NEW CONDITIONS FOR K-ORDERED HAMILTONIAN GRAPHS

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Abstract. We show that in any graph G on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y, we can fix the order of k vertices on a given cycle and find a hamiltonian cycle encountering these vertices in the same order, as long as $k < n/12$ and G is $\lceil (k+1)/2 \rceil$-connected. Further we show that every $\lfloor 3k/2 \rfloor$-connected graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y is k-ordered hamiltonian, i.e. for every ordered set of k vertices we can find a hamiltonian cycle encountering these vertices in the given order. Both connectivity bounds are best possible.

1. Introduction

One of the most widely studied classes of graphs are hamiltonian graphs. In this paper we are interested in the following question: When can we guarantee a certain set S of vertices to appear on a hamiltonian cycle in a given order? In [**?**], Ng and Schultz first explored the following related concept introduced by Chartrand. A graph is called *k-ordered hamiltonian*, if for every vertex set S of size k there is a hamiltonian cycle encountering the vertices in S in a given order. Clearly, every hamiltonian graph is 3-ordered hamiltonian. Ng and Schultz [**?**] showed that k-ordered hamiltonian graphs must be $(k-1)$-connected. Further, they showed the following theorem.

Theorem 1. [**?**] *Let G be a graph of order n and let k be an integer with $3 \leq k \leq n$. If $d(u) + d(v) \geq n + 2k - 6$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

This bound was later improved in [**?**] and [**?**] by Faudree et al. for small values of k.

Theorem 2. [**?**] *Let G be a graph of order n and let k be an integer with $3 \leq k \leq n/2$. If $d(u) + d(v) \geq n + (3k - 9)/2$ for every pair u, v of nonadjacent vertices of G, then G is k-ordered hamiltonian.*

Instead of increasing the bound on the degree sum from the Ore-bound for hamiltonicity as in these papers, we choose to ask for a higher connectivity with the resultant effect of being able to lower the degree sum condition. We will first prove the following theorem.

Theorem 3. *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k < n/12$ be an integer, and let C be a cycle encountering a vertex sequence $S = \{x_1, \ldots, x_k\}$ in the given order. If G is $\lceil (k+1)/2 \rceil$-connected, then G has a hamiltonian cycle encountering S in the given order.*

Corollary 4. *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k < n/12$ be an integer, and let C be a cycle encountering a vertex sequence $S = \{x_1, \ldots, x_k\}$ in the given order. If G is $\lceil (k+1)/2 \rceil$-connected, then G has a hamiltonian cycle encountering S in the given order.*

The connectivity bound is best possible, as illustrated by the following graph G_1. Let L, K, R be complete graphs with $|R| = \lceil (2n - k)/4 \rceil$, $|K| = \lfloor k/2 \rfloor$, $|L| = n - |K| - |R|$. Let G_1 be the union of the three graphs, adding all possible edges containing vertices of K. Clearly, $\delta(G_1) > n/2$, and G_1 is $\lfloor k/2 \rfloor$-connected. Let $S = \{x_1, \ldots, x_k\}$ with $x_i \in K$ if i is even and $x_i \in R$ otherwise. The cycle $C = x_1 x_2 \ldots x_k x_1$ contains S in the right order, but no cycle containing S in the right order can contain any vertices of L.

A graph is called *k-ordered*, if for every vertex sequence S of size k there is a cycle encountering the vertices in S in the given order. Now observe that every k-ordered graph is $(k-1)$-connected. Thus, we get the following corollaries (these are very similar to theorems used in [**?**] and [**?**]).

Corollary 5. *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k < n/12$ be an integer, and suppose that G is k-ordered. Then G is k-ordered hamiltonian.*

Corollary 6. *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k < n/12$ be an integer, and suppose that G is k-ordered. Then G is k-ordered hamiltonian.*

We further prove the following theorem.

