DEGREE CONDITIONS FOR H-LINKED DIGRAPHS

MICHAEL FERRARA, MICHAEL JACOBSON AND FLORIAN PFENDER

Abstract. Given a (multi)digraph H, a digraph D is *H-linked* if every injective function $\iota : V(H) \to V(D)$ can be extended to an H-subdivision. In this paper, we give sharp degree conditions that assure a sufficiently large digraph D is H-linked for arbitrary H. The notion of an H-linked digraph extends the classes of m-linked, m-ordered and strongly m-connected digraphs.

First, we give sharp minimum semi-degree conditions for H-linkedness, extending results of Kühn and Osthus on m-linked and m-ordered digraphs. It is known that the minimum degree threshold for an undirected graph to be H-linked depends on a partition of the (undirected) graph H into three parts. Here, we show that the corresponding semi-degree threshold for H-linked digraphs depends on a partition of H into as many as nine parts.

We also determine sharp Ore-Woodall-Type degree-sum conditions assuring that a digraph D is H-linked for general H. As a corollary, we obtain (previously undetermined) sharp degree-sum conditions for m-linked and m-ordered digraphs.

All graphs and digraphs in this paper are finite. For notation not defined here we refer the reader to [1]. We write $V(D)$ and $E(D)$ for the set of vertices and edges of a digraph D, and will often write $|D|$ as shorthand for $|V(D)|$. For any two vertices $v, w \in V(D)$, there is at most one edge from v to w and at most one edge from w to v, denoted by vw and wv, respectively. In multidigraphs, there may be more than one edge in each direction. An (undirected) graph or multigraph G can be viewed as a directed (multi)graph D, where for every undirected edge vw in G, both edges vw and wv in D are present. If $X \subseteq V(D)$ is a vertex set, we will often just write X for the induced subdigraph $D[X]$ if the context is clear. We write X^c for the complement $V(D) \setminus X$. We let C_k and P_k denote the directed cycle and path of order k, respectively, and let T_k denote the transitive tournament of order k. For a set X of vertices and a vertex v (possibly in X), let $d^+_X(v)$ denote the number of edges from v to a vertex in X and $d^-_X(v)$ denote the number of edges from vertices in X to v. For sets of vertices X and Y in $V(D)$, we let $\delta^+_X(Y)$ denote the minimum of $d^+_X(v)$ taken over all $v \in Y$, and we define $\delta^-_X(Y)$ similarly. If $X = V(D)$, then we will write $\delta^+(Y)$ and $\delta^-(Y)$ in the interest of concision. The *semi-degree* of a vertex v, denoted $d^0(v)$, is the minimum of $d^+(v)$ and $d^-(v)$. We let $\delta^0(D)$ denote the minimum semi-degree of a digraph D and let $\sigma_2(D)$ denote the minimum of $d^+(v) + d^-(w)$ taken over all pairs of distinct vertices v and w such that vw is not in $E(D)$, a quantity Woodall studied in [12] to transfer Ore's condition for hamiltonian cycles in graphs to digraphs. The

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number of isolated vertices in D is denoted by $n_0(D)$. In order to emphasize that a path P is a $v-w$ path, we sometimes write vPw instead of P. Also, when convenient, we will write $u(v)w$ for a path which may either be uw or uvw.

A digraph D is *m-linked* if for any ordered set of $2m$ vertices $S = \{s_1, t_1, \dots, s_m, t_m\}$ in $V(D)$, there exist disjoint paths $P_1, \dots, P_m$ such that for each i, P_i starts at s_i and ends at t_i. We will refer to this collection of paths as an *S-linkage* in D and will frequently call S the *terminals* of the S-linkage. Similarly, we say that D is *m-ordered* if for any ordered list of m vertices $v_1, \dots, v_m$ in D, there exists a directed cycle that visits these vertices in the given order. For a fixed digraph H, a digraph D is *H-linked* if any injection $\iota : V(H) \to V(D)$ can be extended to an H-subdivision in D. Such a subdivision will often be referred to as an *H-linkage* in D. We frequently refer to ι as an H-linkage problem, and to the image of ι as the *terminals* of the H-linkage problem. The notion of an H-linked digraph generalizes those of m-linked and m-ordered digraphs, as D is mT_2-linked if and only if D is m-linked and D is C_m-linked if and only if D is m-ordered.

Kühn and Osthus determined the minimum semi-degree necessary to assure that a sufficiently large digraph D is m-linked or m-ordered, respectively.

Theorem 1. [9]

(a) *Let $m \geq 2$, and let D be a digraph with $|D| \geq 1600m^3$. If $\delta^0(D) \geq \frac{1}{2}(|D| + 2m - 2)$, then D is m-linked.*
(b) *Let $m \geq 3$, and let D be a digraph with $|D| \geq 200m^3$. If $\delta^0(D) \geq \frac{1}{2}(|D| + m - 2)$, then D is m-ordered.*

These degree bounds are best possible.

In this paper, we give sharp σ_2 conditions that assure a digraph D with modestly large minimum semi-degree is H-linked for arbitrary H. We then use this result to determine the δ^0-threshold for a digraph to be H-linked, thereby extending Theorem 1. We also obtain a sharp σ_2-condition (with no restriction on δ^0) for a digraph D to be H-linked.

Minimum degree conditions that assure an (undirected) graph G is H-linked for arbitrary connected H were first given (independently) in [2] and [7]. These were subsequently strengthened in [4] to include arbitrary multigraphs H. In order to discuss these results, we must first introduce a useful parameter, where we consider all partitions $V(H) = A \cup B \cup C$ of the vertex set of H.

For a (multi-)graph H, let

$$b(H) = \max_{\substack{A \cup B \cup C = V(H) \\ |E(A,B)| \geq 1}} |E(A,B)| + |C|.$$

When H is connected, it is straightforward to see that we may choose C to be empty in any optimal partition, so that $b(H)$ is equal to the maximum number of edges in a bipartite subgraph of H. As was noted in [3] and [4], when H is disconnected, $b(H)$

depends not only on the maximum size of a bipartite subgraph of H, but also on the number of components of H without even cycles.

The following result of Gould, Kostochka and Yu gives the δ-threshold for H-linkedness and also represents the current best bound on the necessary order of the target graph G. This result was subsequently obtained as a corollary to results in [3] and [5], albeit for larger $|G|$.

Theorem 2. [4] *Let H be a (multi-)graph with $c(H)$ components that do not contain even cycles and G be a graph of order $n \geq 9.5(|E(H)| + c(H) + 1)$. If*

$$\delta(G) \geq \frac{n + b(H) - 2}{2},$$

then G is H-linked. The degree bound is sharp.

Depending on the parity of n and m, many of the sharpness examples for Theorem 1 can be obtained from the associated undirected examples for m-linked and m-ordered graphs from [6] and [8] by replacing edges with directed 2-cycles. This may entice one to conjecture that the minimum semi-degree threshold for H-linked digraphs depends on a partition of $V(H)$ that resembles $b(H)$. We show that, in general, there is a much starker contrast between the minimum semi-degree threshold for H-linked digraphs and the minimum degree threshold for H-linked graphs, as the δ^0-threshold for a digraph to be H-linked depends on a partition of $V(H)$ into as many as *nine* parts.

