\}| = 6$, and $N(a) \subseteq \{a', b, b'\}$;

(8') $|\{a, b, b', a', c, c'\}| = 6$, and $N(a') \subseteq \{a, c, c'\}$.

In fact, these types of rungs are each two rungs of type (1) in a row, so Theorem 2.4 is still valid if we allow them. But we will use this notation in the following arguments.

Without loss of generality we may assume that $v_i \ne v_{i+1}$ for $1 \le i \le m-1$. Otherwise, either $(G[V(R_i \cup R_{i+1})], (v_{i-1}, x_{i-1}, y_{i-1}), (v_{i+1}, x_{i+1}, y_{i+1}))$ is a rung (possibly of type (8) or (8')), $N(x_{i+1}) \subseteq \{v_i, x_i, y_i, y_{i+1}, v_{i+2}, x_{i+2}, y_{i+2}\}$, or $N(y_{i+1}) \subseteq \{v_i, x_i, y_i, x_{i+1}, v_{i+2}, x_{i+2}, y_{i+2}\}$, contradicting $\delta(G) \ge 8$. Similarly, we may assume that $v_0 \ne v_1$.

Assume that $N(v_i) \cap V(J) \subseteq \{v_{i-1}, vi+1, u\}$ for some $1 \le i \le m-1$ and some $u \in V(J)$. Then $\{x_i, y_i\} \not\subseteq \{x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}\}$ since $d(v_i) \ge 8$. This implies that $x_i, y_i \notin \{x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}\}$, since $d(x_i), d(y_i) \ge 8$, and in fact $N(x_i) \setminus y_i = N(y_i) \setminus x_i = \{v_{i-1}, x_{i-1}, y_{i-1}, v_i, v_{i+1}, x_{i+1}, y_{i+1}\}$. But since $d(v_i) \ge 8$, $N(v_i) \cap \{x_{i-1}, y_{i-1}, x_{i+1}, y_{i+1}\} \ne \emptyset$, but this contradicts the fact that L is a ladder. Therefore, every v_i has at least 4 neighbors in $V(J)$. But this impossible by a simple application of Euler's formula.

Case 2.5.5. $k \ge 7$

We only need to show that every 7-connected graph is P^4-linked, the other bounds follow from Case 2.5.4. We show the slightly stronger statement that obstructions are at most 6-connected.

Let $(G, (a, b, b'), (a', c, c'))$ be an obstruction, and suppose that G is 7-connected. Following the same arguments as in Case 2.5.3, we can see that $V(R_i) = \{v_{i-1}, x_{i-1}, y_{i-1}, v_i, x_i, y_i\}$ for $1 \le i \le m$, while $\{x_i, y_i\} = \{b, b'\}$ and $\{x_{m-i}, y_{m-i}\} = \{c, c'\}$ for $0 \le i \le 3$.

Without loss of generality we may assume that $v_i \neq v_{i+1}$ for $3 \leq i \leq m-4$. Otherwise, either $(G[V(R_i \cup R_{i+1})], (v_{i-1}, x_{i-1}, y_{i-1}), (v_{i+1}, x_{i+1}, y_{i+1}))$ is a rung (possibly of type (8) or (8')) or $\{v_i, x_i, y_i, v_{i+2}, x_{i+2}, y_{i+2}\}$ is a cut set.

Assume that $|N_J(v_i)|, |N_J(v_{i+1})| \leq 3$ for some $2 \leq i \leq m-2$. We will consider

$$S = \{x_i, y_i, x_{i+1}, y_{i+1}\} \setminus \{x_{i-1}, y_{i-1}, x_{i+2}, y_{i+2}\}.$$

