CLAW-FREE 3-CONNECTED P_{11}-FREE GRAPHS ARE HAMILTONIAN

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Abstract. We show that every 3-connected claw-free graph which contains no induced copy of P_{11} is hamiltonian. Since there exist non-hamiltonian 3-connected claw-free graphs without induced copies of P_{12} this result is, in a way, best possible.

1. Statement of the main result

A graph G is $\{H_1, H_2, \ldots H_k\}$-free if G contains no induced subgraphs isomorphic to any of the graphs H_i, $i = 1, 2, \ldots, k$. A graph without induced copies of $K_{1,3}$ is called claw-free, and a graph containing no copies of K_3 is triangle-free.

Broersma and Veldman [3] showed the following theorem. (Here and below P_k denotes the path on k vertices.)

Theorem 1. *If G is a 2-connected $\{K_{1,3}, P_6\}$-free graph, then G is hamiltonian.*

Bedrossian [1] characterized all pairs of forbidden subgraphs X, Y, such that every 2-connected $\{X, Y\}$-free graph is hamiltonian. Later, Faudree and Gould [6] extended that list under the extra condition that the graph has at least ten vertices.

In the above results, it is natural to consider 2-connected graphs, as this is a neccessary condition for hamiltonicity. In this paper we study 3-connected graphs instead to see what kind of results we can achieve with this extra condition. We show the following result analogous to Theorem 1.

Theorem 2. *Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is hamiltonian.*

This extends a result from Brousek *et al.* [5], who showed as a corollary of a result about 2-connected claw-free graphs that every 3-connected $\{K_{1,3}, P_7\}$-free graph is hamiltonian.

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Furthermore, in the last section of the paper, we give an example of an infinite family of non-hamiltonian 3-connected $\{K_{1,3}, P_{12}\}$-free graphs.

2. Closure, cycle closure and line graphs

We start with some definitions and notation (for terminology not defined here we refer the reader to [2]). For a graph G which contains at least one cycle the *circumference* of G, denoted by $c(G)$, is the length of a longest cycle contained in G. We denote the *neighborhood* of a set of vertices $X \subseteq V(G)$ in a graph G by $N(X)$. Similarly, the *closed neighborhood* of a set of vertices $X \subseteq V(G)$ is $N[X] = X \cup N(X)$. We write $L(G)$ for the *line graph* of G. A graph G is *essentially k-edge-connected* if the deletion of less than k edges leaves at most one component with more than one vertex. In this paper by *circuit* we mean a closed trail, possibly of length zero. A circuit C is *dominating* if every edge in G is incident to at least one vertex of C.

The *closure* $\mathrm{cl}(G)$ of a graph G is the minimal $(K_4 - e)$-free graph containing G as a spanning subgraph. This notion was introduced by Ryjáček [10], who also characterized basic properties of the closure operation.

Theorem 3. *Let G be a claw-free graph. Then:*

(i) $\mathrm{cl}(G)$ *is uniquely determined by* G,
(ii) *there is a (unique) triangle-free graph* H *such that* $\mathrm{cl}(G) = L(H)$,
(iii) $c(\mathrm{cl}(G)) = c(G)$,
(iv) G *is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

A claw-free graph G is *closed*, if $\mathrm{cl}(G) = G$. By (ii), all closed graphs consist of a collection of maximal cliques, each two of which share at most one vertex. A class $\mathcal{P}$ of graphs is called *stable under* cl, if $G \in \mathcal{P}$ implies $\mathrm{cl}(G) \in \mathcal{P}$ for every claw-free graph G. Brousek *et al.* [5] showed the following theorem.

Theorem 4. *The class of $\{K_{1,3}, P_\ell\}$-free graphs is stable under* cl *for any $\ell \geq 3$.*

Broersma and Ryjáček [4] expanded on the closure operation and introduced the *cycle closure* of a claw-free graph G, $\mathrm{cl}_C(G)$, as follows.

Let G be a closed claw-free graph and let C be an induced cycle of length k. We say that the cycle C is *eligible in* G if $4 \leq k \leq 6$ and if the k-cycle $L^{-1}(C)$ in $H = L^{-1}(G)$ contains at least $k - 3$ nonconsecutive vertices of degree two in H.

For an eligible cycle C in G set $B_C = \{uv \mid u,v \in N_G[C], uv \notin E(G)\}$. The graph G'_C with vertex set $V(G'_C) = V(G)$ and edge set $E(G'_C) = E(G) \cup B_C$ is called the *C-completion of G at C*.

