NEW ORE-TYPE CONDITIONS FOR H-LINKED GRAPHS

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Abstract. For a fixed (multi)graph H, a graph G is *H-linked* if any injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. The notion of an H-linked graph encompasses several familiar graph classes, including k-linked, k-ordered and k-connected graphs. In this paper, we give two sharp Ore-type degree sum conditions that assure a graph G is H-linked for arbitrary H. These results extend and refine several previous results on H-linked, k-linked and k-ordered graphs.

All graphs in this paper are finite. For notation not defined here we refer the reader to [1]. If $X \subseteq V(G)$ is a vertex set, we will often just write X for the induced subgraph $G[X]$ if the context is clear. Given an integer-valued graph parameter p and a graph property $\mathcal{P}$, the *p-threshold for $\mathcal{P}$* is the minimum $k = k(n)$ such that any graph G of order n with $p(G) \geq k$ has property $\mathcal{P}$. We will frequently consider p-thresholds restricted to specific graph classes, such as sufficiently large graphs, or graphs with a prescribed number of edges.

Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G, respectively, and let $\sigma_2(G)$ denote the minimum degree sum of nonadjacent vertices in G. Throughout the paper, we will often refer to σ_2 conditions as *Ore-type* conditions in light of Ore's classical theorem on hamiltonian graphs. We will also let $n_i(G)$ be the number of vertices of degree i in G.

A graph G is *k-linked* if for any ordered subset of $2k$ vertices $S = \{s_1, t_1, \dots, s_k, t_k\}$ there exist disjoint paths $P_1, \dots, P_k$ such that for each i, P_i is an $s_i - t_i$ path. We will refer to this collection of paths as an *S-linkage* in G. We also say that G is *k-ordered* if for any list of k vertices $v_1, \dots, v_k$ in G, there exists a cycle that visits these vertices in the given order.

For a fixed graph H, a graph G is *H-linked* if any injection $f : V(H) \to V(G)$ can be extended to an H-subdivision in G. We refer to the injection f as an *H-linkage*

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problem (in G). The notion of an H-linked graph generalizes those of k-linked, k-ordered and k-connected graphs, as G is kK_2-linked if and only if G is k-linked, G is C_k-linked if and only if G is k-ordered and G is k-connected if and only if G is $K_{1,k}$-linked.

1. Degree Conditions for H-Linked Graphs

Kawarabayashi, Kostochka and Yu [8] determined sharp minimum degree and degree sum conditions for a graph G of order at least $2k$ to be k-linked.

Theorem 1. *Let G be a graph on $n \geq 2k$ vertices. If*

$$\delta(G) \geq \begin{cases} \frac{n+2k-3}{2}, & \text{if } n \geq 4k-1 \\ \frac{n+5k-5}{3}, & \text{if } 3k \leq n \leq 4k-2 \\ n-1, & \text{if } 2k \leq n \leq 3k-1 \end{cases}$$

or

$$\sigma_2(G) \geq \begin{cases} n+2k-3, & \text{if } n \geq 4k-1 \\ \frac{2(n+5k)}{3}-3, & \text{if } 3k \leq n \leq 4k-2 \\ 2n-3, & \text{if } 2k \leq n \leq 3k-1 \end{cases}$$

then G is k-linked. These bounds are best possible.

For sufficiently large graphs, the relevant portion of these conditions were obtained independently in [6]. Sharp minimum degree and degree sum conditions for k-ordered graphs were determined in [2] and [9], respectively.

Theorem 2. *Let G be a graph of order n and $k \geq 2$ be an integer. If*
(a) $n \geq 11k-3$ *and* $\delta(G) \geq \lceil \frac{n}{2} \rceil + \lfloor \frac{k}{2} \rfloor - 1$, *or*
(b) $n \geq 53k^2$ *and* $\sigma_2(G) \geq n + \lceil \frac{3k-9}{2} \rceil$,
then G is k-ordered.

Turning our attention the the broader class of H-linked graphs, minimum degree conditions that assure a graph G is H-linked for arbitrary connected H were first given in [3] and [10]. These were subsequently strengthened in [5] to include arbitrary multigraphs H, thereby extending Theorem 1. Similar conditions concerned with finding (strong) H-immersions in a graph G appear in [4]. In order to discuss these results, we must first introduce a useful parameter.