1. Minimum Semi-Degree Conditions for H-Linkage in Digraphs

Let $\mathcal{P} = (A, B, C, L_1, L_2, L_3, R_1, R_2, R_3)$ be a partition of the vertices of a (multi)digraph H, which from here forward we assume to have at least one edge. Let

$$\begin{aligned}
E^L_{\mathcal{P}}(H) :=& E(A \cup L_1 \cup L_2, B \cup L_1 \cup L_3),\\
E^R_{\mathcal{P}}(H) :=& E(B \cup R_1 \cup R_2, A \cup R_1 \cup R_3),\\
\vec{b}_{\mathcal{P}}(H) :=& |E^L_{\mathcal{P}}(H)| + |E^R_{\mathcal{P}}(H)| + |C| + \min\{|R| - |L_1|, |L| - |R_1|\} \text{ and}\\
\vec{b}^1_{\mathcal{P}}(H) :=& |E^L_{\mathcal{P}}(H)| + |E^R_{\mathcal{P}}(H)| + |C|\\
&+ \min\{|R| - |L_1|, |L| - |R_1|, |L_2| + |R_2|, |L_3| + |R_3|\}.
\end{aligned}$$

If $|E(H)| \geq 2$, let $\vec{b}(H)$ and $\vec{b}_1(H)$ be the maxima of $\vec{b}_{\mathcal{P}}(H)$ and $\vec{b}^1_{\mathcal{P}}(H)$, respectively, over all such partitions $\mathcal{P}$. When $|E(H)| = 1$, we let $\vec{b}(H) = \vec{b}_1(H) = |V(H)| - 1$.

Now we are ready to state the two main results of this section.

Theorem 3. *Let H be a (multi-)digraph and D be a digraph. If*

$$\begin{aligned}
\delta^0(D) &\geq 34|E(H)| + n_0(H), \text{ and}\\
\sigma_2(D) &\geq |D| + \vec{b}(H) - 2,
\end{aligned}$$

then D is H-linked. The bound on σ_2 is best possible.

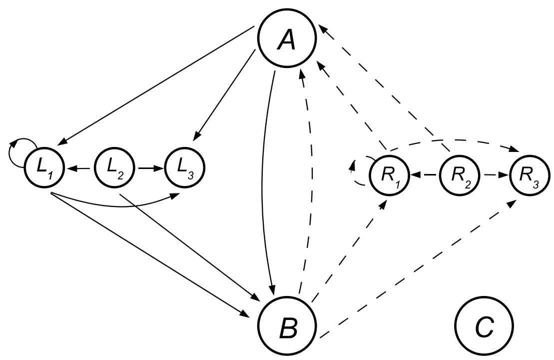

FIGURE 1. The sharpness-determining partition of H. Solid edges are in $E^L_{\mathcal{P}}(H)$ and dashed edges are in $E^R_{\mathcal{P}}(H)$.

Theorem 4. *Let H be a (multi-)digraph and D be a digraph. If*

$$\delta^0(D) \geq \max\left\{\left\lceil \tfrac{1}{2}(|D| + \vec{b}_1(H) - 2)\right\rceil, 34|E(H)| + n_0(H)\right\},$$

then D is H-linked. This result is sharp whenever the first of the two bounds applies.

The bounds in Theorems 3 and 4 are sharp for every H with $E(H) \neq \emptyset$. Choose a partition $\mathcal{P}$ of $V(H)$ with $\vec{b}_{\mathcal{P}}(H) = \vec{b}(H)$ (or $\vec{b}^1_{\mathcal{P}}(H) = \vec{b}_1(H)$, respectively), and suppose that this partition represents the intended ground-set of an H-subdivision. To obtain a graph D on n vertices that demonstrates the sharpness of Theorem 3, we add $|E^L_{\mathcal{P}}(H)| + |E^R_{\mathcal{P}}(H)| - 1$ vertices to C and distribute the remaining $n - |V(H)| - |E^L_{\mathcal{P}}(H)| - |E^R_{\mathcal{P}}(H)| + 1$ vertices arbitrarily between A and B. To construct a sharpness example D' for Theorem 4, we will distribute the vertices between A and B in a way that maximizes $\delta^0(D)$. The edges of D (resp. D') are all possible directed edges, except for

$$\{vw \mid v \in A \cup L_1 \cup L_2, w \in B \cup L_1 \cup L_3\} \cup \{vw \mid v \in B \cup R_1 \cup R_2, w \in A \cup R_1 \cup R_3\}.$$

Effectively, this means that D and D' have structure complementary to the digraph depicted in Figure 1.

The minimum semi-degree of D' is achieved either by a vertex $a \in A$ or a vertex $b \in B$, and we may choose the sizes of A and B such that $|\delta^0_V(A) - \delta^0_V(B)| \leq 1$. Therefore, the minimum semi-degree of D' is the minimum of (1) - (4), where $R =$

$R_1 \cup R_2 \cup R_3$ and $L = L_1 \cup L_2 \cup L_3$:

$$\left\lfloor \tfrac{1}{2}\left(d^+(a) + d^+(b)\right)\right\rfloor = \left\lfloor \tfrac{1}{2}\left(n + |E_{\mathcal{P}}(H)| + |C| + |L_2| + |R_2| - 3\right)\right\rfloor \tag{1}$$
$$\left\lfloor \tfrac{1}{2}\left(d^-(a) + d^-(b)\right)\right\rfloor = \left\lfloor \tfrac{1}{2}\left(n + |E_{\mathcal{P}}(H)| + |C| + |L_3| + |R_3| - 3\right)\right\rfloor \tag{2}$$
$$\left\lfloor \tfrac{1}{2}\left(d^+(a) + d^-(b)\right)\right\rfloor = \left\lfloor \tfrac{1}{2}\left(n + |E_{\mathcal{P}}(H)| + |C| + |R| - |L_1| - 3\right)\right\rfloor \tag{3}$$
$$\left\lfloor \tfrac{1}{2}\left(d^-(a) + d^+(b)\right)\right\rfloor = \left\lfloor \tfrac{1}{2}\left(n + |E_{\mathcal{P}}(H)| + |C| + |L| - |R_1| - 3\right)\right\rfloor \tag{4}$$

Similarly, for any distribution of excess vertices to A and B, $\sigma_2(D)$ is determined by one of (3) or (4).

Therefore, D misses the degree sum condition in Theorem 3 by one, and D' similarly has minimum semi-degree one less than that in Theorem 4. However, neither D nor D' has an H-linkage through the specified vertices, as every path corresponding to an edge in $E^L_{\mathcal{P}}(H) \cup E^R_{\mathcal{P}}(H)$ needs to use one vertex in $C \setminus V(H)$.

The reader may wonder if the complicated definition of $\vec{b}(H)$ and $\vec{b}_1(H)$ is a necessity or if there are simpler definitions. To this end, we can state that these are the simplest deginitions relying on a partition of $V(H)$, as one can see as follows. Let H be a digraph with $V(H)$ partitioned into a partition $\mathcal{P}$ as above, with each part containing at least two vertices. Let $E(H)$ be exactly the set of edges depicted in Figure 1, i.e. the maximal set of edges such that $E_{\mathcal{P}}(H) = E(H)$. Then it is easy to see that $\mathcal{P}$ is, up to the obvious two-fold symmetry, the unique partition maximizing $\vec{b}_{\mathcal{P}}(H)$, and similarly $\vec{b}^1_{\mathcal{P}}(H)$. Thus, both $\vec{b}(H)$ and $\vec{b}_1(H)$ depend on the sizes of all parts of $\mathcal{P}$.

