Number of maximal partial clones

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Abstract

All maximal partial clones on 4-element, 5-element, and 6-element sets have been found and are compared to the case of maximal clones of all total functions. Due to the large numbers of maximal partial clones other criteria to check for generating systems of all partial functions are analyzed.

1 Introduction

In many-valued logic finite basic sets are considered. We only have to consider the set $E_k := \{0, 1, \dots, k-1\}$ with $k \geq 3$ being fixed in the rest of this paper.

The set $P_k := \{f^{(n)} \mid f^{(n)} : E_k^n \to E_k, n \geq 1\}$ is the set of all total functions on E_k. Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called an n-ary partial function on E_k with domain D. We also write $\mathrm{dom}(f) = D$. Let $\widetilde{P}_k^{(n)}$ be the set of all n-ary partial functions on E_k and

$$\widetilde{P}_k := \bigcup_{n \geq 1} \widetilde{P}_k^{(n)}.$$

Let $C_\emptyset := \left\{ f \in \widetilde{P}_k \,\middle|\, \mathrm{dom}(f) = \emptyset \right\}$.

The n-ary function $e_i^{(n)}$ defined by $e_i^{(n)}(x_1, \dots, x_n) := x_i$ with $i \in \{1, \dots, n\}$ is called projection onto the i-th coordinate. Let $J_k := \left\{ e_i^{(n)} \,\middle|\, n \in \mathbb{N}, 1 \leq i \leq n \right\}$ be the set of all projections.

Let $f[g_1, \dots, g_n] \in \widetilde{P}_k^{(m)}$ be the *composition* as given in [2] with $f \in \widetilde{P}_k^{(n)}$ and $g_1, \dots, g_n \in \widetilde{P}_k^{(m)}$, i.e., $x \in \mathrm{dom}(f[g_1, \dots, g_n])$ iff

$$(x \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i)) \wedge (g_1(x), \dots, g_n(x)) \in \mathrm{dom}(f)$$

and

$$f[g_1, \dots, g_n](x) := f(g_1(x), \dots, g_n(x))$$

for all $x \in \mathrm{dom}(f[g_1, \dots, g_n])$.

A *partial clone* (*clone*) on E_k is a composition closed subset of $\widetilde{P}_k$ (P_k) containing J_k. Relations are useful to describe the clones of $\widetilde{P}_k$. We often write the elements of relations as columns and a relation can then be given as a matrix. For example the relation $\varrho = \{(0,1,2), (1,2,0), (3,4,5), (2,3,1)\}$ can also be written as

$$\varrho = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}.$$

Let a matrix be given by $C = (c_{ij})_{h \times n}$. Then c_{i*} are the rows of the matrix with $i \in \{1, \dots, h\}$, i.e., $c_{i*} = (c_{i1}, c_{i2}, \dots, c_{in})$, and c_{*j} are the columns of the matrix with $j \in \{1, \dots, n\}$, i.e., $c_{*j} = (c_{1j}, c_{2j}, \dots, c_{hj})^{\mathrm{T}}$.

Let $\mathcal{R}_k^{(h)}$ be the set of all h-ary relations on E_k and $\mathcal{R}_k := \bigcup_{h \geq 1} \mathcal{R}_k^{(h)}$.

An n-ary function $f^{(n)} \in \widetilde{P}_k$ *preserves* an h-ary relation $\varrho^{(h)} \in \mathcal{R}_k$ iff for all $c_{*1}, c_{*2}, \dots, c_{*n} \in \varrho$ with $c_{1*}, \dots, c_{h*} \in \mathrm{dom}(f)$ holds

$$f(c_{*1}, \dots, c_{*n}) := \begin{pmatrix} f(c_{11}, \dots, c_{1n}) \\ f(c_{21}, \dots, c_{2n}) \\ \vdots \\ f(c_{h1}, \dots, c_{hn}) \end{pmatrix} \in \varrho.$$

Let $\mathrm{pPOL}_k\, \varrho$ be the set of all functions $f \in \widetilde{P}_k$ which preserve the relation $\varrho \in \mathcal{R}_k$.

Let $f \in \widetilde{P}_k^{(1)}$ be a unary function. Define $f^0 := e_1^{(1)}$ and $f^n(x) := f(f^{n-1}(x))$ for all $n \geq 1$.