For a (multi-)graph H, let

$$b(H) = \max_{\substack{A \cup B \cup C = V(H) \\ V(H) \neq C}} |E(A,B)| + |C|.$$

As every graph G has a bipartite subgraph with at least half of the edges in G, $b(H) \geq |E(H)|/2$. When H is connected, it is straightforward to see that we may choose C to be empty in any optimal partition, so that $b(H)$ is equal to the maximum number of edges in a bipartite subgraph of H. As was noted in [4] and [5], when H is disconnected, $b(H)$ depends not only on the maximum size of a bipartite subgraph of H, but also on the number of components of H without even cycles.

The following result of Gould, Kostochka and Yu gives the δ-threshold for H-linkedness and also represents the current best bound on the necessary order of the target graph G.

Theorem 3. *Let H be a (multi-)graph with $c(H)$ components that do not contain even cycles and G be a graph of order $n \geq 9.5(|E(H)| + c(H) + 1)$. If*

$$\delta(G) \geq \tfrac{1}{2}(n + b(H) - 2),$$

then G is H-linked. This result is sharp.

Kostochka and Yu [11] gave Ore-type conditions, dependent on k, implying that a graph G is H-linked for *every* graph H with k edges.

Theorem 4. *Let G be a graph of order n and let H be a simple graph with k edges and minimum degree at least two. If*

$$\sigma_2(G) \geq \begin{cases} \lceil n + \frac{3k-9}{2} \rceil & n > 2.5k - 5.5 \\ \lceil n + \frac{3k-8}{2} \rceil & 2k \leq n \leq 2.5k - 5.5 \\ 2n - 3 & k \leq n \leq 2.5k - 1, \end{cases}$$

then G is H-linked.

In light of Theorem 2, one interesting consequence of Theorem 4 is that amongst those graphs H with k edges, C_k has the largest σ_2-threshold for H-linkedness when n is sufficiently large.

The goal of this paper is to refine Theorem 4 by giving sharp Ore-type conditions that assure a graph G is H-linked for an arbitrarily chosen H. We note here that the σ_2-threshold for H-linkedness is not, in general, twice the minimum degree given in Theorem 3, as Theorem 2 demonstrates that this is not the case for $H = C_k$ when n is sufficiently large. Our first result demonstrates that twice the minimum degree in Theorem 3 *does* suffice if we add only a mild minimum degree condition to G.

Theorem 5. *Let H be a multigraph and G be a graph with $|G| \geq 20|E(H)| + n_0(H)$. If*

$$\delta(G) \geq 4|E(H)| + n_0(H), \text{ and}$$
$$\sigma_2(G) \geq |G| + b(H) - 2,$$

then G is H-linked. This result is sharp.

We also utilize Theorem 5 to give a sharp σ_2 bound that, without any additional minimum degree condition, assures a graph G is H-linked for any simple graph H. Let

$$a(H) = \max_{A \cup B = V(H)} \left(|E(A,B)| + |B| - \Delta_B(A)\right).$$

Theorem 6. *Let H be a simple graph and G be a graph of order $n > 20|E(H)|$. If*

$$\sigma_2(G) \geq n + a(H) - 2,$$

then G is H-linked. This result is sharp.

Observe that for arbitrary H, $a(H) \geq b(H)$. To see this, suppose that $V(H) = A \cup B \cup C$ with $e(A,B) + |C| = b(H)$. Then, if we let $B^* = B \cup C$, it follows that

$$a(H) \geq e(A, B^*) + |B^*| - \Delta_{B^*}(A) \geq e(A,B) + |C| = b(H).$$

There are a number of graphs H, including C_k, for which $a(H) > b(H)$. As such, Theorem 6 demonstrates that there are many choices of H for which the σ_2-threshold for H-linkedness is more than twice the δ-threshold.

2. Preliminary Lemmas

A version of the following Lemma originally appears in [12], pertaining to directed graphs. The proof for undirected graphs is analogous and, hence, omitted.

Lemma 7. *Let G be a graph, $k \geq 1$ and $v \in V(G)$ with $d(v) \geq 2k-1$. If $G - v$ is k-linked, then G is k-linked.*

Thomas and Wollan [14] used the following to prove that every $10m$-connected graph is m-linked, which represents the current best bound on connectivity sufficient to assure linkedness.

Theorem 8. *Let $m \geq 2$ and G be a $2m$-connected graph. If $|E(G)| \geq 5m|G|$, then G is m-linked.*

Corollary 9. *Let $m \geq 2$ and G be a $2m$-connected graph of order n. If $\sigma_2(G) \geq n$ and $n \geq 20m$, then G is m-linked.*

We close with the following straightforward fact and a useful, but equally straightforward, lemma.