2. Technical Lemmas

We now give several facts and technical lemmas that will be utilized to prove our results.

Lemma 5.

$$\vec{b}_1(H) \geq \max\left\{|E^L_{\mathcal{P}}(H)| + |C| + |R| - |L_1|, |E^R_{\mathcal{P}}(H)| + |C| + |L| - |R_1|\right\}.$$

Proof. Let $\mathcal{P}$ be a partition maximizing $|E^L_{\mathcal{P}}(H)| + |C| + |R| - |L_1|$. We may assume that $R = \emptyset$ as otherwise we could move all vertices from R to C without lowering the value of the sum. Then,

$$|E^L_{\mathcal{P}}(H)| + |C| + |R| - |L_1| = |E^L_{\mathcal{P}}(H)| + |C| - |L_1| \leq \vec{b}^1_{\mathcal{P}}(H) \leq \vec{b}_1(H).$$

An identical argument shows that $|E^R_{\mathcal{P}}(H)| + |C| + |L| - |R_1| \leq \vec{b}_1(H)$. ☐

The next three statements have straightforward proofs, so we omit parts of them here. The first can be found in [10].

Lemma 6. *Let D be a digraph, $m \geq 1$ and $v \in V(D)$ with $d^0(v) \geq 2m - 1$. If $D - v$ is m-linked, then D is m-linked.*

Lemma 7. *If H is a multidigraph, and D is an edge maximal non-H-linked digraph, then for all $v, w \in V(D)$ with $v \neq w$, every $m \geq |E(H)|$ and every set X is of at least $2m$ vertices in D such that $D[X]$ is m-linked, the following hold:*

(a) *$D[X]$ is complete,*
(b) *$d_X^+(v) \geq 2m-1 \iff X \subseteq N^+[v]$,*
(c) *$d_X^-(v) \geq 2m-1 \iff X \subseteq N^-[v]$, and*
(d) *$X \subseteq N^+[v] \cap N^-[w] \Longrightarrow vw \in E(D)$.*

Proof. For (a), assume that there are $x, y \in X$ with $xy \notin E(D)$. Then, by the maximality of D, $D + xy$ is H-linked. Taking a solution F of an H-linkage problem in $D + xy$ and all paths P in F corresponding to an edge in H, we disregard the part of P from the first occurrence of a vertex in $X \cap P$ to the last occurrence of a vertex in $X \cap P$. Then, the fact that $D[X]$ is m-linked allows to complete this partial H-linkage in D, contradicting the assumption that D is not H-linked. The proofs for (b)-(d) are similar. □

Fact 8. *Let D be a digraph and H a (multi-)digraph with $|E(H)| = m$ and $n_0(H) = 0$. If D is m-linked, then D is H-linked.*

Next, we give a straightforward degree sum condition that assures D is m-linked.

Lemma 9. *Let D be a digraph and $m \geq 1$. If $\sigma_2(D) \geq |D| + 3m - 4$, then D is m-linked.*

Proof. Let $S = \{s_1, t_1, \ldots, s_m, t_m\}$ be the set of terminals of the linkage problem. If $s_i t_i \notin E(D)$, then $|N^+(s_i) \cap N^-(t_i) \setminus S| \geq m$, so for every such pair we can find a path of length 2. □

Thomassen [11] demonstrated the existence of non-2-linked digraphs with arbitrarily high strong connectivity. However, sufficient strong connectivity can be used to lower the bound on σ_2 in Lemma 9.

Lemma 10. *If D is a strongly $\frac{9}{2}m$-connected digraph with $\sigma_2(D) \geq |D| + \frac{1}{2}m - 2$. Then D is m-linked.*

Proof. Let $S = \{s_1, t_1, \ldots, s_m, t_m\} \subset V(D)$ be the terminals of the linkage problem. Let $1 \leq i \leq m$ and assume that $s_i t_i \notin E(D)$. Then

$$|N^+(s_i) \cap N^-(t_i)| + |N^+(s_i) \cup N^-(t_i)| = d^+(s_i) + d^-(t_i) \geq |D| + \tfrac{1}{2}m - 2 \tag{5}$$

by the degree sum condition.

By Menger's Theorem, there are $\frac{9}{2}m$ internally vertex disjoint $s_i - t_i$ paths, and at least $\frac{5}{2}m$ of them intersect S only in s_i and t_i. We choose a smallest system $\mathcal{P}_i = (P_i^j)_{1 \leq j \leq \lceil \frac{5}{2}m \rceil}$ of internally disjoint paths with $|P_i^j| \leq |P_i^{j+1}|$ such that no path in $\mathcal{P}_i$ intersects $S \setminus \{s_i, t_i\}$. Note that every path contains exactly one vertex in each of $N^+(s_i)$ and $N^-(t_i)$. Let

$$p_i^j := |\{P \in \mathcal{P}_i \ : \ |P| = j\}|.$$

Then we have

$$p_i^3 = \min\left\{\lceil \tfrac{5}{2}m\rceil, |(N^+(s_i)\cap N^-(t_i))\setminus S|\right\}, \tag{6}$$

$$\sum_j p_i^j = \lceil \tfrac{5}{2}m\rceil, \text{ and,} \tag{7}$$

$$\sum_{j\geq 4}(j-4)p_i^j \leq |D| - |S\cup N^+(s_i)\cup N^-(t_i)| \leq p_i^3 + \left\lceil \tfrac{3}{2}m\right\rceil, \tag{8}$$

where (8) follows from (5) and (6), together with the observation that every path of order $j \geq 4$ uses $j-4$ vertices of $V(D)\setminus(N^+(s_i)\cup N^-(t_i)\cup S)$.

Now consider the sum

$$\Sigma_i := \sum_{j\geq 3}\frac{p_i^j}{j-2}.$$

Given (7) and (8), this sum is minimized for $p_i^4 = m$, $p_i^5 = \left\lceil \frac{3}{2}m\right\rceil$, and $p_i^j = 0$ for all other j by convexity. Thus,

$$\Sigma_i \geq \tfrac{1}{2}m + \tfrac{1}{2}m = m.$$

Next, sequentially find two terminals s_j and t_j of minimal distance in the remaining digraph, pick a shortest path P_j from s_j to t_j and delete $V(P_j)$ from D. This may destroy some paths in $\mathcal{P}_i$ for $i\neq j$, and thus alter Σ_i. If $|P_j| = 2$, all the remaining Σ_i remain the same as all the paths in $\mathcal{P}_i$ stay intact. If $|P_j| = t > 2$, and thus all remaining $\mathcal{P}_i$ have no remaining paths of shorter length, then P_j can intersect at most $t-2$ paths in $\mathcal{P}_i$ for $i \neq j$, each contributing at most $\frac{1}{t-2}$ to Σ_i, and thus Σ_i is reduced by at most 1. Therefore, this process can be continued until the linkage is completed. □

3. Proof of Theorems 3 and 4

As large parts of the proofs of Theorems 3 and 4 coincide, we will prove them jointly. Let $m := |E(H)|$, $k := |H|$. Let $\iota : V(H) \to V(D)$ be an H-linkage problem, and for the sake of notation we may assume that $V(H) \subseteq V(D)$ and that ι is the identity. For the sake of contradiction, assume that D satisfies the conditions of Theorem 3 or 4, and assume that ι has no solution in D, and that D is edge maximal under this assumption.

