VERTEX COLORING EDGE WEIGHTINGS WITH INTEGER WEIGHTS AT MOST 6

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Abstract. A weighting of the edges of a graph is called neighbor distinguishing if the weighted degrees of the vertices yield a proper coloring of the graph. In this note we show that such a weighting is possible from the weight set $\{1,2,3,4,5,6\}$ for all graphs not containing components with exactly 2 vertices.

All graphs in this note are finite and simple. For notation not defined here we refer the reader to [3].

For some $k \in \mathbb{N}$, let $f : E(G) \to \{1,2,\dots,k\}$ be an integer weighting of the edges of a graph G. This weighting is called vertex coloring if the weighted degrees $w(v) = \sum_{u\in N(v)} w(uv)$ of the vertices yield a proper coloring of the graph. It is easy to see that for every graph which does not have a component isomorphic to K^2, there exists such a weighting for some k.

In 2002, Karoński, Łuczak and Thomason (see [6]) conjectured that such a weighting with $k = 3$ is possible for all such graphs ($k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4). A first constant bound of $k = 30$ was proved by Addario-Berry et al. in 2007 [1], which was later improved to $k = 16$ in [2] and to $k = 13$ in [7].

In this note we show a completely different approach that improves the bound to $k = 6$. We were able to further improve the bound to $k = 5$ in [5], but this current note exhibits some interesting ideas with their own merit which were not used in the other paper.

Consider a related result by the first author using a total weighting, i.e. we add weights to the vertices as well.

Lemma 1 ([4]). *For any connected graph G with $|G| \geq 3$, there is an edge weighting $f : E(G) \to \{1,2,3\}$, and a vertex weighting $f' : V(G) \to \{0,1\}$ such that the total weight $w(v) := f'(v) + \sum_{w\in N(v)} f(vw)$ gives a proper coloring of $V(G)$.*

With the help of this result, the first author was able to reduce the bound to $k = 10$ after tripling all weights and adjusting some of the resulting edge weights by 1 with a Kempe chain type argument to get a neighbor distinguishing edge weighting.

In this note, we use similar ideas to get down to $k = 6$ in the original problem. We start with a slight generalization of Lemma 1. The proof is almost identical but is included here for completeness.

Lemma 2. *Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Then, for any connected graph G with $|G| \geq 3$, and for any spanning tree T, there is an edge weighting $f : E(G) \to \{\alpha - \beta, \alpha, \alpha + \beta\}$, and a vertex weighting $f' : V(G) \to \{0, \beta\}$ such that the total weight $w(v) := f'(v) + \sum_{w \in N(v)} f(vw)$ gives a proper coloring of $V(G)$. Further, we can choose f in a way that $f(e) = \alpha$ for all edges $e \in E(T)$.*

Proof. We order the vertices $V(G) = \{v_1, v_2, \dots, v_n\}$ such that for $k \geq 2$, every v_k has exactly one edge in T to a vertex in $\{v_1, v_2, \dots, v_{k-1}\}$. We start by assigning the weight α to every edge of G, and then adjust this edge weight at most once to assign a total weight to every v_k in order, which then remains unchanged.

The vertex v_1 gets weight $\alpha d_G(v_1)$. Let us assume for some $k \geq 2$ that we have adjusted edge weights f on $E(G[\{v_1, \dots, v_{k-1}\}]) \setminus E(T)$ and vertex weights f' on $\{v_1, \dots, v_{k-1}\}$ so that the first $k-1$ vertices have their final total weight $w(v_i)$.

For v_k, we can adjust the weights of all edges $E(v_k, \{v_1, \dots, v_{k-1}\}) \setminus E(T)$, by β: If $v_k v_i \in E(G) \setminus E(T)$ and $f'(v_i) = 0$, we can choose between $(f(v_k v_i) = \alpha, f'(v_i) = 0)$ and $(f(v_k v_i) = \alpha - \beta, f'(v_i) = \beta)$ without changing $w(v_i)$. Similarly, if $v_k v_i \in E(G) \setminus E(T)$ and $f'(v_i) = \beta$, we can choose between $(f(v_k v_i) = \alpha, f'(v_i) = \beta)$ and $(f(v_k v_i) = \alpha + \beta, f'(v_i) = 0)$ without changing $w(v_i)$. Finally, we can choose $f'(v_k)$. This gives us a total of $|E(v_k, \{v_1, \dots, v_{k-1}\}) \setminus E(T)| + 2 = |E(v_k, \{v_1, \dots, v_{k-1}\})| + 1$ different possibilities for $w(v_k)$, and we may pick one that is different from all weights in $N(v_k) \cap \{v_1, \dots, v_{k-1}\}$.

