AN ITERATIVE APPROACH TO THE IRREGULARITY STRENGTH OF TREES

MICHAEL FERRARA, RONALD J. GOULD, MICHAŁ KAROŃSKI, AND FLORIAN PFENDER

Abstract. An assignment of positive integer weights to the edges of a simple graph G is called irregular if the weighted degrees of the vertices are all different. The irregularity strength, $s(G)$, is the maximal edge weight, minimized over all irregular assignments, and is set to infinity if no such assignment is possible. In this paper, we determine the exact value $s(T)$ for trees T in which every two vertices of degree not equal to two are at distance at least 8, and we give an iterative algorithm that achieves this value.

1. Introduction and Notation

Let $w : E(G) \to \mathbb{N}$ be an assignment of positive integer weights to the edges of a simple graph G. This assignment yields a weighted degree $w(v) := \sum_{v \in e} w(e)$ for all vertices $v \in V(G)$, and is called irregular if the weighted degrees of the vertices are all different. Let $I(G)$ denote the set of irregular labelings of G. Define the irregularity strength $s(G)$ of a simple graph G to be

$$\min_{f \in I(G)} \max_{e \in E(G)} f(e) = s(G)$$

if $I(G)$ is nonempty and $s(G) = \infty$ otherwise. It is readily seen that $s(G) < \infty$ if and only if G contains no isolated edges and at most one isolated vertex.

Graph irregularity strength was introduced in [?] by Chartrand *et al.*. Upper bounds are known for general graphs of order n (Nierhoff [?] shows the sharp bound $s(G) \le n - 1$), and d-regular graphs (Frieze *et al.* [?] show a bound of $s(G) \le c \cdot n/d$ for $d \le n^{1/2}$, and $s(G) \le c \cdot n \log n/d$ for general d, which was recently improved to $s(G) \le c \cdot n/d$ for all d by Przybyło [?]). The exact irregularity strength is known only for very few classes of graphs.

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Let n_i denote the number of vertices of degree i in a graph G. Then a simple counting argument shows that

$$s(G) \geq \lambda(G) := \max_k \left\lceil \frac{1}{k} \sum_{i=1}^{k} n_i \right\rceil .$$

Kinch and Lehel [?] demonstrated, by considering the irregularity strength of tP_3, that $\lambda(G)$ and $s(G)$ may differ asymptotically. They subsequently conjectured that if G is connected graph, then $\lambda(G)$ and $s(G)$ differ by at most an additive constant.

It is easy to see that for trees, $\frac{1}{k}\sum_{i=1}^{k} n_i$ attains its maximum for $k = 1$ or for $k = 2$. Cammack *et al.* [?] show that $s(T) = \lambda(T)$ for full d-ary trees, and Amar and Togni [?] show that $s(T) = \lambda(T) = n_1$ for all trees with $n_2 = 0$ and $n_1 \geq 3$. For general trees, it is not even the case that $s(T)$ is within an additive constant of n_1. Bohman and Kravitz [?] present an infinite sequence of trees with irregularity strength converging to $\frac{11-\sqrt{5}}{8} n_1 > n_1 > \frac{n_1+n_2}{2}$.

In this paper, we present an iterative algorithm showing that $s(T) = \lambda(T)$ for another class of trees, but this time $n_1 < n_2$, i.e. $s(T) = \left\lceil \frac{n_1+n_2}{2} \right\rceil$. We believe that the methods developed here have the potential to be modified and used to determine the irregularity strength of a broader class of trees or more general graphs. The following is the main result of this paper.

Theorem 1. *Let T be a tree in which every two vertices of degree not equal to two are at distance at least 8, and with $n_1 \geq 3$. Then $s(T) = \lambda(T) = \left\lceil \frac{n_1+n_2}{2} \right\rceil$.*

The reader should note that we may obtain T from a tree containing no vertices of degree 2 by subdividing each edge at least 7 times.

2. Proof of Theorem 1

2.1. A helpful lemma.

Repeated application of the following lemma is at the heart of our algorithm.