For each $m \in \mathbb{N}$ let $\eta_m := (0, 1, \dots, m-1)^{\mathrm{T}}$.

Let $\omega(v)$ be the set of entries of $v = (v_1, \dots, v_h) \in E_k^h$, i.e., $\omega(v) = \omega((v_1, \dots, v_h)) := \{v_1, \dots, v_h\}$. Additionally let $\omega(\varrho) = \bigcup_{v \in \varrho} \omega(v)$.

2 Theorem of Haddad and Rosenberg

Definition 1. Let for all h with $1 \leq h \leq k$

$$\begin{aligned}\varrho_1 &:= \{(a,a,b,b),(a,b,a,b) \mid a,b \in E_k\},\\ \varrho_2 &:= \{(a,a,b,b),(a,b,a,b),(a,b,b,a) \mid a,b \in E_k\},\\ \iota_k^h &:= \left\{(x_1,\ldots,x_h) \in E_k^h \;\middle|\; |\{x_1,\ldots,x_h\}| \leq h-1\right\}.\end{aligned}$$

Definition 2. Let ε be an arbitrary equivalence relation on E_h. Define $\delta_{k,\varepsilon}^{(h)} := \left\{(a_0,\ldots,a_{h-1}) \in E_k^h \;\middle|\; (i,j) \in \varepsilon \Longrightarrow a_i = a_j\right\}$. If h or k can be deduced from the context we just write δ_ε or $\delta_\varepsilon^{(h)}$ or $\delta_{k,\varepsilon}$. If the relation ε is given by the non-singular equivalence classes $\varepsilon_1,\ldots,\varepsilon_r$ then we write $\delta_{k;\varepsilon_1,\ldots,\varepsilon_r}^{(h)}$ or $\delta_{\varepsilon_1,\ldots,\varepsilon_r}$ instead of $\delta_{k,\varepsilon}^{(h)}$. For example $\delta_{k;E_h}^{(h)} = \left\{(x,x,\ldots,x) \in E_k^h \;\middle|\; x \in E_k\right\}$.

Definition 3. Let $\varrho^{(h)} \subseteq E_k^h$. Then we write $\sigma(\varrho) := \varrho \setminus \iota_k^h$ and $\delta(\varrho) := \varrho \cap \iota_k^h = \varrho \setminus \sigma(\varrho)$. If $\delta = \delta_\varepsilon$ for some equivalence relation ε then we write $\varepsilon(\varrho) := \varepsilon$.

Definition 4. Let $\varrho^{(h)} \subseteq E_k^h$. Then ϱ is

- *areflexive*, if $h \geq 2$ and $\delta(\varrho) = \emptyset$, i.e., for each $(x_1,\ldots,x_h) \in \varrho$ we have $x_i \neq x_j$ for all $1 \leq i < j \leq h$.
- *quasi-diagonal*, if $\sigma(\varrho)$ is a non-empty areflexive relation, $\delta(\varrho) = \delta_\varepsilon$ with $\varepsilon \neq \iota_h^2$ an equivalence relation.

Definition 5. Let $\varrho^{(h)} \subseteq E_k^h$, $\sigma := \sigma(\varrho)$ and $\delta := \delta(\varrho)$.

If $r = (r_0, r_1, \ldots, r_{n-1}) \in E_k^n$ is a tuple and $\pi \in S_n$ then we write $r^{[\pi]} := (r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(n-1)})$. Let $\Gamma_\sigma := \left\{\pi \in S_h \;\middle|\; \sigma \cap \sigma^{[\pi]} \neq \emptyset\right\}$, where S_h is the set of all permutations on E_h and $\sigma^{[\pi]} := \left\{s^{[\pi]} \;\middle|\; s \in \sigma\right\}$.

The *model* of ϱ is the h-ary relation $M(\varrho) := \left\{\eta_h^{[\pi]} \;\middle|\; \pi \in \Gamma_\sigma\right\} \cup (\delta \cap E_h^h)$ on E_h.