Note that $b(H) \geq b_1(H) \geq \frac{m}{2}$, so by Lemma 10, D is not strongly $\frac{9}{2}m$-connected. Hence, there is a minimal cut set Z in D with $|Z| < \frac{9}{2}m$. The degree conditions imply that $D - Z$ has exactly two strong components X and Y where, without loss of generality, there are no edges from Y to X. For $x\in X$ and $y \in Y$, we have that

$$n + \tfrac{m}{2} - 2 \leq d^-(x) + d^+(y) \leq |X| + |Y| + 2|Z| - 2 = n + |Z| - 2,$$

so

$$\delta^-_X(X) + \delta^+_Y(Y) \geq |X| + |Y| - (|Z| - \tfrac{m}{2}) - 2. \tag{9}$$

Let

$$X_1 := \{x \in X \mid d^+_X(x) \geq 18m\}, \text{ and } X_2 := X \setminus X_1,$$
$$Y_1 := \{y \in Y \mid d^-_Y(y) \geq 18m\}, \text{ and } Y_2 := Y \setminus Y_1.$$

Then a double count of the non-edges in X and Y gives

$$|X_2|(|X| - 18m) \leq |X|(|X| - \delta^-_X(X) - 1), \text{ and}$$
$$|Y_2|(|Y| - 18m) \leq |Y|(|Y| - \delta^+_Y(Y) - 1),$$

so by (9),

$$\begin{aligned} |X_2| + |Y_2| &\leq (|X| - \delta^-_X(X) - 1)\frac{|X|}{|X| - 18m} + (|Y| - \delta^+_Y(Y) - 1)\frac{|Y|}{|Y| - 18m} \\ &\leq (|X| + |Y| - \delta^-_X(X) - \delta^+_Y(Y) - 2)\frac{\delta^0(D) - |Z|}{\delta^0(D) - |Z| - 18m} \\ &\leq (|Z| - \tfrac{m}{2})\frac{\delta^0(D) - |Z|}{\delta^0(D) - |Z| - 18m} < 4m\frac{59}{23} < \frac{21}{2}m. \end{aligned}$$

Every vertex in X_1 has at least $|X_1| - \frac{9}{2}m$ in-neighbors in X_1 and at least $\frac{15}{2}m$ out-neighbors in X_1, so $\sigma_2(X_1) \geq |X_1| + 3m$. Thus X_1, and by a similar argument Y_1, are m-linked by Lemma 9. Note that since

$$n - |X_1 \cup Y_1| = |Z| + |X_2| + |Y_2| < 15m < \delta^0(D) - 2m,$$

we have $\max\{d^+_{X_1}(v), d^+_{Y_1}(v)\} > 2m$, and $\max\{d^-_{X_1}(v), d^-_{Y_1}(v)\} > 2m$ for every vertex $v \in V(D)$. Let

$$\begin{aligned} X_+ &:= \{v \in V(D) : X_1 \subseteq N^+[v]\}, \\ X_- &:= \{v \in V(D) : X_1 \subseteq N^-[v]\}, \\ Y_+ &:= \{v \in V(D) : Y_1 \subseteq N^+[v]\}, \\ Y_- &:= \{v \in V(D) : Y_1 \subseteq N^-[v]\}. \end{aligned}$$

Then, by Lemma 7, we can partition $V(D)$ into $A, B, C, L_1, L_2, L_3, R_1, R_2, R_3$ as follows:

$$\begin{array}{lcccccccc} A & := & X_+ & \cap & X_- & \cap & Y_+^c & \cap & Y_-^c, \\ B & := & X_+^c & \cap & X_-^c & \cap & Y_+ & \cap & Y_-, \\ C & := & X_+ & \cap & X_- & \cap & Y_+ & \cap & Y_-, \\ L_1 & := & X_+ & \cap & X_-^c & \cap & Y_+^c & \cap & Y_-, \\ L_2 & := & X_+ & \cap & X_- & \cap & Y_+^c & \cap & Y_-, \\ L_3 & := & X_+ & \cap & X_-^c & \cap & Y_+ & \cap & Y_-, \\ R_1 & := & X_+^c & \cap & X_- & \cap & Y_+ & \cap & Y_-^c, \\ R_2 & := & X_+^c & \cap & X_- & \cap & Y_+ & \cap & Y_-, \\ R_3 & := & X_+ & \cap & X_- & \cap & Y_+ & \cap & Y_-^c. \end{array}$$

It is important to note that Lemma 7 implies that D has structure complementary to the digraph depicted in Figure 1. Specifically, if $(v,w) \notin (A \cup L_1 \cup L_2) \times (B \cup L_1 \cup L_3)$ and $(v,w) \notin (B \cup R_1 \cup R_2) \times (A \cup R_1 \cup R_3)$, then by Lemma 7, vw is an edge in D. As an example, consider v in L_2 and w in A. As X_1 is m-linked, $v \in X_+$ and $w \in X_-$, it follows from part (d) of Lemma 7 that $vw \in E(G)$. More generally, if $v \in X_+$ (respectively Y_+) and w is in X_- (resp. Y_-), then $vw \in E(D)$.

It therefore follows that $X_+ \cap X_-$ and $Y_+ \cap Y_-$, and thus both $D[A \cup C \cup L_2 \cup R_3]$ and $D[B \cup C \cup L_3 \cup R_2]$, are complete digraphs. Further, $X_1 \subseteq A \cup R_3$, as every vertex in X_1 is in $X_+ \cap X_- \cap Y_-^c$, and $Y_1 \subseteq B \cup R_2$, as every vertex in Y_1 is in $Y_+ \cap Y_- \cap X_+^c$. In particular, $|A \cup R_3| \geq 19m$ and $|B \cup R_2| \geq 19m$.

For the sake of contradiction we assume that H is a minimal counterexample to either Theorem 3 or 4. Let $\mathcal{P} = (A^H, B^H, C^H, L_1^H, L_2^H, L_3^H, R_1^H, R_2^H, R_3^H)$ be the partition of $V(H)$ induced by the above partition of $V(D)$.

Note that this implies that $n_0(H) = 0$. Otherwise, let $v \in V(H)$ with $d_H(v) = 0$. Then $\vec{b}(H-v) = \vec{b}(H) - 1$ and $\vec{b}_1(H-v) = \vec{b}_1(H) - 1$. Thus, by the minimality of H, there is an $(H-v)$-linkage in $D-v$. This yields an H-linkage in D, a contradiction.