Lemma 2. *Let $P = v_0 v_1 \ldots v_{\ell+1}$, $\ell \geq 1$ be a path, and let $w_1, w_2, \ldots w_\ell$ be a strictly increasing sequence of natural numbers greater than one, so that all even numbers between w_1 and w_ℓ are part of the sequence. Then there exists a weighting w of the edges of P such that*

1. *$w(v_0)$ is odd,*
2. *$w(v_i) = w_i$ for $1 \leq i \leq \ell$,*
3. *$-1 \leq w(v_i v_{i+1}) - w(v_{i-1} v_i) \leq 2$ for $1 \leq i \leq \ell$.*

Proof. We proceed by induction on ℓ. To begin, let $\ell = 1$. Depending on w_1, we assign edge weights as follows:

w_1	$w(v_0v_1)$	$w(v_1v_2)$
$4k$	$2k-1$	$2k+1$
$4k+1$	$2k+1$	$2k$
$4k+2$	$2k+1$	$2k+1$
$4k+3$	$2k+1$	$2k+2$

Assume now that $\ell \geq 2$ and we are given $w_1, \ldots, w_\ell$. By the induction hypothesis, we can assign weights to the edges of $P' = v_0v_1 \ldots v_\ell$ so that $w(v_0)$ is odd, $w(v_i) = w_i$, and $-1 \leq w(v_iv_{i+1}) - w(v_{i-1}v_i) \leq 2$ for $1 \leq i \leq \ell - 1$.

If $w_\ell - w_{\ell-1} = 1$, let $w(v_\ell v_{\ell+1}) = w(v_{\ell-2}v_{\ell-1}) + 1$. Then $w(v_\ell) = w_\ell$ and

$$-1 \leq \underbrace{w(v_\ell v_{\ell+1}) - w(v_{\ell-1}v_\ell)}_{=w(v_{\ell-2}v_{\ell-1})-w(v_{\ell-1}v_\ell)+1} \leq 2.$$

If $w_\ell - w_{\ell-1} = 2$, and thus $w_{\ell-1}$ is an even number, let $w(v_\ell v_{\ell+1}) = w(v_{\ell-2}v_{\ell-1}) + 2$. Then $w(v_\ell) = w_\ell$ and

$$0 \leq \underbrace{w(v_\ell v_{\ell+1}) - w(v_{\ell-1}v_\ell)}_{=w(v_{\ell-2}v_{\ell-1})-w(v_{\ell-1}v_\ell)+2} \leq 2.$$

□

2.2. **Setting up the weighting.** We are given a tree T in which any two vertices of degree not equal to two are at distance at least 8. We decompose $E(T)$ into edge disjoint paths such that the end vertices of the paths correspond to the vertices in T with degree not equal to 2. If one thinks of T as a subdivision of a tree T' with $n_2(T') = 0$, then each path corresponds to an edge of T'. A bottom vertex in a path is called a pendant vertex if it is a leaf of T and we will call any of these paths a *pendant path* if it contains a pendant edge.

We will root T at a vertex *root* of maximum degree, giving each path a top-to-bottom orientation. We then order the paths in our decomposition of T in the following manner. Select any $d_T(root)$ pendant paths to be the final, or bottom, paths in the ordering. We then order the remaining paths such that any path having bottom vertex v with $d_T(v) \geq 3$ will have exactly $d_t(v) - 2$ pendant paths directly above it in the path ordering. For an example, see Figure **??**.

Let $P_1, \ldots, P_t$ denote the paths under this ordering, where P_1 is the top path. We will also allow this path ordering to induce an order on the vertices of the paths, where x in P_i is below y in P_j if either $i > j$ or $i = j$ and x is below y on P_i.