Continuing in this fashion, we can find the desired weighting. □

Now we are ready to proof the main result of this note.

Theorem 3. *For every graph G without components isomorphic to a K^2, there is a weighting $\omega : E(G) \to \{1, 2, \dots, 6\}$, such that the induced vertex weights $\omega(v) := \sum_{u \in N(v)} \omega(uv)$ properly color $V(G)$.*

Proof. We may assume that G is connected as we can treat every component separately. Start with any spanning tree T and consider the weighting (f, f', w) from Lemma 2 with parameters $\alpha = 4$ and $\beta = -2$. At this point, all edges and vertices have even weights. In the remainder of the proof we will adjust f and f', but $w(v)$ will remain unchanged (and even) for every vertex $v \in V(G)$.

Let $H = G[\{v \in V(G) \mid f'(v) = -2\}]$, and find a maximal spanning subgraph H_1 of H with maximum degree at most 2. Add -1 to the weights $f(e)$ of all edges in H_1, and adjust $f'(v)$ accordingly for all vertices in $V(H_1)$ to keep $w(v)$ unchanged. Now all vertices $v \in V(G)$ have $f'(v) \in \{0, -1, -2\}$, all edges $e \in E(G)$ have $f(e) \in \{1, 2, \ldots 6\}$, and all edges $e \in E(T)$ have $f(e) \in \{3, 4\}$.

For $i \in \{0, 1, 2\}$ let $S_i := \{v \in V(G) \mid f'(v) = -i\}$ and $s_i := |S_i|$. Note that all vertices $v \in S_0 \cup S_2$ have even weights $w(v) - f'(v)$, and vertices in S_1 have odd weights. By the maximality of H_1, all edges uv with $u, v \in S_1 \cup S_2$ have $u, v \in S_1$ and $uv \in E(H_1)$. In particular, $w(u) - f'(u) \neq w(v) - f'(v)$ for the end vertices of these edges. Let us denote the set of these edges by E^*.

If $s_2 = 0$, we are done by setting $\omega = f$. If $s_2 = 1$ and $s_1 = 0$, let $u \in S_2$. Note that all edges e incident to u have weights $f(e) \in \{2, 4, 6\}$. If u has a neighbor v with $w(u) + 2 \neq w(v)$, subtract 1 on the edge uv and we are done by setting $\omega = f$ (note that u and v are the only vertices with odd weight ω). If for all neighbors $v \in N(u)$ we have $w(u) + 2 = w(v)$ and $|N(u)| \geq 2$, subtract 1 on two different edges incident to u. Again, this leads to a proper weighting ω. Finally, if the only neighbor $v \in N(u)$ has $w(u) + 2 = w(v)$, we can find a vertex $x \in N_T(v) \setminus \{u\}$, subtract 1 from $f(uv)$ and add 1 to $f(xv)$, and again this leads to a proper weighting ω.

If $s_2 = 1$ and $s_1 \geq 1$, take a T-path between $u \in S_2$ and $v \in S_1$, and, in the manner of a Kempe chain argument, add and subtract 1 in turn to all edges of this path, making sure that we subtract 1 on the edge incident to v. This leads to a proper weighting ω.

If $s_2 \geq 2$, the following inductive argument shows that we can find $\lceil s_2/2 \rceil$ paths in T such that the set of ends of the paths is exactly S_2, and every edge of T is used at most twice. For $2 \leq s_2 \leq 3$, these paths are easy to find. For $s_2 \geq 4$, find an edge $e \in E(T)$ so that both components of $T - e$ contain at least 2 vertices in S_2 and at least one of the components contains an even number of vertices in S_2. Now apply induction on the two components to find the desired paths.

In the manner of a Kempe chain argument, add and subtract 1 in turn to all edges of each of these paths, such that only the weights of the end vertices are affected, and adjust f' for these vertices accordingly. If a vertex $u \in S_2$ is end vertex of two paths (i.e., if s_2 is odd), we make sure to subtract 1 on the edges incident to u of both paths so that we end up with $f'(u) = 0$. Note that we only use edges in $E(T)$, and therefore we do not introduce edge weights less than 1 or greater than 6. After this process, all vertices previously in S_2 now have weights $f'(v) \in \{-3, -1, 0\}$. If we set $\omega = f$, we see that $\omega = w$ for all vertices with $w(v)$ even, and the only edges between vertices with odd weight $\omega(v)$ are in E^*, and therefore their end vertices have different weights. Thus, ω is as desired. $\square$

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