Further, note that $E(H) = E^L_{\mathcal{P}}(H) \cup E^R_{\mathcal{P}}(H)$. Otherwise, if $e = st \in E(H) \setminus (E^L_{\mathcal{P}}(H) \cup E^R_{\mathcal{P}}(H))$, let F be a minimal $(H-e)$-linkage. Then F contains at most $2m$ vertices in each of $A \cup C \cup L_2 \cup R_3$ and $B \cup C \cup L_3 \cup R_2$, so there are $a \in (A \cup C \cup L_2 \cup R_3) \setminus V(F)$ and $b \in (B \cup C \cup L_3 \cup R_2) \setminus V(F)$. But then, either sat or sbt completes the H-linkage.

Choose a 4-tuple (e, F, p, q) consisting of an edge $e = st \in E(H)$, an $(H-e)$-linkage F in D, and two vertices $p \in (A \cup R_3) \setminus V(F)$ and $q \in (B \cup R_2) \setminus V(F)$ such that

(a) Every path in F corresponding to an edge in $E(H)$ has at most one internal vertex in $A \cup L_2 \cup R_3$ and at most one internal vertex in $B \cup L_3 \cup R_2$,
(b) given (a), $|E(D[\{p,q\}]))| + |(C \cup L \cup R) \setminus V(F)|$ is maximized, and
(c) given (a) and (b), $|V(F)|$ is maximized.

By symmetry, we may assume that $e \in E^L_{\mathcal{P}}(H)$. Let $a \in (A \cup R_3) \setminus (V(F) \cup \{p\})$, $b \in (B \cup R_2) \setminus (V(F) \cup \{q\})$. Then $ab, pq \notin E(D)$ as otherwise we can complete F to an H-linkage via $sabt$ or $spqt$. In particular, $\{a,p\} \subset A$ and $\{b,q\} \subset B$. Similarly, $N^+(a) \cap N^-(b) \subseteq V(F)$, as otherwise we can use $saxbt$ for some $x \in V(D) \setminus V(F)$.

In the following, let $u, v \in V(H)$, $x \in (A \cup L_3 \cup R_2) \setminus V(H)$, $y \in (B \cup L_2 \cup R_3) \setminus V(H)$, $c \in C \setminus V(H)$, $l \in L_1 \setminus V(H)$, and $r \in R_1 \setminus V(H)$ if such vertices exist. First suppose that $uv \in E^L_{\mathcal{P}}(H)$. Keeping in mind that D has structure complementary to that pictured in 1, the following assertions hold.

- There is not r' in $R \setminus V(F) = \emptyset$ as otherwise $sr't$ would complete an H-linkage.
- If $uxyv \subseteq F$, then $ub, av \notin E(D)$, as otherwise we can replace $uxyv$ by uav (or ubv) and add $sxyt$ to F to complete the H-linkage. Similarly, $xb \notin E(D)$ or $ay \notin E(D)$, as otherwise we can replace $uxyv$ by $uayv$ and add $sxbt$ to complete the H-linkage.

- If $uxv \subseteq F$, then $xb \notin E(D)$ as otherwise replacing uxv by $uxbv$ would contradict (c). Similarly, if $uyv \subseteq F$, then $ay \notin E(D)$.
- If $uxcyv \subseteq F$ or $uxryv \subseteq F$, then $ub, xb, ay, av \notin E(D)$ as otherwise replacing $uxcyv$ (or $uxryv$) by $u(x)bv$ (or $ua(y)v$, respectively) and adding sct (or srt) would create an H-linkage from F.
- If $u(x)l(y)v \subseteq F$, then $ub, xb, ay, av \notin E(D)$ by (c). If $u(x)l(l_1 \dots l_i)l'(y)v \subseteq F$, then $N^+(a) \cap \{(l_1, \dots, l_i), l', (y), v\} = N^-(b) \cap \{u, (x), l, (l_1, \dots, l_i)\} = \emptyset$ by (c).

Now suppose that for all $vu \in E^R_{\mathcal{P}}(H)$ and $vPu \subseteq F$, $N^+(a) \cap N^-(b) \cap V(P) \subseteq \{u, v\}$. In particular, this is the case if $|E(q, p)| + |L \setminus V(F)| \geq 1$. Otherwise, we could use a vertex in $N^+(a) \cap N^-(b) \cap V(P)$ to connect s and t and use qp or a vertex in $L \setminus V(F)$ to connect v and u, completing an H-linkage. Let

$$\begin{aligned} E^* &:= \{uv \in E^L_{\mathcal{P}}(H) : ub, av \notin E(D)\}, \\ A^* &:= V(H) \cap N^+(a) \cap N^-(b)^c, \\ B^* &:= V(H) \cap N^+(a)^c \cap N^-(b), \\ C^* &:= V(H) \cap N^+(a) \cap N^-(b), \text{ and} \\ L^*_1 &:= V(H) \cap N^+(a)^c \cap N^-(b)^c. \end{aligned}$$

Note that $e \in E^*$. Then

$$n - 2 + \vec{b}_1(H) \leq d^+(a) + d^-(b) \leq n - 2 + |E^*| - 1 + |C^*| - |L^*_1|,$$

so that $\vec{b}_1 \leq E^*| + |C^*| - |L^*_1| - 1$. On the other hand, the partition $V(H) = A^* \cup B^* \cup C^* \cup L^*_1$ shows that

$$\vec{b}(H) \geq \vec{b}_1(H) \geq |E^*| + |C^*| - |L^*_1|,$$

a contradiction.

Therefore, $ba, qp \notin E(D)$, $L \subseteq V(F)$, and $E^R_{\mathcal{P}}(H) \neq \emptyset$. In particular, there is an edge $e' = t's' \in E^R_{\mathcal{P}}(H)$, $t'Ps' \subseteq F$ and a vertex $z \in (N^+(a) \cap N^-(b) \cap V(P)) \setminus \{s', t'\}$. Replacing $t'Ps'$ by $s(a)z(b)t$ gives an $(H - e')$-linkage F' satisfying (a)-(c).

Claim 1. *If $uPv \subseteq F$ for some $uv \in E(H)$, then*

$$\begin{aligned} n^{+-} &:= |(N^+(p) \cap N^-(q) \cap V(P)) \setminus \{u, v\}| \leq 1, \\ n^{-+} &:= |(N^-(p) \cap N^+(q) \cap V(P)) \setminus \{u, v\}| \leq 1, \\ n^{++} &:= |(N^+(p) \cap N^+(q) \cap V(P)) \setminus \{u, v\}| \leq 1, \text{ and} \\ n^{-+} &:= |(N^-(p) \cap N^-(q) \cap V(P)) \setminus \{u, v\}| \leq 1. \end{aligned}$$

Further, if $P = uxv$ or $P = uyv$, then $N(p) \cap N(q) \cap V(P) \subseteq \{u, v\}$.

Proof. Supose first that $uv \neq e'$, and note that $uPv \subseteq F \cap F'$. If $n^{+-} \geq 2$, then we can replace P in F' with some $s - t$ path through p and leave either an extra $A - B$ (or $B - A$) edge incident to q outside F or a vertex in $(C \cup L \cup R) \setminus V(F)$, contradicting (b). If one of the other sets contains two vertices, we can replace P in F'

by some $t'-s'$ path through p or q, and leave an extra $A-B$ (or $B-A$) edge outside F or a vertex in $(C \cup L \cup R) \setminus V(F)$, contradicting (b) again. If $uv = e'$, note that $uPv = t'(y)z(x)s'$. If $x \in V(P)$, then note that $x \notin V(F')$ and thus $qx, xq \notin E(D)$ by (b). Similarly, $py, yp \notin E(D)$ if $y \in V(P)$.