Fact 10. *Let G be a graph and H a (multi-)graph with $|E(H)| = m$ and $n_0(H) = 0$. If G is m-linked, then G is H-linked.*

Lemma 11. *Let H be a multigraph, and let G be an edge maximal non-H-linked graph. Then for every $m \geq |E(H)|$ and $X \subseteq V(G)$ with $|X| \geq 2m$:*

$$G[X] \text{ is } m\text{-linked} \iff G[X] \text{ is complete.}$$

3. Proofs of Theorems 5 and 6

We are now ready to prove our main results.

Proof of Theorem 5. Sharpness is established by the following example, which is identical to the sharpness example for Theorem 3. Let $A \cup B \cup C$ be a partition of $V(H)$ such that $|E(A,B)| + |C| = b(H)$. Create G by first adding $|E(A,B)| - 1$ vertices to C to obtain C^*, and then adding vertices to A and B to create sets A^* and B^*, each of size $\frac{n-|C^*|}{2}$. The edges of G are all possible edges in $(A^* \cup C^*)$ and $(B^* \cup C^*)$. It is straightforward to see that G is not H-linked, as there is not a sufficient number of edges to create paths representing the edges in $E(A,B)$.

For the proof of the main statement of the Theorem, we will in fact show a slightly stronger statement as follows.

Claim 1. *Let H be a multigraph and G be a graph with $|G| \geq 20|E(H)| + n_0(H)$, and let $V(H) \subseteq V(G)$. If*

$$\delta(G) \geq 4|E(H)| + n_0(H), \text{ and}$$

$$d(x) + d(y) \geq |G| + b(H) - 2, \text{ whenever } x, y \in V(G) \setminus V(H) \text{ and } xy \notin E(G),$$

then there is an H-linkage in G.

Let $n = |G|$ and $m = |E(H)|$. Note that the statement is trivial for $m \leq 1$, so we may also assume that $m \geq 2$.

For the sake of contradiction, we assume that there is no H-linkage in G, and furthermore that Claim 1 is true for every proper subgraph $H' \subsetneq H$. Further, assume that G is edge maximal without an H-linkage.

If $v \in V(H)$ is isolated in H, then solving the H-linkage problem in G is equivalent to solving the associated $(H-v)$-linkage problem in $G - v$. As $G - v$ satisfies all of the conditions in Claim 1 (note that $b(H-v) = b(H) - 1$), this yields a contradiction, so H does not contain any isolated vertices.

If G is $2m$-connected, we are done by Corollary 9, so we may assume that there is a minimal cut set Z in G with $|Z| \leq 2m - 1$. The degree conditions on G imply that $G - Z$ has exactly two components, call them X and Y and we assume, without loss of generality, that $|X| \leq |Y|$. Let $x \in X$ and $y \in Y$, then

$$n + b(H) - 2 \leq d(x) + d(y) \leq |X| + |Y| + 2|Z| - 2 \leq n + |Z| - 2,$$

so

$$\delta_X(X) + \delta_Y(Y) \geq |X| + |Y| - |Z| + b(H) - 2.$$

Therefore,

$$\delta_X(X) \geq \max\{|X| - |Z| + b(H) - 1, \delta(G) - |Z|\} \geq |X| - \tfrac{3}{2}m.$$

We now wish to show that both X and Y are m-linked. If $|X| \geq 5m$, then $\delta_X(X) \geq \frac{7|X|}{2}$, so X is m-linked by Theorem 1. Suppose, then, that $|X| < 5m$, so $2(|X| + |Z|) < |G|$ and X is complete by the degree sum condition. Since $|X| \geq$

$\delta(G) + 1 - |Z| \geq 2m + 2$, the fact that X is complete implies that X is m-linked. Analogously, we also conclude that Y is m-linked.