The relation ϱ is *coherent*, if the following conditions hold:

1. $\varrho \neq E_k^h$, $\varrho \neq \emptyset$,
2. (a) ϱ is a unary relation, i.e., $h = 1$, or
   (b) ϱ is areflexive with $2 \leq h \leq k$, or
   (c) ϱ is quasi-diagonal with $2 \leq h \leq k$, or
   (d) $\delta = \iota_k^h$ with $3 \leq h \leq k$, or
   (e) $\delta = \varrho_i$ with $i \in \{1,2\}$ (see Definition 1) and $h = 4$,
3. $r^{[\pi]} \in \sigma$ for all $r \in \sigma$ and all $\pi \in \Gamma_\sigma$,
4. for every σ' with $\emptyset \neq \sigma' \subseteq \sigma$ there is a relational homomorphism $\varphi : E_k \to E_h$ from σ' to $M(\varrho)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., $(\varphi(r_0),\ldots,\varphi(r_{h-1})) = (0,\ldots,h-1)$ for some $r = (r_0,\ldots,r_{h-1}) \in \sigma'$,
5. (a) if $\delta = \iota_k^h$ and $h \geq 3$ then $\Gamma_\sigma = S_h$,
   (b) if $\delta = \varrho_1$ then $\Gamma_\sigma = \langle(0231),(12)\rangle$ (Γ_σ is the permutation group generated by the cycles (0231) and (12)),
   (c) if $\delta = \varrho_2$ then $\Gamma_\sigma = S_4$.

Let $\widetilde{\mathcal{R}}_k^{\max}$ be the set of all coherent relations with $\mathrm{pPOL}_k\,\varrho \neq \mathrm{pPOL}_k\,\chi$ for all $\varrho,\chi \in \widetilde{\mathcal{R}}_k^{\max}$ and $\varrho \neq \chi$. Let

$$p\mathscr{M}_k := \{P_k \cup C_\emptyset\} \cup \left\{\mathrm{pPOL}_k\,\varrho \;\middle|\; \varrho \in \widetilde{\mathcal{R}}_k^{\max}\right\}.$$

Theorem 6 (of Haddad and Rosenberg; [3, 4])**.** *Let $k \geq 2$. For each $A \subset \widetilde{P}_k$ with $A = [A]_{\mathrm{P}}$ there is a maximal partial clone M_A with $A \subseteq M_A$. A clone M is a maximal partial clone of $\widetilde{P}_k$ if and only if $M \in p\mathscr{M}_k$, i.e., $p\mathscr{M}_k$ is the set of all maximal partial clones of $\widetilde{P}_k$.*

Theorem 7 (Completeness criterion for $\widetilde{P}_k$; [4])**.** *Let $C \subseteq \widetilde{P}_k$. Then $[C]_{\mathrm{P}} = \widetilde{P}_k$ if and only if $C \not\subseteq M$ for all $M \in p\mathscr{M}_k$.*

Definition 8. The set of coherent relations $\widetilde{\mathcal{R}}_k^{\max}$ can be divided into the following sets:

$$\begin{aligned}\mathcal{U} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 1\},\\ \mathcal{A} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is areflexive}\},\\ \mathcal{Q} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is quasi-diagonal}\},\\ \mathcal{S} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 3 \wedge \delta(\chi) = \iota_k^\mu\},\\ \mathcal{L} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 4 \wedge \delta(\chi) \in \{\varrho_1,\varrho_2\}\}.\end{aligned}$$

3 Number of maximal partial clones and maximal clones

Definition 9. Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$.

Define the *relation-class* $\mathrm{class}(\varrho)$ by

$$\mathrm{class}(\varrho) := \{\{(g(v))^{[\pi]} \mid v \in \varrho\} \mid g \in S_k, \pi \in S_h\}.$$

Let $p\mathscr{M}_k^{\mathrm{C}} := \{\mathrm{class}(\varrho) \mid \varrho \in \widetilde{\mathcal{R}}_k^{\max}\}$.

For the number of maximal partial clones $|p\mathscr{M}_k|$ in k-valued logic with $k \in \{2,3,4,5,6\}$ see Table 3. These are compared to the number of maximal (total) clones $|\mathscr{M}_k|$ as given in [7]. See Tables 1 and 2 for more detailed information about which types of relations contribute to the total number of coherent relations. The relations are split by arity which is shown in the second row of the header of the Tables. One can see that the ternary quasi-diagonal and areflexive relations contribute the biggest part to the number of maximal partial clones for $k \geq 4$. Especially the ternary relations $\varrho \in \mathcal{Q}$ with $\delta(\varrho) = \delta_{\{0,1\}}^3$ contribute 292440 for

$k = 5$ and 5008453443 for $k = 6$. That means these relations alone determine the magnitude of $|p\mathscr{M}_k|$, at least for these cases.