Now, if $P = uxv$ and $xq \in E(D)$ (or $qx \in E(D)$), we can use xq (or qx) in $F-P$ (or $F'-P$) to create an $(H-uv)$-linkage F'' with $|V(F'')| > |V(F)|$, contradicting (c). Similar arguments apply when $P = uyv$. $\square$

Let $\mathcal{P}^*$ be the following partition of $V(H)$ depending on the neighborhoods of p and q in D.

$$\begin{array}{lclcllcllcllcl}
A^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p) & \cap & N^+(q)^c & \cap & N^-(q)^c, \\
B^* & := & V(H) & \cap & N^+(p)^c & \cap & N^-(p)^c & \cap & N^+(q) & \cap & N^-(q), \\
C^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p) & \cap & N^+(q) & \cap & N^-(q), \\
L_1^* & := & V(H) & \cap & N^+(p)^c & \cap & N^-(p) & \cap & N^+(q) & \cap & N^-(q)^c, \\
L_2^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p) & \cap & N^+(q) & \cap & N^-(q)^c, \\
L_3^* & := & V(H) & \cap & N^+(p)^c & \cap & N^-(p) & \cap & N^+(q) & \cap & N^-(q), \\
R_1^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p)^c & \cap & N^+(q)^c & \cap & N^-(q), \\
R_2^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p)^c & \cap & N^+(q) & \cap & N^-(q), \\
R_3^* & := & V(H) & \cap & N^+(p) & \cap & N^-(p) & \cap & N^+(q)^c & \cap & N^-(q).
\end{array}$$

Also, let

$$\begin{aligned}
E_1^* &:= \{uv \in E(H) : uPv \subseteq F, (N^+(p) \cap N^-(q) \cap V(P)) \setminus \{u,v\} \neq \emptyset\}, \\
E_2^* &:= \{uv \in E(H) : uPv \subseteq F, (N^-(p) \cap N^+(q) \cap V(P)) \setminus \{u,v\} \neq \emptyset\}, \\
E_3^* &:= \{uv \in E(H) : uPv \subseteq F, (N^+(p) \cap N^+(q) \cap V(P)) \setminus \{u,v\} \neq \emptyset\}, \\
E_4^* &:= \{uv \in E(H) : uPv \subseteq F, (N^-(p) \cap N^-(q) \cap V(P)) \setminus \{u,v\} \neq \emptyset\}, \\
E^* &:= E_1^* \cup E_2^* \cup E_3^* \cup E_4^*.
\end{aligned}$$

Claim 2.

$$E^* \cup e \subseteq E^L_{\mathcal{P}^*}(H) \cup E^R_{\mathcal{P}^*}(H).$$

Proof. First consider $e' = t's'$ and $t'(y)z(x)s' \subseteq F$ with $z \in N^+(a) \cap N^-(b)$. If $t'p \in E(D)$ or $qs' \in E(D)$, we can use this edge to connect t' and s', and complete the H-linkage via $sazbt$. Thus, $t'p, qs' \notin E(D)$ and $e' \in E^R_{\mathcal{P}^*}(H)$.

Now let $uv \in E^L_{\mathcal{P}}(H) \setminus e$ and $uPv \subseteq F$. Then $up, qv \in E(D)$ as $A \cup L_1 \cup L_2 \subseteq N^-[p]$ and $B \cup L_1 \cup L_3 \subseteq N^+[q]$ by Lemma 7. If $uq \in E(D)$, then $uPv = uxv$ or $uPv = uyv$, so $N(p) \cap N(q) \cap V(P) \subseteq \{u, v\}$ by Claim 1, so $uv \notin E^*$. On the other hand, if $uq \notin E(D)$, then $u \in A^* \cup L_1^* \cup L_2^*$. Similarly, if $v \notin B^* \cup L_1^* \cup L_3^*$, then $uv \notin E^*$. Thus, $(E^L_{\mathcal{P}}(H) \setminus e) \cap E^* \subseteq E^L_{\mathcal{P}^*}(H)$.

Symmetric arguments using F' show that $e \in E^L_{\mathcal{P}^*}(H)$ and $(E^R_{\mathcal{P}}(H) \setminus e') \cap E^* \subseteq E^R_{\mathcal{P}^*}(H)$. $\square$

Claim 2 implies that $|E^*| \leq |E^L_{\mathcal{P}^*}(H)| + |E^R_{\mathcal{P}^*}(H)| - 1$. This yields

$$\begin{aligned}
d^+(p) + d^-(q) &\leq n - 2 + |E^*_1| + |C^*| + |R^*| - |L^*_1| \\
&\leq n - 2 + |E^L_{\mathcal{P}^*}(H)| + |E^R_{\mathcal{P}^*}(H)| - 1 + |C^*| + |R^*| - |L^*_1|, \qquad (10)\\
d^-(p) + d^+(q) &\leq n - 2 + |E^*_2| + |C^*| + |L^*| - |R^*_1| \\
&\leq n - 2 + |E^L_{\mathcal{P}^*}(H)| + |E^R_{\mathcal{P}^*}(H)| - 1 + |C^*| + |L^*| - |R^*_1|, \qquad (11)\\
d^+(p) + d^+(q) &\leq n - 2 + |E^*_3| + |C^*| + |L^*_2| + |R^*_2| \\
&\leq n - 2 + |E^L_{\mathcal{P}^*}(H)| + |E^R_{\mathcal{P}^*}(H)| - 1 + |C^*| + |L^*_2| + |R^*_2|, \qquad (12)\\
d^-(p) + d^-(q) &\leq n - 2 + |E^*_1| + |C^*| + |L^*_3| + |R^*_3| \\
&\leq n - 2 + |E^L_{\mathcal{P}^*}(H)| + |E^R_{\mathcal{P}^*}(H)| - 1 + |C^*| + |L^*_3| + |R^*_3|. \qquad (13)
\end{aligned}$$

Inequalities (10) and (11) now yield Theorem 3, and inequalities (10)-(13) yield Theorem 4. $\square$

4. Ore-Woodall-Type Conditions for H-Linkage in Directed Graphs

In [5], sharp Ore-type degree conditions were given that assure an undirected graph G is H-linked for an arbitrary simple graph H. Let

$$a(H) := \max_{A \cup B = V(H)} \left(|E(A, B)| + |B| - \Delta_B(A)\right).$$

Theorem 11. [5] *Let H be a simple graph and G be a graph of order $n > 20|E(H)|$. If*

$$\sigma_2(G) \geq n + a(H) - 2,$$

then G is H-linked. This degree bound is sharp.