Definition 1. *Let G be a claw-free graph. We say that a graph H is a* cycle closure *of G, denoted $H = \mathrm{cl}_C(G)$, if there is a sequence of graphs $G_1, \dots, G_t$ such that*

(i) $G_1 = \mathrm{cl}(G)$,
(ii) $G_{i+1} = \mathrm{cl}((G_i)'_C)$ *for some eligible cycle C in G_i, $i = 1, \dots, t-1$,*
(iii) $G_t = H$ *contains no eligible cycles.*

For the cycle closure, the following is true.

Theorem 5. [4] *Let G be a claw-free graph. Then*

(i) $\mathrm{cl}_C(G)$ *is well defined (i.e. uniquely determined),*
(ii) $c(G) = c(\mathrm{cl}_C(G))$.

We will start by showing the following theorem about the cycle closure.

Theorem 6. *The class of $\{K_{1,3}, P_\ell\}$-free graphs is stable under cl_C for any $\ell \geq 3$.*

Proof. By Theorems 4 and 5, it is sufficient to show that G'_C is P_ℓ-free for every $\{K_{1,3}, P_\ell\}$-free graph G, and any eligible cycle C.

Suppose, to the contrary, that G'_C contains an induced P_ℓ, $P = x_1x_2\dots x_\ell$. Since G is P_ℓ-free and $E(G'_C) = E(G) \cup B_C$, $E(P)$ contains at least one edge of B_C. Since $G'_C[N[C]]$ is complete, $E(P)$ contains at most two vertices in $N[C]$. Thus, $E(P)$ contains exactly one edge $e \in B_C$, say $e = x_ix_{i+1}$, and $V(P) \cap N[C] = \{x_i, x_{i+1}\}$. Take a shortest path R in G from x_i to x_{i+1} using only vertices from $V(C)$ as internal vertices to create a path $P' = x_1 \dots x_iRx_{i+1} \dots x_\ell$. As $V(P) \cap N[C] = \{x_i, x_{i+1}\}$, P' is induced, contradicting the fact that G is P_ℓ-free. This proves the theorem. □

Let G be a 3-connected claw-free graph closed under cl_C. Let $L^{-1}(G)$ be the unique line graph original, i.e. the unique graph whose line graph is identical with G, guaranteed by Theorem 3(ii). Similarly, let F be a claw-free graph closed under cl_C.

The following are well known facts about line graphs:

Fact 7. *If G is a line graph, the following are true:*

(i) *G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected.*

(ii) [8] *G is hamiltonian if and only if $L^{-1}(G)$ has a dominating circuit.*
(iii) *F is an induced subgraph of G if and only if $L^{-1}(F)$ is a (not necessarily induced) subgraph of $L^{-1}(G)$.*

Let $\bar{L}(G)$ be the graph obtained from $L^{-1}(G)$ after deleting all vertices of degree one and after replacing all vertices of degree two by edges between their two neighbors. Let $\mathcal{M} = \mathcal{M}(\bar{L}(G)) \subseteq V(\bar{L}(G))$ be the set of vertices which were neighbors of vertices of degree less or equal two in $L^{-1}(G)$. From Fact 7, we get the following statements about $\bar{L}(G)$.

Fact 8. *The following are true:*
(i) *$\bar{L}(G)$ is well defined.*
(ii) *$\bar{L}(G)$ is triangle-free.*
(iii) *$\bar{L}(G)$ is 3-edge-connected.*
(iv) *G is hamiltonian if and only if $\bar{L}(G)$ has a dominating circuit covering all vertices in $\mathcal{M}$.*

Proof. By Fact 7(i), $L^{-1}(G)$ is essentially 3-edge-connected, therefore the vertices of degree less than 3 form an independent set in $L^{-1}(G)$, and the graph $\bar{L}(G)$ resulting from their deletion/replacement contains no vertices of degree less than three. Furthermore, there are no triangles or multiple edges in $\bar{L}(G)$ as G is closed under cl_C, and $L^{-1}(G)$ thus contains no induced k-cycles with at least $k-3$ vertices of degree two, where $3 \le k \le 6$. This establishes (i) and (ii).