These numbers were found by a computer program which is described in [11]. The program is written in Haskell, is single-threaded and took about 52 hours on a SunFire V490 to compute all coherent relations for $k = 6$. The numbers found coincide with previous results; see [1] for $k = 2$, [6] and [8] (independent from each other) for $k = 3$, and [5] with corrections in [10] for $k = 4$.

The size of $p\mathscr{M}_k^{\mathrm{C}}$ seems to be related to $|p\mathscr{M}_k|$ in an interesting way as can be seen in Table 4. It seems reasonably to assume that

$$|p\mathscr{M}_k| \sim k! \cdot |p\mathscr{M}_k^{\mathrm{C}}|$$

for $k \geq 6$. Because $\mathrm{pPOL}_k\, \varrho = \mathrm{pPOL}_k\, \varrho^{[\pi]}$ for any $\varrho^{(h)}$ and $\pi \in S_h$, there are at most $|S_k| = k!$ different partial clones for every $\mathrm{class}(\varrho)$. Thus clearly $|p\mathscr{M}_k| \leq k! \cdot |p\mathscr{M}_k^{\mathrm{C}}|$.

Thinking about automatic checking for completeness of sets $C \subset \widetilde{P}_k$ the following idea might reduce the memory size needed for these checks.

Definition 10. Let $f^{(n)} \in \widetilde{P}_k$ and $g \in S_k$. Then define $F^{(n)} := f^g$ by

$$\mathrm{dom}(F) := \{(g^{-1}(x_1), \ldots, g^{-1}(x_n)) \mid (x_1, \ldots, x_n) \in \mathrm{dom}\, f\}$$

and

$$f^g(y_1, \ldots, y_n) := g^{-1}\left(f(g(y_1), \ldots, g(y_n))\right)$$

for all $(y_1, \ldots, y_n) \in \mathrm{dom}(F)$.

For $U \leq S_k$ let $f^U := \{f^g \mid g \in U\}$ and $f^\star := f^{S_k}$. For $C \subseteq \widetilde{P}_k$ let $C^U := \{f^U \mid f \in C\}$ and $C^\star := C^{S_k}$.

Theorem 11. *Let $C \subseteq \widetilde{P}_k$ and $T \subseteq \widetilde{\mathcal{R}}_k^{\max}$ with*

$$\widetilde{\mathcal{R}}_k^{\max} \subseteq \bigcup_{\varrho \in T} \mathrm{class}(\varrho).$$

Then $[C]_{\mathrm{P}} = \widetilde{P}_k$ if and only if

- *$C \not\subseteq P_k \cup C_\infty$, and*
- *$C^g \not\subseteq \mathrm{pPOL}_k\, \varrho$ for all $\varrho \in T$ and all $g \in S_k$.*

Proof. Let $M \in p\mathscr{M}_k \setminus \{P_k \cup C_\infty\}$. Then there is some $\chi^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$ with $M = \mathrm{pPOL}_k\, \chi$ by Theorem 6, and by assumption there is some $\varrho \in T$ with $\chi \in \mathrm{class}(\varrho)$. Thus we have some $g \in S_k$ and $\pi \in S_h$ with

$$\chi = \{g(v)^{[\pi]} \mid v \in \varrho\},$$

where we can assume w.l.o.g. $\pi = \mathrm{id}$. We have $C^g \not\subseteq \mathrm{pPOL}_k\, \varrho$, thus there is some $f^{(n)} \in C$ and $v_1, \ldots, v_n \in \varrho$ with

$$f^g(v_1, \ldots, v_n) := w \in E_k^h \setminus \varrho.$$

Thus

$$f(g(v_1), \ldots, g(v_n)) = g(w) \in g(E_k^h) \setminus g(\varrho) = E_k^h \setminus \chi$$

with $g(v_i) \in g(\varrho) = \chi$ for all $i \in \{1, \ldots, n\}$. That means $f \notin \mathrm{pPOL}_k\, \chi = M$.