For directed graphs, we achieve a similar, yet more complicated, bound. As it is the case for undirected graphs, multiedges yield complications to such a degree that a bound depending mostly on a partition of the vertex set appears infeasible. Thus, the results in this section apply only to digraphs. Let $\mathcal{P} = (A, B, L, R)$ be a partition of $V(H)$ with $B \neq V(H)$, and let

$$E_{\mathcal{P}}(H) := E(A \cup L, B \cup L) \cup E(B \cup R, A \cup R).$$

Let

$$\vec{a}_{\mathcal{P}}(H) := |E_{\mathcal{P}}(H)| + |B| + \min\left\{|R| - \Delta^+_{B \cup L}(A \cup L), |L| - \Delta^-_{B \cup R}(A \cup R)\right\},$$

and let

$$\vec{a}(H) := \max_{\mathcal{P}} \vec{a}_{\mathcal{P}}(H).$$

Note that for all H, $\vec{a}(H) \geq \vec{b}(H)$, as for a 9-part partition $\mathcal{P}$ maximizing $\vec{b}_{\mathcal{P}}(H)$, you can take $\mathcal{P}'$ with $A' = A$, $B' = B \cup C \cup L_3 \cup R_2$, $L' = L_1 \cup L_2$ and $R' = R_1 \cup R_3$ and observe that $\vec{a}_{\mathcal{P}'}(H) \geq \vec{b}_{\mathcal{P}}(H)$. Further, note that $\vec{a}(H') \leq \vec{a}(H)$ whenever H' is a subdigraph of H. The proof of these simple facts is left to the reader.

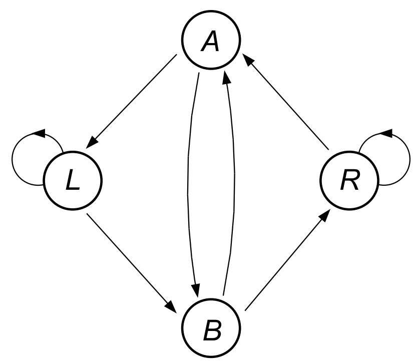

FIGURE 2. The sharpness-determining partition of H.

Theorem 12. *Let H and D be digraphs with $|D| > n_0(H) + 222|E(H)|^2$. If*

$$\sigma_2(D) \geq |D| + \vec{a}(H) - 2,$$

then D is H-linked. This degree bound is sharp.

Proof. We begin by showing that the degree sum condition is sharp. Let $\mathcal{P} = (A, B, L, R)$ be a partition of $V(H)$ with $\vec{a}_{\mathcal{P}}(H) = \vec{a}(H)$. For $n \geq |H| + |E_{\mathcal{P}}(H)| - 1$, let B' be a set of $n - |H| - |E_{\mathcal{P}}(H)| + 1$ new vertices, let C be a set of $|E_{\mathcal{P}}(H)| - 1$ new vertices, and let D be the directed graph with

$$V(D) = V(H) \cup B' \cup C, \text{ and}$$
$$E(D) = (V(D) \times V(D)) \setminus (E_{\mathcal{P}}(H) \cup ((A \cup L) \times B') \cup (B' \times (A \cup R))).$$

Then, every path corresponding to an edge in $E_{\mathcal{P}}(H)$ in an H-linkage must use a vertex in C, but C contains too few vertices, so D is not H-linked. Yet, the degree bound from Theorem 12 is missed by 1.

We now come to the proof of the positive statement. For the sake of contradiction, we assume that H is a minimal counterexample and that D is edge maximal without containing an H-linkage. This implies that H does not contain isolated vertices. Further, if $e \in E(H) \cap E(D)$, an $(H - e)$-linkage in D can trivially be extended to an H-linkage, so $E(H) \cap E(D) = \emptyset$.

Let $m := |E(H)|$, and we may assume that $m \geq 2$ as the statement is trivial for $m \leq 1$. If $\delta^0(D) \geq 34m$, we are done by Theorem 3, thus we may assume that there is a vertex v with $d^0(v) < 34m$, we may assume by symmetry that $d^+(v) < 34m$. Let $Y = V(D) \setminus N^+[v]$ and $y \in Y$. Then

$$d^+(v) + d^-(y) \geq |N^+[v|| + |Y| + \vec{a}(H) - 2,$$

and thus

$$\delta_Y^-(Y) > |Y| - 34m.$$

So at most $34m|Y|$ edges are missing inside Y. Setting

$$Y_1 := \{y \in Y : d_Y^+(y) \geq 80m\}, \text{ and } Y_2 := Y \setminus Y_1,$$

we get

$$|Y_2| < \frac{34m|Y|}{|Y| - 80m} \leq 43m,$$

as $|Y| \geq 222m^2 - 34m \geq 410m$. Thus, for any pair $y_1, y_2 \in Y_1$, we have $d_{Y_1}^+(y_1) + d_{Y_1}^-(y_2) > |Y_1| + 3m$, and therefore Y_1 is m-linked by Lemma 9. Let $B \supseteq Y_1$ be a maximal m-linked set. If $B = V(D)$ we are done, so we may assume that $B \neq V(D)$. By Lemma 6 , all vertices $x \in V(D) \setminus B$ have $d_B^0(x) < 2m$. Let

$$\begin{aligned}
A :=& \{v \in V(D) \setminus B : d_B^+(v) < 2m, d_B^-(v) < 2m\},\\
L :=& \{v \in V(D) \setminus B : d_B^+(v) < 2m, d_B^-(v) \geq 2m\},\\
R :=& \{v \in V(D) \setminus B : d_B^+(v) \geq 2m, d_B^-(v) < 2m\}.
\end{aligned}$$

Then $A \cup R \subseteq N^+[v]$ and $L \subseteq N^+[v] \cup Y_2$. We have

$$((A \cup L) \times (A \cup R)) \cup ((B \cup R) \times (B \cup L)) \subseteq E(D)$$

by the degree sum condition and by Lemma 7, respectively. Let $\mathcal{P} = (A^H, B^H, L^H, R^H)$ be the partition induced on H. Then $E(H) = E_{\mathcal{P}}(H)$ since $E(H) \cap E(D) = \emptyset$. We have

$$\begin{aligned}
|B \setminus (N^+(A) \cup N^-(A) \cup N^+(L) \cup N^-(R))| &> |D| - 2m(2|A| + |L| + |R|)\\
&\geq |D| - 2m(2|N^+[v]| + |Y_2|)\\
&\geq |D| - 222m^2 \geq 0,
\end{aligned}$$

so there is a vertex $b \in B \setminus (N^+(A) \cup N^-(A) \cup N^+(L) \cup N^-(R))$.

We will show that D not only contains an H-linkage (which would suffice for the desired contradiction), but it contains an H-linkage F, where every path in F corresponding to an edge in $E(H)$ has length exactly 2. We will consider partitions $\mathcal{P}^* = (A^*, B^*, L^*, R^*)$ of $V(H)$ with $A^* \subseteq A^H$, $L^* \subseteq A^H \cup L^H$ and $R^* \subseteq A^H \cup R^H$ and use induction on $2|A^*| + |L^*| + |R^*|$ to show that D contains internally disjoint paths of length 2 corresponding to all edges in $E_{\mathcal{P}^*}(H) \subseteq E_{\mathcal{P}}(H) = E(H)$.