Let $z \in Z$, and suppose there are vertices $x \in X$ and $y \in Y$ such that $xz, yz \notin E(G)$. Then

$$n + |Z| + 2d(z) \geq d(x) + 2d(z) + d(y) \geq 2n + m - 4,$$

so

$$d(z) \geq \tfrac{1}{2}(n + m - |Z| - 4) \geq \tfrac{1}{2}(n - m - 4) > 6m.$$

Thus, for every $z \in Z$, we have $d_X(z) \geq 2m$ or $d_Y(z) \geq 2m$. Let

$$\begin{aligned} B :=& \{v \in V(G) \ : \ d_Y(v) \geq 2m - 1\}, \text{ and} \\ A :=& V(G) \setminus B. \end{aligned}$$

Then, $A \supseteq X$ and $B \supseteq Y$ are m-linked by Lemma 7, and therefore complete by Lemma 11. Let A^H, B^H be the partition of $V(H)$ induced by this partition of $V(G)$.

Choose $ab \in E(H)$, let $H' = H - ab$ and let $F \subseteq G$ be a solution of the H'-linkage problem of minimum order. In particular, this implies that $|F \cap A| \leq 2m$ and $|F \cap B| \leq 2m$, so $A \setminus F \neq \emptyset$ and $B \setminus F \neq \emptyset$. Since A and B are complete, we conclude that $a \in A$ and $b \in B$, and in particular, $E(H) = E_H(A, B)$. By the minimality of F we have $|E_F(A, B)| = |E_{H'}(A, B)| = |E(H)| - 1$.

Let $v \in A \setminus F$ and $w \in B \setminus F$. If $vw \in E(G)$, then we can extend F to a solution of the H-linkage problem using the path $avwb$, so we conclude that $vw \notin E(G)$. Similarly, if there exists an $x \in (N(v) \cap N(w)) \setminus F$, we can extend F to a solution of the H-linkage problem using $avxwb$, so $N(v) \cap N(w) \subseteq F$.

It is our goal to show that $|N(v) \cap N(w)| \leq |E_F(A, B)|$. Consider first $xy \in E(F) \setminus E(H)$ with $x \in A$ and $y \in B$. If $x \in N(w)$ and $y \in N(v)$, then we can replace xy by xw and vy in F and solve the H-linkage problem, using one of the new edges instead of xy and the other to connect a and b. So $|N(v) \cap N(w) \cap \{x, y\}| \leq 1$ for all $xy \in E_F(A, B) \setminus E(H)$.

Now, let $xy \in E(F) \cap E(H)$ with $x \in A$ and $y \in B$. If $\{x, y\} = \{a, b\}$ (so that a and b are joined by at least two edges in H), then by the same argument as above, $|N(v) \cap N(w) \cap \{x, y\}| \leq 1$. If, instead $\{x, y\} \neq \{a, b\}$, then there is another edge $x'y' \in E(F)$ with $x' \in A$ and $y' \in B$ that lies on an $x - y$ path in F. Now, if $x = x'$ (or, nearly identically, if $y = y'$) then as above, $|N(v) \cap N(w) \cap \{x, y'\}| \leq 1$, and so $|N(v) \cap N(w) \cap \{x, y, x', y'\}| \leq 2$. Also, if $x \neq x', y \neq y'$, and $vy, x'w \in E(G)$, then we can replace $x'y'$ by vy in F and use $x'w$ to connect a and b. Similarly, we can't have $x \neq x', y \neq y'$ and both of $xw, vy' \in E(G)$, so again, $|N(v) \cap N(w) \cap \{x, y, x', y'\}| \leq 2$.

We therefore conclude that $|N(v) \cap N(w)| \leq |E_F(A, B)|$. This yields a contradiction, as then

$$a(H) \leq |N(v) \cap N(w)| \leq |E_F(A, B)| = |E(H)| - 1 \leq a(H) - 1.$$

□

Proof of Theorem 6. Sharpness follows from the following example. Starting from a partition $A \cup B$ of $V(H)$ with $(|E(A,B)| + |B| - \Delta_B(A)) = a(H)$, add a set C of $|E(A,B)| - 1$ vertices. Blow up B to B^* by adding $n - |A| - |B| - |C|$ vertices to B and then add all edges in $A \cup C$, $B^* \cup C$, and all edges between A and B except for the edges in H. This graph is not H-linked, as there is not a sufficient number of vertices in C to create paths representing the edges in $E(A,B)$, and has $\sigma_2 = n + a(H) - 3$.

As in the proof of Theorem 5, we may assume that $n_0(H) = 0$ as isolated vertices in H contribute 2 to $|G| + a(H)$ and at most 2 to $\sigma_2(G)$.

For the sake of contradiction, we assume that G is not H-linked, and furthermore that G is edge maximal with this property. Let $m = |E(H)|$ and $n = |G|$.