To start with, $S \neq \emptyset$, otherwise either $d(v_i) < 7$, $d(v_{i+1}) < 7$, or L is not a ladder. If there is an edge from v_{i-1} into S, then $v_i x_{i-1}, v_i y_{i-1} \notin E(G)$, otherwise R_i is not a rung. As $d(v_i) \geq 7$, $|\{x_i, y_i, x_{i+1}, y_{i+1}\}| = 4$ and $\{x_i, y_i, x_{i+1}, y_{i+1}\} \subset N(v_i)$. This implies that there is no edge from v_{i+1} to $\{x_i, y_i\}$, otherwise R_{i+1} is not a rung. But now, $\{v_{i-1}, x_{i-1}, y_{i-1}, v_i, x_{i+1}, y_{i+1}\}$ is a cut set, a contradiction. Thus, there is no edge from v_{i-1} into S. Similarly, there is no edge from v_{i+2} into S. But this implies that $\{x_{i-1}, y_{i-1}, v_i, v_{i+1}, x_{i+2}, y_{i+2}\}$ is a cut set, a contradiction. Therefore, at least one of $|N_J(v_i)|, |N_J(v_{i+1})|$ must be greater than 3 for $2 \leq i \leq m-2$.

Now consider J and $C = J \cap L$. Without loss of generality we may assume that C is in fact a cycle, otherwise we may add the missing edges, and the resulting graph is still an obstruction. Since G is 7-connected, C has no chords, and $J \setminus C$ is connected. Let B be an end block of $J \setminus C$, and $x \in V(B)$ the only cut vertex of $J \setminus C$ in B (if $B \neq J \setminus C$). B inherits a plane embedding from the embedding of J, and all the vertices on the outer face of this embedding (other than x) have degree $d_B(v) \geq 4$ in B by the argument in the last paragraph (and thus $|V(B)| \geq 5$ and $d_B(x) \geq 2$). Suppose there are k (including x) vertices on the outer face, and ℓ vertices not on the outer face. For those internal vertices, we have $d_B(v) = d(v) \geq 7$. If we now connect all vertices on the outer face with an additional vertex y, the resulting graph B' is still planar. But

$$|E(B')| \geq \frac{4k + 7\ell - 2}{2} + k \geq 3(k + \ell + 1) - 4 > 3|V(B')| - 6,$$

contradicting the planarity of B'. □

3 $K^2 \cup P^3$

Theorem 3.1. *Let* $N \geq 29$. *Then*

$$\begin{array}{rcl} \delta(k, K^2 \cup P^3, N) &=& \lceil \frac{N+2}{2} \rceil, \text{ if } k \leq 3, \\ \delta(4, K^2 \cup P^3, N) &=& \lceil \frac{N+1}{2} \rceil, \\ \sqrt{\frac{N-1}{2}} + 2.25 \; < \; \delta(5, K^2 \cup P^3, N) &<& \sqrt{3N} + 4, \\ 6 \leq 8 - o(1) \; \leq \; \delta(6, K^2 \cup P^3, N) &\leq& 10, \\ k \; \leq \; \delta(k, K^2 \cup P^3, N) &\leq& \max\{k, 10\}, \text{ if } k \geq 7. \end{array}$$

Proof.

Case 3.1.1. $k \leq 3$

By Fact 1.4, every $K^2 \cup P^3$-linked graph is 4-connected. This implies that $\delta(k, K^2 \cup P^3, N) \geq \lceil \frac{N+2}{2} \rceil$. Equality follows from the next case as every graph with minimum degree $\lceil \frac{N+2}{2} \rceil$ is 4-connected.

Case 3.1.2. $k = 4$

To show that $\delta(4, K^2 \cup P^3, N) > \lfloor \frac{N}{2} \rfloor$ consider a graph G consisting of two complete graphs G_1 and G_2 with $|G_1| = \lceil \frac{N+2}{2} \rceil$, $|G_2| = \lfloor \frac{N+2}{2} \rfloor$, and $|G_1 \cap G_2| = 2$, and two additional edges $p_2 b$, $p_1 a$ with $p_2, a \in V(G_1 \setminus G_2)$ and $p_1, b \in V(G_2 \setminus G_1)$. If we choose $p_3 \in V(G_2)$, then G contains no $(K^2 \cup P^3)$-linkage consisting of a $a - b$ path and a $p_1 - p_2 - p_3$ path.

Now let G be a 4-connected graph on N vertices with minimum degree $\delta(G) \geq \lceil \frac{N+1}{2} \rceil$. If G is 5-connected, then G is $K^2 \cup P^3$-linked by the next case, so we may assume that G has a 4-separation (A, B). If N is even, then $|A| = |B| = \frac{N+4}{2}$, and $G[A]$ and $G[B]$ can have missing edges only inside $A \cap B$. Such a graph can easily be seen to be $(K^2 \cup P^3)$-linked.