Clearly, $\bar{L}(G)$ is essentially 3-edge-connected, since $L^{-1}(G)$ is essentially 3-edge-connected, and each edge cut in $\bar{L}(G)$ induces an edge cut of the same size in $L^{-1}(G)$. Again, there are no vertices of degree less than three in $\bar{L}(G)$, so this implies (iii).

Finally, it is easy to see that every dominating circuit in $L^{-1}(G)$ induces a dominating circuit covering all vertices in $\mathcal{M}$ in $\bar{L}(G)$ and vice versa, together with Fact 7(ii) this establishes (iv). $\square$

Fact 9. *If G is P_ℓ-free for some $\ell \ge 3$ and G is non-hamiltonian, then $\bar{L}(G)$ contains none of the following as a (not necessarily induced) subgraph:*
(i) *$P_{\ell+1}$,*
(ii) *$P_\ell = x_1x_2\dots x_\ell$ with $x_1 \in \mathcal{M}$,*
(iii) *$P_{\ell-1} = x_1x_2\dots x_{\ell-1}$ with $x_1, x_{\ell-1} \in \mathcal{M}$.*

Proof. If $\bar{L}(G)$ contains a $P_{\ell+1}$ or a $P_\ell = x_1x_2\dots x_\ell$ with $x_1 \in \mathcal{M}$, then $L^{-1}(G)$ contains a $P_{\ell+1}$, which contradicts the fact that G is P_ℓ-free by Fact 7(iii).

Thus, assume that $\bar{L}(G)$ contains a path $P_{\ell-1} = x_1x_2\dots x_{\ell-1}$ with $x_1, x_{\ell-1} \in \mathcal{M}$. Let v be a vertex in $N_{L^{-1}(G)}(x_1)$ with $d(v) \le 2$, and let u be a vertex in $N_{L^{-1}(G)}(x_{\ell-1})$ with $d(u) \le 2$. If $u \ne v$, then the path in $L^{-1}(G)$ which corresponds to $vx_1x_2\dots x_{\ell-1}u$ contains a path of length $\ell + 1$, which, again, is not possible. Therefore, $u = v$, $d(u) = 2$ and $x_1x_2\dots x_{\ell-1}x_1$ is a cycle in $\bar{L}(G)$. As G is not hamiltonian, $x_1x_2\dots x_{\ell-1}x_1$ is not a dominating circuit covering $\mathcal{M}$ in $\bar{L}(G)$ by Fact 8(iv). Thus, there is another vertex $y \in V(\bar{L}(G))$, connected to some x_k. Now the path in $L^{-1}(G)$ corresponding to $yx_k\dots x_{\ell-1}ux_1\dots x_{k-1}$ contains a path of length $\ell + 1$, the final contradiction. □

Thus, Theorem 2 will follow from Fact 9 and the following lemma.

Lemma 10. *Let G be a triangle-free 3-edge-connected graph and let $\mathcal{M} \subseteq V(G)$ be a subset of its vertices. Then G contains one of the following:*

(i) *a dominating circuit containing all vertices in $\mathcal{M}$,*
(ii) P_{12},
(iii) $P_{11} = v_1v_2\dots v_{11}$ *with* $v_1 \in \mathcal{M}$,
(iv) $P_{10} = v_1v_2\dots v_{10}$ *with* $v_1, v_{10} \in \mathcal{M}$.

3. Graphs without long paths

In this section we prove Lemma 10. Our argument includes an elementary but laborious analysis of cases, so we start with stating a few simple facts we shall repeatedly use in this part of the paper.

Fact 11. *Let $P = v_1v_2\dots v_\ell$ be a longest path in a connected graph G.*

(i) $N(v_1) \subseteq V(P)$. *Moreover, $v_\ell \notin N(v_1)$ unless P is a hamiltonian path.*
(ii) *If some v_i, $2 \le i \le \ell - 2$, has a neighbor outside $V(P)$, then $v_{i+1} \notin N(v_1)$.*
(iii) *If $w \notin V(G) \setminus V(P)$ is adjacent to v_2 and v_j for some $2 \le i < j \le \ell - 1$, then $v_{j-1} \notin N(v_1)$.*

Proof. It is easy to check that if any of the conditions (i)–(iii) fails, then G contains a path longer than P. □

Fact 12. *Let $P = v_1\dots v_\ell$ be a longest path in a 2-connected, 3-edge-connected, triangle-free graph G, and let H denote the graph induced in G by $V(G) \setminus V(P)$.*

(i) *If $\ell \le 10$, then $V(G) \setminus V(P)$ is an independent set.*

(ii) *If* $\ell = 11$, *then all components of* H *which contain more than one vertex are stars, with vertices* $z, y_1, \dots, y_k$, *such that the neighborhood of each of vertices* y_i, $i = 1, 2, \dots, k$, *consists of* z, v_4, *and* v_8.