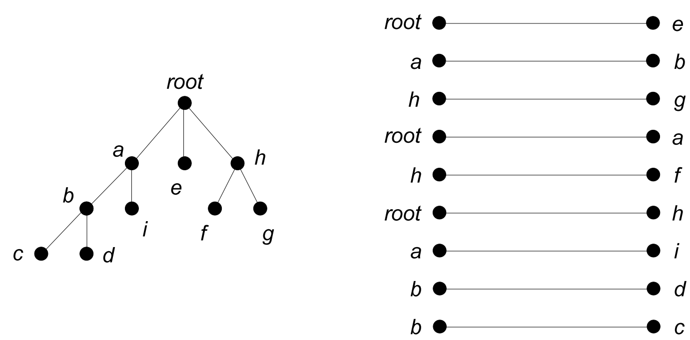

FIGURE 1. A tree and its associated path ordering

Let $\bar{M}$ be the set of vertices of degree two which are right below the top vertex in a path. Let M be the set of all other vertices of degree two. Next, we apply Lemma 2 to the paths in order. We will initially assign the weights $2, \ldots, |M| + 1$ to the vertices in M from the bottom to the top. In applying the lemma, we will require that the lowest internal vertex in each path receive the lowest weight and so on. Finally, for $\lambda = \lambda(T) = \lceil \frac{n_1+n_2}{2} \rceil$, we will label each top edge with weight λ or $\lambda - 1$ so that each vertex in $\bar{M}$ has an odd weight. We call this initial weighting w_0.

Clearly this is not an irregular weighting, as each pendant vertex will certainly have the same weight as some vertex in M preceding it in the ordering. We will attempt to improve our weighting. Let H_0 denote the set of weights of pendant and $\bar{M}$-vertices under w_0. Note that these weights are all different and odd.

We apply Lemma 2 to the paths P_i. In particular, starting with the bottomost vertex in M, we assign to each vertex in M the lowest weight that is neither in H_0 nor assigned to a lower vertex in M. Again, we will conclude this new weighting by labeling each top edge of each path with either λ or $\lambda - 1$ such that the weight of the corresponding vertex in $\bar{M}$ is odd. We will call this new weighting w_1. If this is not an irregular weighting with maximum edge weight at most λ, then we will repeat this process by constructing a weighting that avoids the (odd) weights H_1 of the pendant and $\bar{M}$-vertices in w_1, and so on.

Throughout this process, the following facts hold.

Fact 3. *Let $m_j, m_k \in M$ such that m_j is below m_k. Then $w_i(m_j) < w_i(m_k)$ for any $i > 0$.*

Fact 4. *The weights of the pendant and $\bar{M}$-vertices depend on the weight of their neighbor in M as follows:*

neighbor	*pendant*	*vertex in $\bar{M}$*	
		λ *even*	λ *odd*
$4k$	$2k-1$	$\lambda+2k-1$ *or* $\lambda+2k+1$	$\lambda+2k$
$4k+1$	$2k+1$	$\lambda+2k-1$ *or* $\lambda+2k+1$	$\lambda+2k$
$4k+2$	$2k+1$	$\lambda+2k+1$	$\lambda+2k$ *or* $\lambda+2k+2$
$4k+3$	$2k+1$	$\lambda+2k+1$	$\lambda+2k$ *or* $\lambda+2k+2$

Each of the paths in our decomposition have length at least eight and as such, the second lowest and third highest vertices on each path are distance at least five apart. Consequently when applying Lemma 2 these vertices receive weights at least five apart in any iteration. This observation, along with Fact **??** yields the following lemmas.

Lemma 5. *Let p_j and p_k denote the bottom vertices of P_j and P_k respectively, where $j > k$ and $d_T(p_j) = d_T(p_k) = 1$. Then for any $i \geq 0$, $w_i(p_k) - w_i(p_j) \geq 2$.*

Lemma 6. *Let p_j and p_k denote the bottom vertices of P_j and P_k respectively, where $j > k$ and $d_T(p_j) = d_T(p_k) \geq 3$. Then for any $i \geq 0$,*

$$w_i(p_k) - w_i(p_j) \geq 2d_T(p_j) - 2.$$

Lemma 7. *Let $\{p_j\} = \bar{M} \cap V(P_j)$ and $\{p_k\} = \bar{M} \cap V(P_k)$, where $j > k$. Then for any $i \geq 0$, $w_i(p_k) - w_i(p_j) \geq 2$.*

As a corollary of Lemma **??**, we get the following statement.