The statement is trivially true for $2|A^*| + |L^*| + |R^*| = 0$ as then $E_{\mathcal{P}^*}(H) = \emptyset$. Next consider a partition $\mathcal{P}^*$ with $2|A^*| + |L^*| + |R^*| \geq 1$ and assume that the statement is true for all partitions with smaller values. Let $a \in A^* \cup L^*$ maximizing $d_{\mathcal{P}^*}^+(a) := |E_{\mathcal{P}^*}(a, L^* \cup B^*)|$, and $a' \in A^* \cup R^*$ maximizing $d_{\mathcal{P}^*}^-(a') := |E_{\mathcal{P}^*}(B^* \cup R^*, a')|$.

We may assume that $d^+_{\mathcal{P}^*}(a)+|L^*| \geq d^-_{\mathcal{P}^*}(a')+|R^*|$, as the other case can be handled with a symmetric argument. If $a \in A^* \subseteq A^H$, let $\mathcal{P}^{**}$ be the partition of $V(H)$ with

$$A^{**} = A^* - a,\ B^{**} = B^*,\ L^{**} = L^*,\ \text{and}\ R^{**} = R^* + a.$$

If $a \in L^* \subseteq A^H \cup L^H$, let $\mathcal{P}^{**}$ be the partition of $V(H)$ with

$$A^{**} = A^*,\ B^{**} = B^* + a,\ L^{**} = L^* - a,\ \text{and}\ R^{**} = R^*.$$

Note that in either case, $A^{**} \subseteq A^H$, $L^{**} \subseteq A^H \cup L^H$ and $R^{**} \subseteq A^H \cup R^H$. Then $2|A^{**}| + |L^{**}| + |R^{**}| = 2|A^*| + |L^*| + |R^*| - 1$, and by induction D contains a subgraph F consisting of internally disjoint paths of length 2 corresponding to all edges in $E_{\mathcal{P}^{**}}(H) = E_{\mathcal{P}^*}(H) \setminus E_{\mathcal{P}^*}(a, V(H))$. Then

$$\begin{aligned}|D| - 2 + |E_{\mathcal{P}^*}(H)| + |B^*| + |R^*| - d^+_{\mathcal{P}^*}(a)&\\ &\leq d^+(a) + d^-(b)\\ &\leq_{(*)} |D| - 2 - |E_H(a, L^H)| + |N^+(a) \cap N^-(b)|\\ &\leq |D| - 2 - d^+_{\mathcal{P}^*}(a) + |N^+(a) \cap N^-(b)|,\end{aligned}$$

where $(*)$ is true since $E(H) \cap E(D) = \emptyset$ and thus $N^+_H(a) \cap (N^+(a) \cup N^-(b)) = \emptyset$. So, as $N^-(b) \cap (A^* \cup L^*) = \emptyset$,

$$\begin{aligned}|(N^+(a) \cap N^-(b)) \setminus V(H)| &\geq |(N^+(a) \cap N^-(b))| - |B^*| - |R^*|\\ &\geq |E_{\mathcal{P}^*}(H)| = |E_{\mathcal{P}^{**}}(H)| + d^+_{\mathcal{P}^*}(a).\end{aligned}$$

Thus, there are at least $d^+_{\mathcal{P}^*}(a)$ vertices in $N^+(a) \cap ((B^* \cup R^*) \setminus V(F))$, and we can use these as internal vertices in paths of length 2 corresponding to all edges in $E_{\mathcal{P}^*}(a, V(H))$.

□

Remark: The bound on $|D|$ in Theorem 12 can be lowered some, but we have elected not to make the necessary modifications to the proof in the interest of concision.

5. m-Linked and m-Ordered Digraphs

We now turn our attention to m-linked and m-ordered graphs, and demonstrate how we may obtain Theorem 1 from Theorem 4. To do so, we explain how the partition $\mathcal{P}$ is constructed for mT_2 and C_m, respectively. We get $\vec{b}^1_{\mathcal{P}}(mT^2) = \vec{b}_{\mathcal{P}}(mT^2) = 2k$ by setting $L_2 = \{s_1\}$, $L_3 = \{t_1\}$, $R_2 = \{s_2\}$, $R_3 = \{t_2\}$, and $C = \{s_3, t_3, \ldots, s_m, t_m\}$, where s_it_i are the i edges of the matching. We get $\vec{b}^1_{\mathcal{P}}(C^m) = \vec{b}_{\mathcal{P}}(C^m) = m$ by setting $L_2 = \{v_1\}$, $B = \{v_2\}$, $R_3 = \{v_3\}$, and $C = \{v_4, \ldots, v_m\}$, where $C^k = v_1v_2\ldots v_mv_1$. In both cases it is fairly straightforward to check that the given partition is optimal. Thus we obtain the following corollaries which improve significantly upon the order bounds in Theorem 1.

Corollary 13. *Let D be a digraph, and $m \geq 2$.*

(a) *If $\delta^0(D) \geq 40m$ and $\sigma_2(D) \geq |D| + 2m - 2$, then D is m-linked.*

(b) *If* $|D| \geq 80m$ *and* $\delta^0(D) \geq \frac{1}{2}(|D| + 2m - 2)$*, then* D *is* m*-linked.*

Corollary 14. *Let* D *be a digraph, and* $m \geq 3$.

(a) *If* $\delta^0(D) \geq 40m$ *and* $\sigma_2(D) \geq |D| + m - 2$*, then* D *is* m*-ordered.*

(b) *If* $|D| \geq 80m$ *and* $\delta^0(D) \geq \frac{1}{2}(|D| + m - 2)$*, then* D *is* m*-ordered.*

In a similar way, we obtain previously unknown tight Ore-Woodall-type bounds for m-linked and m-ordered digraphs. First, $\vec{a}_{\mathcal{P}}(mT_2) = \left\lfloor \frac{5}{2}m \right\rfloor - 1$ by setting $A = \emptyset$, $L = \{s_1, s_2, \ldots, s_{\lfloor \frac{m}{2} \rfloor}\}$, $R = \{t_{\lfloor \frac{m}{2} \rfloor + 1}, \ldots, t_m\}$, and $B = V(mT_2) \setminus (L \cup R)$. Similarly, $\vec{a}_{\mathcal{P}}(C_m) = \left\lfloor \frac{3}{2}m \right\rfloor - 1$ by setting $A = \{v_2, v_4, \ldots\}$, $B = \{v_1, v_3, \ldots\}$, and $L = R = \emptyset$. In both cases it is fairly straightforward to check that the given partition is optimal (but not unique). Thus we obtain the following corollaries.

Corollary 15. *If* $|D| \geq 222m^2$ *and* $\sigma_2(D) \geq |D| + \left\lfloor \frac{5}{2}m \right\rfloor - 3$*, then* D *is* m*-linked.*

Corollary 16. *If* $|D| \geq 222m^2$ *and* $\sigma_2(D) \geq |D| + \left\lfloor \frac{3}{2}m \right\rfloor - 3$*, then* D *is* m*-ordered.*

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Current address, M. Ferrara, M. Jacobson: Department of Mathematical and Statistical Sciences, University of Colorado Denver, Denver, CO 80217

E-mail address: `michael.ferrara@ucdenver.edu;michael.jacobson@ucdenver.edu`

Current address, F. Pfender: Universität Rostock, Institut für Mathematik, 18057 Rostock, Germany

E-mail address: `Florian.Pfender@uni-rostock.de`