Theorem 7. *Let G be a graph on n vertices with $d(x) + d(y) \geq n$ for any two nonadjacent vertices x and y. Let $k \leq n/176$ be an integer. If G is $\lfloor 3k/2 \rfloor$-connected, then G is k-ordered hamiltonian.*

The connectivity bound is best possible, as illustrated by the following graph G_2. Let L_2, K_2, R_2 be complete graphs with $|R_2| = \lfloor k/2 \rfloor$, $|K_2| = 2\lfloor k/2 \rfloor - 1$, $|L_2| = n - |K_2| - |R_2|$. Let G_2' be the union of the three graphs, adding all possible edges containing vertices of K_2. Let $x_i \in L_2$ if i is odd, and let $x_i \in R_2$ otherwise. Add all edges $x_i x_j$ whenever $|i-j| \notin \{0, 1, k-1\}$, and the resulting graph is G_2. The degree sum condition is satisfied and G_3 is $(\lfloor 3k/2 \rfloor - 1)$-connected. But there is no cycle containing the x_i in the right order, since such a cycle would contain $2\lfloor k/2 \rfloor$ paths through K_2.

For the analogous theorem with a bound on the minimum degree we get a slight improvement on the connectivity bound for odd k.

Theorem 8. *Let G be a graph on n vertices with minimum degree $\delta(G) \geq n/2$. Let $k \leq n/176$ be an integer. If G is $3\lfloor k/2 \rfloor$-connected, then G is k-ordered hamiltonian.*

Again, the connectivity bound is best possible, as illustrated by the following graph G_3. Let L_3, K_3, R_3 be complete graphs with $|R_3| = \lceil (n-k)/2 \rceil$, $|K_3| = 2\lfloor k/2 \rfloor - 1$, $|L_3| = n - |K_3| - |R_3|$. Let G_3' be the union of the three graphs, adding all possible edges containing vertices of K_3. Let $x_i \in L_3$ if i is odd, and let $x_i \in R_3$ otherwise. Add all edges $x_i x_j$ whenever $|i - j| \notin \{0, 1, k-1\}$, and the resulting graph is G_3. The degree condition is satisfied, and G_3 is $(3\lfloor k/2 \rfloor - 1)$-connected. But there is no cycle containing the x_i in the right order, since such a cycle would contain $2\lfloor k/2 \rfloor$ paths through K_3.

2. Proof of Theorem ??

Assume that C is a maximal cycle encountering S in the given order. If C is hamiltonian, we are done. So, assume $|C| < n$, and let H be a component of $G - C$, say $|H| = r$. The sequence S splits C into k segments $[x_1 C x_2], \ldots, [x_k C x_1]$.

Claim 1. *There is at most one adjacency of H in each segment $[x_i C x_{i+1}]$.*

Suppose the contrary. Let x, y be two adjacencies of H inside $[x_i C x_{i+1}]$ with no other adjacencies of H in (xCy). Let $v \in H \cap N(x)$. Let $|(xCy)| = s$. Since v is not insertible in C we get

$$d(v) \leq r - 1 + \frac{n - r - s + 1}{2}.$$

Insert the vertices of (xCy) one by one into $[yCx]$. If all of them can be inserted, we can extend C through v, so there is a vertex w that can not be inserted. We get

$$d(w) \leq s - 1 + \frac{n - r - s + 1}{2},$$

so
$$d(v) + d(w) \leq n - 1,$$
a contradiction. This proves the claim. □

By claim **??**, C has at most k adjacencies to H. Let $v \in H$, and $w \in C$ be a vertex not adjacent to H. Then
$$n \leq d(v) + d(w) \leq (r - 1 + k) + (n - r - 1) = n + k - 2.$$
Thus, w is adjacent to all but at most $k - 2$ vertices of $G - H$. Further, v is adjacent to all but at most $k - 2$ vertices in H. We claim that H is hamiltonian connected as follows: Either H is complete and we are done, or two vertices $v, u \in H$ are not adjacent. Then $|H| \geq \frac{d(v)+d(u)}{2} - k \geq \frac{n}{2} - k$, using Claim **??** and the degree sum condition. Now $\delta_H(H) \geq |H| - k + 2 > |H|/2 + 1$, which implies hamiltonian connectedness.