If $\delta(G) \geq 4m$, we are done by Theorem 5 (as $b(H) \leq a(H)$), so there is a vertex v with $d(v) < 4m$. Let $Y := V(G) \setminus N[v]$. Then $|Y| > 16m$ and

$$\delta_Y(Y) > |Y| - 4m > \tfrac{1}{2}|Y| + m,$$

and therefore Y is m-linked by Theorem 1. Let $B \supseteq Y$ be maximal such that B is m-linked, and $A := V(G) \setminus B \subseteq N[v]$. If $A = \emptyset$, we are done so we assume that $A \neq \emptyset$. By Lemma 7 no vertex in A has $2m$ neighbors in B, so $\Delta_G(A) < 6m$ and therefore A is complete by the degree sum condition. We now continue in a manner similar to the proof of Theorem 5.

We may assume that G is H'-linked for every proper subgraph $H' \subsetneq H$, as otherwise we could continue with a minimal subgraph H' of H for which G is not H'-linked and observe that $a(H') \leq a(H)$. Let $A^H \cup B^H$ be the partition of $V(H)$ induced by A and B. Note that B is complete by Lemma 11. If there is an edge $e \in E(H) \cap E(G)$, we can extend any solution of the $(H - e)$-linkage problem trivially to a solution of the H-linkage problem, so we conclude that $E(H) \cap E(G) = \emptyset$, and in particular, $E(H) = E_H(A,B)$.

Let $a \in A^H$ maximize $|E_H(a,B)|$, and let $ab \in E(H)$. Let $H' = H - ab$ and let $F \subseteq G$ be a solution of the H'-linkage problem of minimum order, so in particular $|E_F(A,B)| = |E(H')|$.

Let $w \in B \setminus F$. If $aw \in E(G)$, then we can extend F to a solution of the H-linkage problem using the path awb, so we conclude that $aw \notin E(G)$. Similarly, if there exists an $x \in (N(a) \cap N(w)) \setminus F$, we can extend F to a solution of the H-linkage problem using $axwb$, so $N(a) \cap N(w) \subseteq F$. Now let $xy \in E(F)$ with $x \in A$ and $y \in B \setminus B^H$. If $x \in N(w)$ and $y \in N(a)$, then we can replace xy by xw and ay in F and solve the H-linkage problem. Thus, all edges $xy \in E(F)$ with $\{x,y\} \subset N(a) \cap N(w)$ have $y \in B^H \setminus N_H(a)$. But this yields a contradiction, as then

$$\begin{aligned} a(H) \leq |N(a) \cap N(w)| &\leq |E(F)| + |B^H \setminus N_H(a)| \\ &= |E(H)| - 1 + |B^H| - \Delta_{B^H}(A^H) \leq a(H) - 1. \end{aligned}$$

□

We note here that Theorem 6 does not extend to arbitrary multigraphs H. To see this, let $k \geq 6$, $r = 2(k-1)$, and let H be the disjoint union of a star having center c and leaves $\ell_1, \ldots, \ell_r$ with an edge uv of multiplicity k. As defined above, $a(H) = 3k - 1$ (let B consist of u and all of the ℓ_i). However, consider the following example. Let $A = \{c, u, v\}$ be a triangle and X be a clique of order $n-3$ containing disjoint subsets L, X_u and X_v of X with $|X_v| = r, |X_u| = r-1$ and $L = \{\ell_1, \ldots, \ell_r\}$.

Construct G from A and X by adding all edges from u to $X_u \cup L$, v to $X_v \cup L$ and c to $X_u \cup X_v$ and note that $\sigma_2(G) = n + (4k-4) - 2 > n + a(H) - 2$. If we let the vertex labels in G define an H-linkage problem ρ, then we require at least one vertex from $X_u \cup X_v$ to construct the r desired paths from c to L and at least two vertices from $X_u \cup X_v$ to construct each of the remaining $k-1$ paths from u to v. This is a total of at least $2k-4$ additional vertices, which exceeds the $2k-5$ vertices in $X_u \cup X_v$. Hence G is not H-linked.

Theorems 5 and 6 also allow us to obtain a number of interesting results on k-linked and k-ordered graphs as corollaries. In particular, we obtain the degree conditions for sufficiently large k-linked, k-ordered and H-linked graphs found in Theorems 2, 3 and 4, respectively. In most cases, our bounds on $|G|$ are reasonable, but slightly larger than those in the original theorems due to the more general nature of our results.

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