Proof. Suppose there exists a vertex z lying at distance two from P. Then, since G is 2-connected, there are two vertex-disjoint paths which join z with two different vertices of P, each of length at least two. Hence, for some $k \geq 3$ and $2 \leq i < j \leq \ell - 1$, there exists a path $P' = v_i w_1 \dots w_k v_j$ such that $w_1, \dots, w_k \notin V(P)$. Note that $i \geq 3$, since otherwise the path $w_k w_{k-1} \dots w_1 v_i v_{i+1} \dots v_\ell$ is longer than P. Similarly, $j \leq \ell - 3$. But then the path $v_1 \dots v_i w_1 \dots w_k v_j \dots v_\ell$ is longer than P unless $k = 3$ and $\ell = 11$. Hence, if a vertex z lies at distance two from P, then $\ell = 11$ and all paths from z to P have length two and join z with one of the vertices v_4, v_8. All other vertices are within distance one from P.

Let F be a component of H. If it contains a vertex which lies at distance two from P, then, as we have just proved, it must be a star of the type described above. Thus, let us assume that all vertices of F have at least one neighbor on P. Note also that F cannot contain a cycle. Indeed, since G is triangle-free, such a cycle would have at least four vertices; this would imply that two different vertices of P are connected by an "external" path P' of length at least five, which, as we have seen above, is impossible. Thus, since the minimum degree of G is three, at least two vertices of F, say, x and y, have at least two neighbors each on P. Furthermore, if x and y are not adjacent, one can argue as above that F must be a star of the type described in (ii), so we may assume that xy is an edge of G. Let W denote the set of the vertices of P which are adjacent to one of the vertices x and y. Since G is triangle-free the neighborhoods of x and y are disjoint, and so $|W| \geq 4$. Note also that no two vertices of W are consecutive vertices of P, and neighbors of x and y must lie at distance at least three on P, since this will lead to a longer path. Thus, at least one of the vertices v_2 and $v_{\ell-1}$ must belong to W, say, v_2 is adjacent to x. But then the path $yxv_2v_3 \dots v_\ell$ is longer than P, contradicting the choice of P. □

We call a graph G super-eulerian if it contains a circuit which goes through every vertex of G, i.e., if it has a spanning Eulerian subgraph. The following two facts are easy consequences of the above definition.

Fact 13. *Let* G *be a complete bipartite graph with bipartition* (V_1, V_2), *where* $|V_1| = 3$ *and* $|V_2| = k$. *Then, if* $k \geq 2$, G *contains a circuit which covers all vertices of* V_2. *Moreover, if* $k \geq 3$, *then for every two* *different vertices $v, v' \in V_1$ there is a trail in G which starts at v, ends in v', and covers every vertex of G.* □

Fact 14. *Let $H_1, \dots H_m$ be edge-disjoint subgraphs of a graph G, and let F denote the graph with vertices $H_1, \dots, H_m$ in which two vertices H_i, H_j are adjacent if and only if $V(H_i) \cap V(H_j) \neq \emptyset$. If each H_i, $i = 1, \dots, m$, is super-eulerian, $V(G) = \bigcup_i V(H_i)$, and F is connected, then G is super-eulerian.*

In particular, if each block of a connected graph G is super-eulerian, then G is super-eulerian as well. □

We shall also use the following result of Favaron and Fraisse [7], which is a consequence of the nine-point theorem by Holton *et al.* [9].

Lemma 15. *If a graph G is 3-edge-connected, then for every nine vertices of G there is a circuit going through all these vertices.*

In particular, each 3-edge-connected graph on at most nine vertices is super-eulerian.

Before we prove Lemma 10 we show the following lemma.

Lemma 16. *Every triangle-free 3-edge-connected graph which does not contain a P_{10} as a subgraph is super-eulerian.*

Proof. Let G be a triangle-free 3-edge-connected graph without a P_{10}. From Fact 14 and Lemma 15 it follows that we may assume that G is a 2-connected graph on at least ten vertices. Let $P = v_1 \dots v_\ell$, $\ell \leq 9$, denote a longest path in G. Fact 12 implies that all vertices $x \in V(G) \setminus V(P)$ have at least three neighbors on P. Note that since G has at least ten vertices the set $V(G) \setminus V(P)$ is non-empty.