Lemma 8. *Let x, y be two different vertices of degree three or more in T, then $w_i(x) \neq w_i(y)$ for all $i \geq 0$.*

Proof. All that is left to show is the inequality for vertices x, y with $d_T(x) > d_T(y) \geq 3$. As *root* is a vertex of maximum degree in T and x is on P_i for some $i \geq d_T(root) + 1 > d_T(x)$, we have

$$w_i(y) \leq \lambda d_T(y) < (\lambda - 1)(d_T(x) - 1) + 3d_T(x) \leq w_i(x).$$

□

Note that w_{i+1} is completely determined by the set $H_i \subset \{1, 2, \ldots, 2\lambda\}$. As there are only finitely many such sets, this process will eventually stabilize in a loop with some period p, i.e. $H_i = H_{i+p}$ for $i \geq i_0$ and minimal $p \geq 1$. If $p = 1$, then w_{i_0+1} is an irregular weighting with maximum edge weight at most λ and we are done, so we assume in the following that $p > 1$.

For $0 \leq i < p$, let $\hat{w}_i = w_{i_0+1+i}$, where all index calculations regarding $\hat{w}$ will be modulo p. Define $\hat{H}_i$ in a similar manner. Let

$$m = \max_{\substack{i,k \\ x \in M}} \hat{w}_i(x) - \hat{w}_{i-k}(x).$$

The parameter m is the maximum amount that the weight of an M-vertex can vary as we iteratively modify the edge labels throughout one period.

The following lemma is crucial for the proof of Theorem 1.

Lemma 9. *$m = 1$.*

Proof. If $m = 0$ then $p = 1$, which we already excluded, so $m \geq 1$ and we need to show that $m \leq 1$. Let $x \in M$ be the lowest vertex in the ordering for which we can find i and k such that $\hat{w}_i(x) - \hat{w}_{i-k}(x) = m$.

The fact that the weight of x has increased by m implies that there are m pendant vertices or $\bar{M}$-vertices $x_1, x_2, \ldots, x_m$, such that for all $t \geq 0$

$$\hat{w}_t(x_1) \leq \hat{w}_t(x_2) - 2 \leq \ldots \leq \hat{w}_t(x_m) - 2m + 2, \tag{1}$$

$$\hat{w}_{i-k-1}(x_1) > \hat{w}_{i-k}(x), \tag{2}$$

and

$$\hat{w}_{i-1}(x_m) < \hat{w}_i(x). \tag{3}$$

Here (**??**) follows from the fact pendants and $\bar{M}$ vertices receive distinct odd weights, while (**??**) and (**??**) follow from the assumption that the weight of x has changed by exactly m. For $m \leq 3$, this implies that

$$\begin{aligned} \hat{w}_{i-1}(x_1) &\leq_{(??)} \hat{w}_{i-1}(x_m) - 2m + 2 \\ &\leq_{(??)} \hat{w}_i(x) - 2m + 1 = \hat{w}_{i-k}(x) - m + 1 \\ &\leq_{(??)} \hat{w}_{i-k-1}(x_1) - m. \end{aligned}$$

Both $\hat{w}_{i-1}(x_1)$ and $\hat{w}_{i-k-1}(x_1)$ are odd, so in fact

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 2 \left\lceil \frac{m}{2} \right\rceil. \tag{4}$$

For $m \geq 4$, we get

$$\begin{aligned} \hat{w}_{i-1}(x_1) &\leq_{(??)} \hat{w}_{i-1}(x_m) - 2m \\ &\leq_{(??)} \hat{w}_i(x) - 2m - 1 = \hat{w}_{i-k}(x) - m - 1 \\ &\leq_{(??)} \hat{w}_{i-k-1}(x_1) - m - 2. \end{aligned}$$

Both $\hat{w}_{i-1}(x_1)$ and $\hat{w}_{i-k-1}(x_1)$ are odd, so in fact

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 2 \left\lceil \frac{m}{2} \right\rceil + 2.$$

Let $y_1 \in M$ be a neighbor of x_1, and suppose that $m \geq 4$. If $d_T(x_1) = 1$, i.e. $\hat{w}_i(x_1) < \lambda$. Fact **??** then yields that

$$\hat{w}_{i-1}(y_1) \leq 2\hat{w}_{i-1}(x_1) + 2$$

and

$$\hat{w}_{i-k-1}(y_1) \geq 2\hat{w}_{i-k-1}(x_1) - 1.$$

Thus,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 2(\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1)) - 3 \geq 4\left\lceil \frac{m}{2} \right\rceil + 1 > m,$$

a contradiction to (**??**).