Claim 2. *$G - C$ has at most one component.*

Suppose the contrary, let H' be another component with $|H'| = r'$. Let $v \in H$, $v' \in H'$. Since G is $\lceil (k+1)/2 \rceil$-connected, H can be adjacent to at most $\lfloor (k-1)/2 \rfloor$ vertices from S, else there is a contradiction with Claim **??**. The same is true for H'. Thus, for some i, $x_i \notin N(H) \cup N(H')$. But now,
$$3n \leq 2(d(x_i) + d(v) + d(v')) \leq$$
$$2((n - r - r' - 1) + (r - 1 + k) + (r' - 1 + k)) =$$
$$2n + 4k - 6,$$
a contradiction that proves the claim. □

Since G is $\lceil (k+1)/2 \rceil$-connected, there is a segment $[x_j C x_{j+2})$ with two adjacencies y, z of H. By claim **??**, we may assume that $y \in [x_j C x_{j+1})$, and $z \in (x_{j+1} C x_{j+2})$. If $|H| \geq k$ we can even guarantee that $|(N(y) \cup N(z)) \cap H| \geq 2$.

Claim 3. $|C| \geq n/2$.

Suppose $|C| < n/2$. Then $|H| \geq n/2$, and y, z could be picked such that $uy, vz \in E(G)$ for two vertices $u, v \in H$. Find a hamiltonian path P in H from u to v. Observe that $N(x_{j+1}) \cup N(x_{j+2}) \subseteq C$. If $x_{j+1}x_{j+2} \in E(G)$, then the cycle $uPvzC^- x_{j+1}x_{j+2}Cx_j u$ is longer than C, a contradiction. Thus, $x_{j+1}x_{j+2} \notin E(G)$. But now
$$|C| \geq \frac{d(x_{j+1}) + d(x_{j+2})}{2} + 2 > \frac{n}{2},$$
the contradiction proving the claim. □

For the final contradiction we differentiate two cases.

Case 1. *There exists a vertex $w \in (yCx_{j+1}) \cup (zCx_{j+2})$.*

Let $N = N(x_{j+1}) \cap N(x_{j+2}) \cap N(w)$. Since none of the vertices x_{j+1}, x_{j+2}, w is adjacent to H, each is adjacent to all but at most $k-2$ vertices of the cycle. Thus, $|N| \geq |C| - 3k + 6$.

Claim 4. *For some i, $|N \cap [x_i C x_{i+1}]| \geq 4$.*

Suppose not, then

$$n/2 \leq |C| \leq 3k + |C| - |N| \leq 6k - 6,$$

a contradiction for $n \geq 12k$. □

Let i be as in the last claim, and let $v_1, v_2, v_3, v_4 \in N \cap [x_i C x_{i+1}]$ be the first four of these vertices in that order.
If $v_4 \in (yCx_{j+1}]$, define a new cycle as follows:$C' = zC^- v_4 x_{j+2} CyuPvz$ (see Figure **??**).

.42k4.eps

Figure 1. a possible C'

If $v_4 \in (zCx_{j+2}]$, let $C' = zC^- x_{j+2} v_4 CyuPvz$.
Otherwise observe that by claim **??**, there is at most one adjacency x of H in $[v_1 C v_4]$.
For $i \neq j+1$, define the new cycle C' as follows:
If $x \in [v_1 C v_2]$, let $C' = zC^- x_{j+1} v_3 x_{j+2} C v_2 w v_4 CyuPvz$ (see Figure **??**).