Since G is triangle-free, and v_1, v_ℓ have no neighbors outside P (Fact 11(i)), we must have $\ell \geq 7$. Let us first consider the case $\ell = 7, 8$. Let $x \in V(G) \setminus V(P)$ and v_i, v_j, v_k, $2 \leq i < i+1 < j < j+1 < k \leq \ell - 1$ be neighbors of x on P. It is easy to check using Fact 11 that then the only three neighbors of v_1 on P are v_2, v_j and v_k, and v_ℓ can be adjacent only to v_i, v_j and $v_{\ell-1}$. Consequently, all vertices in $V(G) \setminus V(P)$ must have the same neighborhood v_i, v_j and v_k. Since $|V(G) \setminus V(P)| \geq 2$, G contains a circuit K which covers all vertices of $V(G) \setminus V(P)$ and uses no edges joining two vertices of P (see Fact 13 above). Note also that the circuit $K' = v_1 v_2 \dots v_\ell v_j v_1$ contains all vertices of P. Combining K and K' we get a circuit which goes through all vertices of G, and so G is super-eulerian.

Now suppose that $\ell = 9$. Then we split all vertices of $V(G) \setminus V(P)$ into two sets, S_1 and S_2. The set S_1 consists of all the vertices which are adjacent to at least one of the "odd" vertices v_3, v_5, v_7, while

$S_2 = V(G) \setminus (V(P) \cup S_1)$. It is easy to verify that for any vertex $x \in S_1$ which has neighbors v_i, v_j, v_k, $i < j < k$, v_1 is adjacent to v_k, v_9 must be adjacent to v_i, and at least one of the vertices v_1 and v_9 is adjacent to v_j. If $x \in S_2$, then we claim only that v_1 is adjacent to at least two of the vertices v_4, v_6 and v_8, and at least two of the vertices v_2, v_4, v_6 are neighbors of v_ℓ. Note however, that the above observation implies that one of the sets S_1, S_2 is empty. Hence, let us consider the two following cases.

Case 1. $S_2 = \emptyset$.

As we have already observed each $x \in S_1$ determines uniquely if v_1 is adjacent to v_7 or v_8, and if v_9 is adjacent to v_2 or v_3. Thus, there are two vertices $v_i, v_k \in V(P)$ such that for every $x \in S_1$, v_i is adjacent to x and v_9, while v_k is a neighbor of both x and v_1.

Consider first the case that $|S_1| = |V(G) \setminus V(P)|$ is odd. Then, we can cover all but one element of S_1, say x, by a circuit K which contains no edges with both ends in $V(P)$ (Fact 13). Combining K with $v_1v_2 \ldots v_9v_ixv_jv_1$ proves that G is super-eulerian (Fact 14).

If $|S_1|$ is even, then again we apply Fact 13 to find a circuit which contains all but two, say x, x', vertices of S_1, and uses only edges incident to S_1. Now it is enough to find a circuit K on vertices $V(P) \cup \{x, x'\}$. Assume that x has neighbors v_i, v_j and v_k, $i < j < k$, and v_j is adjacent to, say, v_1. Then $K = v_1 \ldots v_9v_ix'v_kxv_jv_1$.

Case 2. $S_1 = \emptyset$.

As before our aim is to show that one can cover all vertices of G by a number of edge-disjoint circuits (note that each circuit must contain at least two vertices from $V(P)$).

Let us partition S_2 into sets S_2' and S_2'', where S_2' consists of all vertices which are adjacent to both vertices v_4 and v_6, while $S_2'' = S \setminus S_2'$. We show first that for every $x \in S_2$ there exists a circuit with vertex set $V(P) \cup \{x\}$. Let us consider two subcases.

Case 2a. $x \in S_2'$.

One can verify using Fact 11 that there are two neighbors $v', v'' \in V(P)$ of x such that v_1 is adjacent to v' and v_9 is adjacent to v''. Hence $v_1v_2 \ldots v_9v''xv'v_1$ is a circuit we are looking for.

Case 2b. $x \in S_2''$.