If N is odd, then we may assume that $|A| = \frac{N+3}{2}$ and $|B| = \frac{N+5}{2}$. Again, $G[A]$ is complete up to some missing edges inside $A \cap B$. Further, $G[B]$ can only miss a matching and then some edges inside $A \cap B$. In particular, there exists a matching with four edges between $B \setminus A$ and A. Such a graph can easily be seen to be $(K^2 \cup P^3)$-linked: given vertices $a, b, p_1, p_2, p_3 \in V(G)$, find a shortest $p_1 - p_2 - p_3$ path using the fewest possible vertices in $A \cap B$ and none of a, b such that the remaining graph is still connected.

Case 3.1.3. $k = 5$

For the lower bound, we will construct a graph similar to those in the proof of Theorem 2.5. Let $\delta \geq 5$. Let Z_i, $1 \leq i \leq 2\delta - 7$ be complete graphs with $|V(Z_i)| = \delta + 1$. Let $\{v_i, x_{i-1}, y_{i-1}, x_i, y_i\} \subset V(Z_i)$, and otherwise the $V(Z_i)$ are disjoint. Let $V(G) = \{a, p_2\} \cup \bigcup V(Z_i)$, let $p_1 = x_{2\delta-7}$, $p_3 = y_{2\delta-7}$ and $b = x_0$. Add the edges ap_i, bp_i for $1 \leq i \leq 2$, p_2y_0, $av_{2\delta-6-j}$, p_2v_j for $1 \leq j \leq \delta - 3$, and v_jv_{j+1} for $1 \leq j \leq 2\delta - 8$ (see Figure 4).

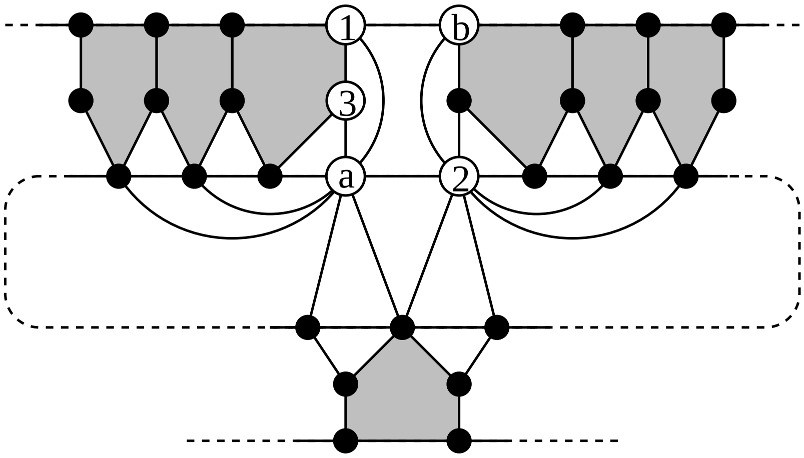


Figure 4: The graph G in Case 3.1.3

Then $\delta(G) = \delta$, G is 5-connected, and $N = |V(G)| = 2\delta^2 - 9\delta + 11 < 2(\delta - 2.25)^2 + 1$. Further, there is not an $a - b$ path and a $p_1 - p_2 - p_3$ path, which are disjoint. Therefore, $\delta(5, K^2 \cup P^3, N) > \sqrt{\frac{N-1}{2}} + 2.25$.

For the upper bound, we will first show the following claim.

Claim 1. *Let G be a graph with minimum degree $\delta = \delta(G) \geq 8$, let $X = \{a, b, p_1, p_2, p_3\} \subset V(G)$. Suppose that G has no 4-separation (A, B) with $X \subseteq A$. Suppose that G does not contain disjoint $a - b$ and $p_1 - p_2 - p_3$ paths, and suppose that no edge can be added without destroying this property.*

Then for every 5-separation (A, B) with $X \subseteq A$ and $p_2 \notin B$, $A \cap B$ induces a K^5.