.42k3

Figure 2. a possible C'

If $x \in [v_3 C v_4]$, let $C' = zC^- x_{j+1} v_2 x_{j+2} C v_1 w v_3 CyuPvz$.
Otherwise, let $C' = zC^- x_{j+1} v_2 C v_3 x_{j+2} C v_1 w v_4 CyuPvz$.
For $i = j+1$, a very similar construction works:
let $C' = zC^- v_4 w v_1 C^- x_{j+1} v_2 C v_3 x_{j+2} CyuPvz$.
In any case, no vertex in $C - C'$ is adjacent to H, so all of them have high degree to C and thus high degree to $C \cap C'$. Therefore, we can insert them one by one into C' creating a longer cycle, a contradiction.

Case 2. *Suppose* $(yCx_{j+1}) \cup (zCx_{j+2}) = \emptyset$.

Let $N' = N(x_{j+1}) \cap N(x_{j+2})$. Then $|N'| \geq |C| - 2k + 4$.

Claim 5. *For some* l, $|N' \cap [x_l C x_{l+1}]| \geq 5$.

Suppose not. Then

$$n/2 \leq |C| \leq 4k + |C| - |N'| \leq 6k - 4,$$

a contradiction for $n \geq 12k$. □

Let l be as in the last claim, and let $z_1, z_2, z_3, z_4, z_5 \in N' \cap [x_l C x_{l+1}]$ be the first five of these vertices in that order. At most one of them is adjacent to H, say z_2. Now a very similar argument as in the last case gives the desired contradiction, just replace x_{j+1} by z_1, x_{j+2} by z_5, and w by z_4. One possible cycle would then be (for $l < i < j$): $C' = zC^- x_{j+1} z_2 C z_3 x_{j+2} C z_1 v_2 C v_3 z_5 C v_1 z_4 v_4 C y u P v z$ (see Figure **??**). □

.5kord2.eps

FIGURE 3. a possible C'

3. Proof of Theorems ?? and ??

By Corollary **??**, all we need to show is that G is k-ordered. For this purpose, we will use a slightly stronger concept.

We will say that a graph G on at least $2k$ vertices is k-*linked*, if for every vertex set $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ of $2k$ vertices, there are k disjoint $x_i y_i$-paths. The property remains the same if we allow repetition in T, and ask for k internally disjoint $x_i y_i$-paths. Thus, as an easy consequence, every k-linked graph is k-ordered.

An important theorem about k-linked graphs is the following theorem of Bollobás and Thomason:

Theorem 9. [?] *Every* $22k$*-connected graph is* k*-linked.*

The following lemmas will be used later.

Lemma 10. *If a* $2k$*-connected graph* G *has a* k*-linked subgraph* H*, then* G *is* k*-linked.*

Proof: Let $T = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ be a set of $2k$ vertices in $V(G)$. Since G is $2k$-connected, there are $2k$ disjoint paths from T to $V(H)$ (trivial paths for vertices in $T \cap H$). Now we can connect these paths in the desired way inside H, since H is k-linked. □

Lemma 11. *If* G *is a graph,* $v \in V(G)$ *with* $d(v) \geq 2k - 1$*, and if* $G - v$ *is* k*-linked, then* G *is* k*-linked.*

Proof: Let $T = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$ be a set of $2k$ vertices in $V(G)$. If $v \notin T$, we can find disjoint x_iy_i-paths inside $G - v$. Thus assume that $v \in T$, without loss of generality we may assume that $v = x_1$. If $y_1 \in N(v)$, we can find disjoint x_iy_i-paths for all $i \geq 2$ in $G - v - y_1$, since $G - v - y_1$ is $(k-1)$-linked. Adding the path vy_1 completes the desired set of paths in G. If $y_1 \notin N(v)$, then there exists a vertex $x_1' \in N(v) - T$, since $d(v) \geq 2k - 1$. We can find disjoint x_iy_i-paths for $i \geq 2$ and a $x_1'y_1$-path in $G - v$, which we can then extend to an x_1y_1-path in G. $\square$

Further, we will use a theorem of Mader about dense graphs:

Theorem 12. [?] *Every graph G with $|V(G)| = n \geq 2k - 1$, and $|E(G)| \geq (2k-3)(n-k+1)+1$ has a k-connected subgraph.*

Corollary 13. [?] *Every graph G with $|V(G)| = n \geq 2k - 1$, and $|E(G)| \geq 2kn$ has a k-connected subgraph.*

Proof of Theorem ??. Let G be a graph fulfilling the stated conditions. Let $S = \{x_1, \dots, x_k\}$ be a set of k vertices. To show that G is k-ordered we need to find a cycle C including the vertices of S in the given order. Corollary **??** will then provide Theorem **??**. Let K be a minimal cutset of G. Let L and R be two components of $G - K$ with $|L| \leq |R|$.

Case 1. *Suppose $|K| \geq 2k$.*

The degree sum condition forces $|E(G)| \geq n^2/4 \geq 44kn$. By Corollary **??**, G has a $22k$-connected subgraph H, which is k-linked by Theorem **??**. By Lemma **??**, G is k-linked and thus k-ordered.

Case 2. *Suppose $3\lfloor k/2 \rfloor \leq |K| \leq 2k - 1$.*

First note that L and R are the only components of $G - K$. Otherwise, let $x \in L$, $y \in R$, $z \in G - (K \cup L \cup R)$, then

$$\begin{aligned} 3n &\leq 2d(x) + 2d(y) + 2d(z) \\ &\leq 2|L| + 2|K| + 2|R| + 2|K| + 2(n - |L| - |R|) \\ &\leq 2n + 4|K| < 2n + 8k, \end{aligned}$$

a contradiction.

Claim 1. *R is k-linked, and L is k-linked or complete.*

Let $v \in L, w \in R$. Then

$$n \leq d(v) + d(w) \leq |L| - 1 + |K| + |R| - 1 + |K| \leq n + 2k - 3.$$

Thus w is connected to all but at most $2k - 3$ vertices in R. Therefore, R is $2k$-connected. Again,

$$|E(R)| \geq |R|(|R| - 2k + 2) \geq |R|(n/2 - 3k + 2) \geq 44k|R|.$$

Thus, R has a $22k$-connected and therefore k-linked subgraph, and so R is k-linked by Corollary **??**, Theorem **??** and Lemma **??**.

If L is complete we are done. Otherwise, let $x, y \in L$ with $xy \notin E$, then

$$|L| \geq \frac{d(x)+d(y)}{2} - |K| \geq \frac{n}{2} - 2k + 1.$$

Every vertex in L is connected to all but at most $2k-3$ vertices in L, therefore L is $2k$-connected. By a similar argument as before, L is k-linked, establishing the claim. □

Claim 2. *For every vertex $v \in K$, at least one of the following holds:*

(1) $d_R(v) \geq 2k$,
(2) $d_L(v) \geq 2k$,
(3) $d_L(v) = |L|$.

Suppose the claim is false for some vertex $v \in K$. Let $x \in L - N(v)$, $y \in R - N(v)$. Then

$$\begin{aligned} 2n &\leq d(x) + 2d(v) + d(y) \\ &< |L| + |K| + 2(|K| + 4k) + |R| + |K| \\ &\leq n + 3|K| + 4k < n + 10k, \end{aligned}$$

a contradiction. □

The last claim yields a partition of K as follows:

$$\begin{aligned} K_R &= \{v \in K \mid d_R(v) \geq 2k\}, \\ K_{L1} &= \{v \in K \mid d_L(v) \geq 2k\} - K_R, \\ K_{L2} &= \{v \in K \mid d_L(v) = |L|\} - (K_R \cup K_{L1}). \end{aligned}$$

Note that either $K_{L1} = \emptyset$ or $K_{L2} = \emptyset$, and that the graph induced on K_{L2} is complete, since all vertices in K_{L2} have degree less than $4k$.