Let us assume that x is adjacent to v_2, v_4 and v_8 (the symmetric case in which x is adjacent to v_2, v_6 and v_8 can be dealt with in a similar way). If there are two neighbors v' and v'' of x such that v_1 is adjacent to v' and v_9 is adjacent to v'' we can proceed as in the previous case.

Thus, assume that it is not the case. Then both vertices v_1 and v_9 are adjacent to both v_4 and v_6. Hence $v_1v_6v_7v_8xv_2\dots v_6v_9v_4v_1$ is a circuit we are looking for.

Now suppose $|S_2| = |V(G)\setminus V(P)| \geq 2$. Each two vertices x, y, from S_2 share at least two neighbors, hence, they lie on a cycle of length four. Consequently, if $|S_2|$ is odd, then we can cover all but one vertex (say, x) of S_2 by edge-disjoint cycles and combine them with a circuit with vertex set $V(P)\cup\{x\}$ to show that G is super-eulerian. An analogous argument can be used to prove that G is super-eulerian if $|S_2|$ is even and the vertices v_1 and v_9 have a common neighbor on P. Thus, let us assume that $|S_2| \geq 2$ is even and the vertices v_1 and v_9 share no neighbors. We cover all but two, say x_1, x_2, vertices of S_2 by edge-disjoint cycles of length four. Then it is easy to see that among the vertices v_2, v_4, v_6 and v_8 we find three, say v', v'', and v''', such that for some $\alpha \in \{1, 2\}$, v' is adjacent to both v_1 and x_α, v'' is adjacent to both x_1 and x_2, and v''' is adjacent to both $x_{3-\alpha}$ and v_9. Then, the circuit $v_1v_2\dots v_9v'''x_{3-\alpha}v''x_\alpha v'v_1$ covers all vertices from $V(P)\cup\{x_1, x_2\}$, and so G is super-eulerian. □

Proof of Lemma 10. Let G be a 3-edge-connected graph such that the vertex set of G is partitioned into two classes: the set $\mathcal{M}$ (the *major* vertices) and the set $V(G)\setminus\mathcal{M}$ (the *minor* vertices). Let ℓ be the number of vertices in a longest path in G and let $P = v_1\dots v_\ell$ denote a longest path for which the set $\{v_1, v_\ell\}$ contains the maximum number of major vertices. We show that if either

- $\ell \leq 10$ and at least one of the vertices v_1, v_ℓ is minor,

or

- $\ell = 11$ and both vertices v_1, v_ℓ are minor,

then there exists a dominating circuit K which contains all major vertices of G.

Note that we may assume that G is 2-connected (Fact 14) and $\ell \geq 10$ (Lemma 16).

Case 1. $\ell = 10$ and at least one of the vertices v_1, v_{10}, say v_1, is minor.

Note first that Lemma 15 implies that there is a circuit K covering the vertices $\{v_2, \dots, v_9\}$. Since it follows from Fact 11(i) and Fact 12 that the set $V(G)\setminus\{v_2, \dots, v_{10}\}$ is independent, either $V(G)\setminus V(K)$ is an independent set which consists of minor vertices and we are done, or the set S of all major vertices in $V(G)\setminus V(P)$ is non-empty. Since the minimum degree of G is three, each vertex $x \in S$ is adjacent to at least three vertices on P. Note however, that x is not adjacent to

v_2 since otherwise the path $xv_2v_3\dots v_{10}$ has the same length but more major ends than P. Furthermore, if v_i is a neighbor of x, not only v_{i+1} is not adjacent to v_1 (see Fact 11(iii)) but v_{i+2} is not a neighbor of v_1 either. Indeed, in this case P can be replaced by a path $xv_iv_{i-1}\dots v_1v_{i+2}v_{i+3}\dots v_{10}$ which starts at the major vertex x. Finally, if x is adjacent to $v_{\ell-1} = v_9$, then v_2 cannot be a neighbor of $v_\ell = v_{10}$, since otherwise the path $v_{10}v_2v_3\dots v_9x$ has one more major end than P.

There are ten possible ways of choosing three neighbors of x among the vertices $v_3, v_4, \dots, v_9$ in such a way that none of them are consecutive. However, using Fact 11 and the observations mentioned above, one can check by a direct inspection that in seven of these cases connecting the vertex v_1 with two vertices in $\{v_4, \dots, v_9\}$ immediately leads either to a longer path, or to a path of the same length as P but with more major ends. The three remaining cases are as follows:

- x is adjacent to v_3, v_7 and v_9. This forces v_1 to be adjacent to v_7 and v_9, while v_{10} is adjacent to v_3 and v_7.
- x is adjacent to v_4, v_6 and v_9. Then v_1 is adjacent to v_4 and v_9, while v_4 and v_6 are neighbors of v_{10}.
- x is adjacent to v_4, v_7 and v_9. Then v_4 and v_7 are neighbors of v_1, while v_{10} is adjacent to v_4 and v_7.