For the sake of contradiction, choose a 5-separation (A, B) with $X \subseteq A$ and $p_2 \notin B$, for which $G[A \cap B]$ is not complete, such that B is minimal. But now it is easy to see that $(B, A \cap B)$ is linked (you may apply Theorem 1.7 to $G[B] + x$, where the added vertex x is joined to every vertex, concluding that $(G[B] + x, (A \cap B) \cup \{x\})$ is linked). This shows that adding edges within $A \cap B$ will not create disjoint $a - b$ and $p_1 - p_2 - p_3$ paths, showing the claim.

Next, let $N \geq 29$, $\delta = \delta(G) \geq \sqrt{3N} + 5$ and consider the set $\mathcal{S}$ of all 5-separations (A, B) with $X \subseteq A$, $p_2 \notin B$, and B is maximal (i.e., there is no such separation (A', B') with $B \subsetneq B'$). For every such separation, $|B| \geq \delta + 1$, and for every pair of two such separations, $B \cap B' \subset A \cap A'$.

Let

$$\mathcal{A} := V(G) \setminus \bigcup_{(A,B)\in\mathcal{S}} B \; , \; \mathcal{B} := \bigcup_{(A,B)\in\mathcal{S}} A \cap B.$$

Note that $|\mathcal{B}| \leq 5|\mathcal{S}| \leq 5\frac{N-|\mathcal{A}|-|\mathcal{B}|}{\delta-4}$. Consider the graph G' obtained from $G[\mathcal{A} \cup \mathcal{B}]$ by adding a vertex p_2' with $N(p_2') = N[p_2]$. Note that G' has no 5-separation (A, B) with $X \cup p_2' \subseteq A$.

If $|\mathcal{A}| \geq \delta$, then (using that every vertex in $\mathcal{B}$ has at least 6 neighbors in G', $|\mathcal{B}| \leq 5|\mathcal{S}| \leq 5\frac{N}{\delta-4} < \frac{5}{\sqrt{3}}\sqrt{N}$ and $\delta \geq \sqrt{3N} + 4$)

$$|E(G')| \geq \tfrac{\delta}{2}(|\mathcal{A}| + 1) + 3|\mathcal{B}| \geq 5|\mathcal{A}| + 5|\mathcal{B}| - 9 = 5|V(G')| - 14.$$

If $6 \leq |\mathcal{A}| \leq \delta$, then

$$\begin{aligned}|E(G')| &\geq (\delta - \tfrac{1}{2}|\mathcal{A}|)(|\mathcal{A}| + 1) + 2|\mathcal{B}| \\ &\geq (|\mathcal{A}| - 4)\sqrt{3N} - \tfrac{|\mathcal{A}|}{2}(|\mathcal{A}| + 1) + 5|\mathcal{A}| + 5|\mathcal{B}| - 9 \\ &\geq 5|\mathcal{A}| + 5|\mathcal{B}| - 9 = 5|V(G')| - 14.\end{aligned}$$

Therefore, $(G', X \cup p_2')$ is linked by Theorem 1.7, and we can find the desired linkage in G, a contradiction. Thus, $|\mathcal{A}| \leq 5$.

Finally, if $|\mathcal{A}| \leq 5$, note that if p_2 has more than 3 neighbors in some B, then G' contains a K^6 and so $(G', X \cup p_2')$ is linked, a contradiction. Thus,

$$\delta - 4 \leq |N(p_2) \cap \mathcal{B}| \leq 3|\mathcal{S}| \leq 3\frac{N - |\mathcal{A} \cup \mathcal{B}|}{\delta - 4},$$

contradicting the fact that $\delta \geq \sqrt{3N} + 4$. This shows that $\delta(5, K^2 \cup P^3, N) < \sqrt{3N} + 4$ for $N \geq 29$.

Case 3.1.4. $k \geq 6$

The lower bound for $\delta(k, K^2 \cup P^3, N)$ follows from Theorem 2.5, the upper bound follows from Theorem 1.7.