Now let $R' = \langle R \cup K_R \rangle, L' = \langle L \cup K_{L1} \cup K_{L2} \rangle$. By Claim **??**, Claim **??** and Lemma **??**, R' is k-linked and L' is k-linked or complete.

For the last part of the proof, let $S_L = L' \cap S$, $S_R = R' \cap S$. Create a new graph G' as follows: For every i with $x_i \in S_L$ and $x_{i-1}, x_{i+1} \in S_R$, add a vertex x_i' with $N(x_i') = N(x_i) \cup \{x_i\}$. It is easy to see that G' is $\lfloor 3k/2 \rfloor$-connected. Therefore, $G' - S_R$ is $(\lfloor 3k/2 \rfloor - |S_R|)$-connected. Using this fact, we can find independent paths in $G' - S_R$ from each of the vertices in $S_L \cup \bigcup x_i'$ into $R' - S_R$, since $|S_L \cup \bigcup x_i'| \leq \min\{k, 2|S_L|\} \leq 3k/2 - |S_R|$. Denote the set of last edges of these paths by M. Now contract the edges $x_i x_i'$ to get back to G.

The existence of the cycle C is now guaranteed, since we can pick appropriate vertices in $S_L \cup (M \cap L')$ and in $S_R \cup (M \cap R')$, and then use the fact that R' is k-linked and L' is k-linked or complete to find the necessary connections. This completes the proof of Theorem **??**.

□

Proof of Theorem ??. Observe that the connectivity only played a role in the last part of the previous proof. Let G be a graph as in Theorem **??**. If G is $\lfloor 3k/2 \rfloor$-connected, we are done by Theorem **??**. Thus, we may assume that k is odd and G has a minimal cut set of size $3\lfloor k/2 \rfloor$. Further, we know that G splits in two parts L' and R', each of which is k-linked (observe that the degree condition forces $|L'| > 2k$) by the proof of Theorem **??**.

Since k is odd, there are two consecutive vertices in S on the same side, we may assume x_1 and x_k is such a pair. Since G is $(3(k-1)/2)$-connected, there exists a matching $M = \{e_1, \ldots, e_{3(k-1)/2}\}$ of edges between R' and L'. We can renumber the edges of M such that $e_i \cap S \subseteq \{x_i\}$ for all $i \leq k-2$, and $e_{k-1} \cap S \subseteq \{x_{k-1}, x_k\}$. Let $x_{k+1} = x_1$. To construct the cycle C, we need to find $x_i x_{i+1}$-paths for all $i \leq k$. If $x_i \in L'$ and $x_{i+1} \in R'$, or if $x_i \in R'$ and $x_{i+1} \in L'$, we want to find a path from x_i to e_i through $L'(R')$ and a path from e_i to x_{i+1} through $R'(L')$. Note that this case can only occur if $i \leq k-1$. If $x_i, x_{i+1} \in L'(R')$, we want to find a $x_i x_{i+1}$-path in $L'(R')$. The simultanuous existence of all these paths is guaranteed since R' and L' are k-linked. This completes the proof of Theorem **??**.

□

References

[1] B. Bollobás, C. Thomason, Highly Linked Graphs, *Combinatorica* **16** (1996), no.3, 313–320.

[2] G. Chartrand, L. Lesniak, “Graphs & Digraphs”, Chapman and Hall, London, 1996.

[3] J. Faudree, R. Faudree, R. Gould, M. Jacobson, L. Lesniak, On k-Ordered Graphs, *J. Graph Theory* **35** (2000), no.2, 69–82.

[4] R. Faudree, R. Gould, A. Kostochka, L. Lesniak, I. Schiermeyer, A. Saito, Degree Conditions For k-ordered Hamiltonian Graphs, preprint.

[5] W. Mader, Existenz von n-fach zusammenhängenden Teilgraphen in Graphen genügend grosser Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97.

[6] L. Ng, M. Schultz, k-Ordered Hamiltonian Graphs, *J. Graph Theory* **24** (1997), no.1, 45–57.