Thus $C \not\subseteq M$ for all $M \in p\mathscr{M}_k$ and by Theorem 7 we have $[C]_{\mathrm{P}} = \widetilde{P}_k$. □

Example 12. Let $k = 6$ and we want to check if $C = \{f_1, \ldots, f_l\}$ is complete, i.e., $[C]_{\mathrm{P}} = \widetilde{P}_k$. We assume the following setting.

- Every tuple in the domain of an $f \in C$ needs $t = 10$ Bytes on average to store.
- There are less than 1000 tuples in all domains combined, i.e.,

$$d := \sum_{f \in C} |\,\mathrm{dom}\, f| < 1000.$$

  Because there exist binary partial Sheffer functions for $k = 6$, i.e., with less than 36 tuples in the domain, the restriction to 1000 tuples is not too restrictive.
- Every coherent relation takes about $r = 10$ Bytes to store in a convenient format for testing preservation of relations.

If we use the direct approach of storing all coherent relations in memory, then we need at least $r \cdot |p\mathscr{M}_k| = 10 \cdot 5242621816$ Bytes, approximately 50 Gigabytes.

If we use Theorem 11 instead, then we need about $t \cdot d \cdot k! + r \cdot |p\mathscr{M}_k^{\mathrm{C}}| < 10 \cdot 1000 \cdot 720 + 10 \cdot 7322017 = 80420170$ Bytes, approximately 80 Megabytes.

As $|p\mathscr{M}_k| \approx k!|p\mathscr{M}_k^{\mathrm{C}}|$ for $k = 6$ we have to make about the same number of tests in either case. Furthermore it is fast to generate the set C^g for all $g \in S_k$. Thus the use of Theorem 11 reduces the memory consumption considerably while leaving the number of tests nearly constant.

For $k = 5$ we have $r \cdot |p\mathscr{M}_k| = 3257220$ and $t \cdot d \cdot k! + r \cdot |p\mathscr{M}_k| < 1232870$ Bytes for the different approaches, respectively. This is not so important with the todays memory sizes, and we would make about 20% more tests with the second approach compared to the first. Thus the second approach would be worse for $k = 5$ in the given scenario.

Unfortunately even the number of relation classes $|p\mathscr{M}_k^{\mathrm{C}}|$ will propably grow very fast since $|p\mathscr{M}_k^{\mathrm{C}}| > |\mathscr{M}_k|$ for all $k \leq 6$. Thus a list of coherent relations seems not practical for bigger k. Under these circumstances we try to use the description of coherent relations directly without generating all coherent relations. We cannot hope to find precise criteria for completeness but sufficient and necessary conditions for complete systems should help in practice. The next chapter states such conditions.

4 Other completeness criteria

The number of coherent relations is quite large and it is computationally difficult to find all of them. Thus we give a list of coherent relations which are easy to enumerate and functions are easy to check for preservation against these relations. These give some necessary conditions for complete function systems.

Example 13. Let $C \subseteq \widetilde{P}_k$ with $[C]_{\mathrm{P}} = \widetilde{P}_k$. Then $C \not\subseteq \mathrm{pPOL}_k\,\varrho$ for all $\varrho \in \mathcal{U} \cup \{\iota_k^h \mid 3 \leq h \leq k\}$.

If $\varrho \in \mathcal{U}$ then we can check if $C \subseteq \mathrm{pPOL}_k\,\varrho$ in at most $\sum_{f \in C} |\operatorname{dom} f|$ steps, because every tuple in $\operatorname{dom} f$ is independent of each other with regard to ϱ in this case. Additionally, the set $\mathcal{U} = \{\chi \subset E_k \mid \chi \neq \emptyset\}$ can be enumerated fast. For $k = 6$ with the setting from the example above there are at most $|\mathcal{U}| \cdot \sum_{f \in C} |\operatorname{dom} f| < 2^6 \cdot 1000 = 64000$ single tuple tests to be done. This is very small with respect to checking all coherent relations.

If $\varrho \in \{\iota_k^h \mid 3 \leq h \leq k\}$ then the test is easy, and most expensive if $C = \{f\}$ for some function $f \in \widetilde{P}_k$, so just assume this is the case. Just take any h tuples $s_{1*}, \ldots, s_{h*} \in \operatorname{dom} f$ with $|\{f(s_{i*}) \mid i \in \{1, \ldots, h\}\}| = h$ and check that for every column s_{*j} we have at most $h-1$ different entries, i.e. $|\{s_{ij} \mid i \in \{1, \ldots, h\}\}| \leq h - 1$.