□

4 $K^2 \cup C^2$ and $P^3 \cup P^3$

By Fact 1.2, every $(K^2 \cup P^3)$-linked graph is $(K^2 \cup C^2)$-linked. Thus, all the upper bounds in Theorem 3.1 apply to $\delta(k, K^2 \cup C^2, N)$ as well. As for lower bounds, note that all the examples in the proof of Theorem 3.1 with $k \leq 5$ yield the same lower bounds for $\delta(k, K^2 \cup C^2, N)$ (none of them contains a disjoint $a - b$ path and a cycle through p_1 and p_2). For $k = 6$, we can employ again the example in Case 2.5.4 in the proof of Theorem 2.5, and note that this graph does not contain a disjoint $p_1 - p_2$ path and a cycle through p_3 and p_4. Therefore, we have the following theorem.

Theorem 4.1. *Let $N \geq 29$. Then*

$$\begin{aligned} & & \delta(k, K^2 \cup C^2, N) &= \left\lceil \tfrac{N+2}{2} \right\rceil, & & \text{if } k \leq 3, \\ & & \delta(4, K^2 \cup C^2, N) &= \left\lceil \tfrac{N+1}{2} \right\rceil, & & \\ \sqrt{\tfrac{N-1}{2}} + 2.25 &< & \delta(5, K^2 \cup C^2, N) &< \sqrt{3N} + 4, & & \\ 6 \leq 8 - o(1) &\leq & \delta(6, K^2 \cup C^2, N) &\leq 10, & & \\ k &\leq & \delta(k, K^2 \cup C^2, N) &\leq \max\{k, 10\}, & & \text{if } k \geq 7. \end{aligned}$$

Now that we have considered all multigraphs with up to 3 edges, let us consider graphs H with 4 edges. We can prove the following theorem about $H = P^3 \cup P^3$.

Theorem 4.2. *Let N be large enough.Then*

$$\begin{array}{rcccl} & & \delta(6, P^3 \cup P^3, N) & = & \left\lceil \frac{N+2}{2} \right\rceil, \\ \sqrt{\frac{N-2}{2}} + 3.25 & < & \delta(7, P^3 \cup P^3, N) & < & \sqrt{5N} + 6, \\ 8 & \leq & \delta(8, P^3 \cup P^3, N) & \leq & 40 \end{array}$$

Proof. For the upper bounds, we use Theorem 5.1 and that $\delta(k, P^3 \cup P^3, N) \leq \delta(k, 2K^2 \cup P^3, N)$ by Fact 1.2. For the lower bounds we find examples.

Case 4.2.1. $k = 6$

Let G consist of two complete graphs G_1 and G_2 with $|G_1| = \lceil \frac{N+3}{2} \rceil$, $|G_2| = \lfloor \frac{N+3}{2} \rfloor$, and $|G_1 \cap G_2| = 3$, and three additional edges $p_1q_1,\ p_2q_2,\ p_3q_3$ with $p_i \in V(G_1 \setminus G_2)$ and $q_i \in V(G_2 \setminus G_1)$. Then G contains no $(P^3 \cup P^3)$-linkage consisting of a $p_1 - q_2 - p_3$ path and a $q_1 - p_2 - q_3$ path.

Case 4.2.2. $k = 7$

Let $\delta \geq 7$. Let Z_i, $1 \leq i \leq 2\delta - 9$, be complete graphs with $|V(Z_i)| = \delta + 1$.
Let $\{v_i, x_{i-1}, y_{i-1}, z_{i-1}, x_i, y_i, z_i\} \subset V(Z_i)$, and otherwise the $V(Z_i)$ are disjoint. Let $V(G) = \{p_2, q_2\} \cup \bigcup V(Z_i)$, let $p_1 = x_{2\delta-9},\ p_3 = y_{2\delta-9},\ q_1 = x_0$ and $q_3 = y_0$. Add the edges p_iq_i for $1 \leq i \leq 3$, p_1p_2, p_2p_3, $p_2z_{2\delta-9}$, q_1q_2, q_2q_3, q_2z_0, $q_2v_j,\ p_2v_{2\delta-8-j}$ for $1 \leq j \leq \delta - 4$, and v_jv_{j+1} for $1 \leq j \leq 2\delta - 10$ (see Figure 5).