Now suppose that $d_T(x_1) = 2$. Here, Fact **??** yields that

$$\hat{w}_{i-1}(y_1) \leq 2(\hat{w}_{i-1}(x_1) - \lambda + 1) + 1$$

and

$$\hat{w}_{i-k-1}(y_1) \geq 2(\hat{w}_{i-k-1}(x_1) - \lambda) - 2.$$

Thus,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 4\left\lceil \frac{m}{2} \right\rceil - 1 > m,$$

a contradiction. This implies that $m \leq 3$.

It then follows that between $\hat{w}_i$ and $\hat{w}_{i-k}$, the weight of a vertex in $\bar{M}$ can change by at most four, and the weight of a pendant vertex can change by at most two. This is a consequence of Fact **??**, which shows that if the weight of some pendant vertex were to change by three or more, then the weight of its neighbor in M would change by at least four. A similar analysis shows that no vertex in $\bar{M}$ can have its weight change by more than four.

Next assume that $m = 3$. From (**??**), we know that

$$\hat{w}_{i-k-1}(x_1) - \hat{w}_{i-1}(x_1) \geq 4,$$

thus $x_1, x_2, x_3 \in \bar{M}$. Further,

$$\hat{w}_{i-k-1}(y_1) - \hat{w}_{i-1}(y_1) \geq 3,$$

as otherwise $\hat{w}(x_1)$ could not decrease by four. But since $\hat{w}_{i-1}(x_1) < \hat{w}_{i-1}(x)$, we know that $\hat{w}_{i-1}(y_1) < \hat{w}_{i-1}(x)$, contradicting the choice of x. This shows that $m \leq 2$ and in turn that weights of vertices in $\bar{M}$ can change by at most two, again by Fact **??**.

Finally assume that $m = 2$. All the inequalities in (**??**) are in fact equalities and we get that for some t,

$$\begin{aligned} \hat{w}_{i-k-1}(x_1) = \hat{w}_{i-1}(x_2) &= 2t + 3, \\ \hat{w}_{i-k-1}(x_2) &= 2t + 5, \\ w_{i-1}(x_1) &= 2t + 1. \end{aligned}$$

Let y_2 be the neighbor of x_2 in M. If x_1 is a pendant vertex, then Fact **??** shows that $\hat{w}_{i-1}(y) \geq 4t+1$ and

$$\hat{w}_{i-1}(y_2) \leq \hat{w}_{i-k-1}(y_2) + 2 \leq 4t+6,$$

a contradiction as at least five vertices from M lie between y_1 and y_2 in the path-ordering, so $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \geq 6$.

Thus, $x_1 \in \bar{M}$. As y_1 and y_2 come before x in the ordering,

$$|\hat{w}_{i-1}(y_1) - \hat{w}_{i-k-1}(y_1)| \leq 1 \text{ and } |\hat{w}_{i-1}(y_2) - \hat{w}_{i-k-1}(y_2)| \leq 1.$$

Fact **??** shows that $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \leq 6$ with equality only if

$$|\hat{w}_{i-1}(y_1) - \hat{w}_{i-k-1}(y_1)| = |\hat{w}_{i-1}(y_2) - \hat{w}_{i-k-1}(y_2)| = 1.$$

But this last equality implies that H_{i-2} contains a number between $\hat{w}_{i-1}(y_1)$ and $\hat{w}_{i-1}(y_2)$, and therefore $\hat{w}_{i-1}(y_2) - \hat{w}_{i-1}(y_1) \geq 7$, the final contradiction proving the lemma. $\square$

The vertices outside of M are sufficiently far apart in T to immediately yield the following corollary to Lemma **??**.