Furthermore, in all the cases, the degree of both v_1 and v_{10} is three. Thus, since in each of the above cases v_1 has a different neighborhood, all vertices of S must have the same neighbors on P.

Suppose that $|S| \geq 2$. Then, Fact 13 implies the existence of a circuit K which uses only edges incident to S and covers all vertices of S. Moreover, v_1 and v_{10} have a common neighbor $v' \in V(P)$, so all vertices of P lie at the circuit $K' = v_1 \dots v_{10}v'v_1$. Combining K and K' we obtain a dominating circuit which contains all major vertices of G.

Now suppose that $S = \{x\}$. Then, from the description of the three cases we deal with, we infer that x has two different neighbors on P, say v' and v'', such that v' is adjacent to v_1, while v'' is a neighbor of v_{10}. Hence the circuit $v_1 \dots v_{10}v''xv'v_1$ contains all major vertices of G and, since it contains all vertices of P, is dominating in G.

Case 2. $\ell = 11$ and both vertices v_1, v_{11}, are minor.

It follows from Lemma 15 that G contains a circuit K which goes through all the vertices $v_2, \dots, v_{10}$. Observe that without loss of generality we may assume that K contains all vertices of G which belong to non-trivial components of the graph H induced by $V(G) \setminus V(P)$. Indeed, it is enough to note that a graph induced by such a component and the vertices v_4 and v_8 contains both a spanning circuit as well as a spanning trail which starts at v_4 and ends at v_8 (Fact 14), which is easy to see with Fact 12(ii) (with the notation from Fact 12(ii), $v_4y_1zy_2v_4y_3v_8y_4v_4\dots$ and $v_4y_1zy_2v_8y_3v_4y_4v_8\dots$ would be a spanning trail and a spanning circuit, respectively). Thus, the set S of all major vertices of G which have at least three neighbors on P must be non-empty; otherwise K would be a dominating circuit which contains all major vertices.

Similarly as in the previous case one needs to examine all possible neighborhoods of $x \in S$, but now we can make use of the fact that both v_1 and v_{11} are minor so, for instance, no vertex from S is adjacent to v_{10}. It turns out that inspecting all possible candidates for neighbors of v_1 and v_{11} one can eliminate all but one case and infer that all vertices $x \in S$ must be adjacent to v_3, v_6 and v_9. This, in turn, forces v_1 to be adjacent to v_6 and v_9, and v_{11} to have v_4 and v_6 as its neighbors. But then the argument identical to that given in Case 1 shows that there exists in G a dominating circuit K which contains all major vertices. This completes the proof of Case 2 and Lemma 10. □

4. A non-hamiltonian 3-connected P_{12}-free claw-free graph

We conclude the paper by giving an example of a graph F which is claw-free and contains no induced copy of P_{12}, yet it is not hamiltonian, which shows that Theorem 2 is, in a way, best possible.

Let H be the graph obtained from the Petersen graph by attaching a pendant edge to each of its vertices. Let $F = L(H)$.

Fact 17. *The graph F is claw-free, 3-connected and non-hamiltonian. Moreover, it contains no induced copy of P_{12}.*

Proof. Clearly, F is claw-free like every line graph. Furthermore, F is 3-connected since H is essentially 3-edge-connected. As the Petersen graph is 3-regular, a dominating circuit of H would be in fact a dominating cycle. Since the Petersen graph is non-hamiltonian, such a cycle can not exist, and thus, F is non-hamiltonian by Fact 7(ii).

Moreover, H does not contain P_{13} as a subgraph, and therefore, F contains no induced copy of P_{12} by Fact 7(iii). □

Finally we remark that in the construction of H one can add more pendant edges to each of the ten vertices of the Petersen graph without making the graph $F = L(H)$ hamiltonian or creating any induced $K_{1,3}$ or P_{12}'s in F. Therefore, there are non-hamiltonian 3-connected $\{K_{1,3}, P_{12}\}$-free graphs on n vertices for every $n \geq 25$.

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