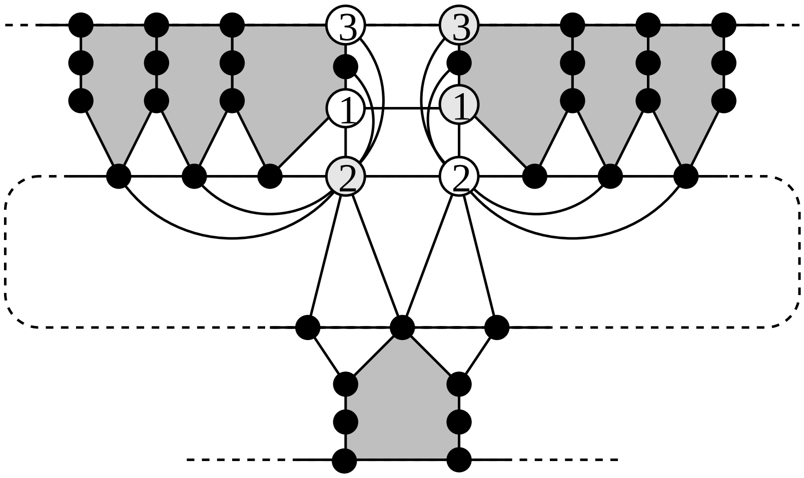


Figure 5: The graph G in Case 4.2.2

Then $\delta(G) = \delta$, G is 7-connected, and $N = |V(G)| = 2\delta^2 - 13\delta + 23 < 2(\delta - 3.25)^2 + 2$. Further, there is not an $p_1 - q_2 - p_3$ path and a $q_1 - p_2 - q_3$ path, which are disjoint. Therefore, $\delta(7, P^3 \cup P^3, N) > \sqrt{\frac{N-2}{2}} + 3.25$.

□

5 Bipartite H with small components

Very similarly to Theorems 3.1 and 4.1, we obtain the following result.

Theorem 5.1. *Let N be large enough, $\ell \geq 1$ and $H \in \{\ell\, K^2 \cup P^3, \ell\, K^2 \cup C^2\}$. Then*

$$\begin{array}{rcccl} & & \delta(2\ell+1, H, N) & = & \left\lceil \frac{N+\ell+1}{2} \right\rceil, \\ & & \delta(2\ell+2, H, N) & = & \left\lceil \frac{N+\ell}{2} \right\rceil, \\ \sqrt{\frac{N-2\ell+1}{2}} + 2\ell + 0.25 & < & \delta(2\ell+3, H, N) & < & \sqrt{(2\ell+1)N} + 2\ell + 2, \\ 2\ell+4 & \leq & \delta(2\ell+4, H, N) & \leq & 10(\ell+2). \end{array}$$

Proof. The proof follows arguments very similar to the proofs of Theorems 3.1 and 4.1, and is left to the reader. The only inequality we elaborate on here is the lower bound for $k = 2\ell + 3$. For this, add $2\ell - 2$ vertices to the bounding graph in Theorem 4.1, and connect them with all other vertices. Making these new vertices the terminals of the extra K^2s it is easy to see that this graph is not $(\ell\, K^2 \cup C^2)$-linked. □

6 Conclusion and open questions

We have determined $\delta(k, H, N)$ for all H with up to three edges, up to some small constant factors. In every case, $\delta(k, H, N) = \Theta(N^{1/\ell})$. Is this the case for all k and H?

We know $\delta(k, H, N)$ only for few H with more than three edges. Very interesting should be the cases $H = C^4$ (as almost always), $H = K^2 \cup K_{1,3}$ and $H = K^2 \cup P^4$. In the last case, we know for sufficiently large N (with a proof similar to Theorem 4.2) that

$$\begin{array}{rcccl} & & \delta(6, K^2 \cup P^4, N) & = & \left\lceil \frac{N+2}{2} \right\rceil, \\ \sqrt[3]{N-2} + 4.7 & \leq & \delta(7, K^2 \cup P^4, N) & < & \sqrt{5N} + 6, \\ 8 & \leq & \delta(8, K^2 \cup P^4, N) & \leq & 40, \end{array}$$

but this leaves quite a gap between the bounds for $\delta(7, K^2 \cup P^4, N)$.

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