Lemma 10. *For two vertices $x, y \notin M$, $\hat{w}_i(x) > \hat{w}_i(y)$ implies $\hat{w}_j(x) > \hat{w}_k(y)$ for all i, j, k.*

2.3. **Clean up.** We will now modify the weighting $\hat{w}_1$ to get an irregular weighting $\hat{w}$ with $\hat{H} = \hat{H}_1$. Let $x \in M$ such that $\hat{w}_1(x) \in \hat{H}_1$. By Lemma **??**, either $\hat{w}_2(x) = \hat{w}_1(x) + 1$ or $\hat{w}_2(x) = \hat{w}_1(x) - 1$. Let $y \in M$ be the neighbor of x with $\hat{w}_1(y) = \hat{w}_2(x)$. Note that there is no vertex z with $\hat{w}_1(z) = 2\hat{w}_2(x) - \hat{w}_1(x)$ (i.e. $\hat{w}_1(z) = \hat{w}_1(x) \pm 2$), as $2\hat{w}_2(x) - \hat{w}_1(x) \in \hat{H}_0 \setminus \hat{H}_1$. We differentiate four cases.

Case 1. *$xy \in E$ and $\hat{w}_2(x) = \hat{w}_1(x) + 1$.*

Set $\hat{w}(xy) = \hat{w}_1(xy) + 1$.

Case 2. *$xy \in E$ and $\hat{w}_2(x) = \hat{w}_1(x) - 1$.*

Set $\hat{w}(xy) = \hat{w}_1(xy) - 1$.

Case 3. *$xy \notin E$ and $\hat{w}_2(x) = \hat{w}_1(x) + 1$.*

Let $x_1 \in M$ be the neighbor of x with $\hat{w}_1(x_1) = \hat{w}_1(x) - 1$, and $y_1 \in M$ be the neighbor of y with $\hat{w}_1(y_1) = \hat{w}_1(x) + 3$. Set $\hat{w}(xx_1) = \hat{w}_1(xx_1) + 3$ and $\hat{w}(yy_1) = \hat{w}_1(yy_1) - 2$.

Case 4. *$xy \notin E$ and $\hat{w}_2(x) = \hat{w}_1(x) - 1$.*

Let $x_1 \in M$ be the neighbor of x with $\hat{w}_1(x_1) = \hat{w}_1(x) + 1$, and $y_1 \in M$ be the neighbor of y with $\hat{w}_1(y_1) = \hat{w}_1(x) - 3$. Set $\hat{w}(xx_1) = \hat{w}_1(xx_1) - 3$ and $\hat{w}(yy_1) = \hat{w}_1(yy_1) + 2$.

Repeating the above for every $x \in M$ with $\hat{w}_1(x) \in \hat{H}_1$ will result in an irregular weighting. Observe that if x and x' both fall in Cases **??** or **??**, then $|\hat{w}_1(x) - \hat{w}_1(x')| \geq 6$, and therefore the weight changes used to correct the weighting do not affect each other. For all other cases, Lemma **??** guarantees that the weight changes stemming from different vertices will not interfere. It is easy to check that none of the edge weights in $\hat{W}$ are below one or above $\lambda + 1$.

If there is an edge with $\hat{w}(xy) = \lambda + 1$, then x and y are the second and third to last vertices of the last path, $\hat{w}(x) = 2\lambda+1$ and $\hat{w}(y) = 2\lambda$, and there is no vertex z with $\hat{w}(z) = 2\lambda - 1$. Change the weight of xy to $\hat{w}'(xy) = \lambda$, and the resulting weighting $\hat{W}'$ is irregular and does not use edge weights above λ. This finishes the proof of Theorem 1.

3. Conclusions

We have extended the known classes of trees with $s(T) = \lambda(T)$. More importantly, however, we have given an explicit algorithm that will generate an irregular weighting for trees in the class under consideration. We are hopeful that this iterative approach will be adaptable to a larger class of trees or more general graphs. For instance, it may be possible to show, via a modification of our algorithm, that there is some absolute constant c such that if T is any tree with $n_2(T) \geq cn_1(T)$, then $\lambda(T) = s(T) = \lceil \frac{n_1+n_2}{2} \rceil$.

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Department of Theoretical and Applied Mathematics, The University of Akron, Akron, OH 44325-4002
E-mail address: <mjf@uakron.edu>

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322
E-mail address: <rg@mathcs.emory.edu>

Department of Discrete Mathematics, Adam Mickiewicz University, 60-769 Poznań, Poland
and
Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322
E-mail address: <karonski@amu.edu.pl>

Institut für Mathematik, Universität Rostock, 18051 Rostock, Germany
E-mail address: <florian.pfender@uni-rostock.de>