The next theorem gives a sufficient condition for completeness.

Theorem 14. *Let*

- $C := \{f_1^{(h_1)}, \ldots, f_l^{(h_l)}\} \subseteq \widetilde{P}_k$,
- $\varphi_1, \ldots, \varphi_m \in S_k$,
- $U_j := \{\varphi(v) \mid v \in U_{j-1}, \varphi \in \langle \varphi_1, \ldots, \varphi_j \rangle\}$ *for* $j \in \{1, \ldots, m\}$ *and* $U_0 := \{\eta_k\}$.

Let

1. $\langle \varphi_1, \ldots, \varphi_m \rangle = S_k$,
2. *for all* $j \in \{1, \ldots, m\}$ *there are* $i \in \{1, \ldots, l\}$ *and* $v_1, \ldots, v_{h_i} \in U_{j-1}$ *with* $f_i(v_1, \ldots, v_{h_i}) = \varphi_j(\eta_k)$,
3. *there are* $i \in \{1, \ldots, l\}$ *and* $v_1, \ldots, v_{h_i} \in U_m$ *with* $f_i(v_1, \ldots, v_{h_i}) \in \delta_\varepsilon^{(k)}$ *for some non-trivial equivalence relation* ε *such that there is some* $x \in E_k$ *with* $(x, y) \notin \varepsilon$ *for all* $y \in E_k \setminus \{x\}$*, i.e.,* ε *has a singular equivalence class,*
4. *for all* $\chi \in \{\iota_k^h \mid h \in \{3, \ldots, k\}\} \cup \{\varrho_1, \varrho_2\}$ *there is some* $f \in C$ *with* $f \notin \mathrm{pPOL}_k\,\chi$*, and*
5. *there is some* $f \in C$ *with* $f \notin P_k \cup C_\infty$.

Then $[C]_{\mathrm{P}} = \widetilde{P}_k$.

Proof. Assume $[C]_{\mathrm{P}} \neq \widetilde{P}_k$. Then there is some $M \in p\mathscr{M}_k$ with $C \subseteq M$. Because there is some $f \in C$ with $f \notin P_k \cup C_\infty$ we know that there is some coherent relation $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$ with $M = \mathrm{pPOL}_k\,\varrho$.

- We first consider $\sigma(\varrho) = \emptyset$. Then $\varrho \in \{\iota_k^h \mid h \in \{3, \ldots, k\}\} \cup \{\varrho_1, \varrho_2\}$ and thus there is some $f \in C$ with $f \notin \mathrm{pPOL}_k\,\varrho$, i.e. $C \not\subseteq M$.

- Now we see that there is some $v \in \sigma(\varrho)$ and we can assume $v = \eta_h$. That means $\{v\} = \mathrm{pr}_h\,U_0 := \mathrm{pr}_{0,1,\ldots,h-1}\,U_0$ and by (2) there is some function $f \in C$ such that $\varphi_1(v) = f(v, \ldots, v) \in \varrho$ because $f \in \mathrm{pPOL}_k\,\varrho$. Doing this repeatedly we get $\mathrm{pr}_h\,U_1 = \{\varphi(v) \mid \varphi \in \langle \varphi_1 \rangle\} \subseteq \varrho$.

  By iteration we get $\mathrm{pr}_h\,U_2, \mathrm{pr}_h\,U_3, \ldots, \mathrm{pr}_h\,U_m \subseteq \varrho$. Thus $\sigma(E_k^h) \subseteq \sigma(\varrho)$ because $\langle \varphi_1, \ldots, \varphi_m \rangle = S_k$ and therefore $U_m = \sigma(E_k^k)$. This implies that $G_{\sigma(\varrho)} = S_h$, i.e. ϱ is totally-symmetric.

  If $h = 1$ then $\sigma(E_k^h) = E_k$ and thus $\varrho = E_k$, i.e. $\varrho \notin \widetilde{\mathcal{R}}_k^{\max}$, in contradiction to the assumption.

  For $h \geq 2$ we know by (3) that $\delta(\varrho) \neq \emptyset$. This means that $\delta(\varrho) = \iota_k^2$ because ϱ is coherent. But then $\varrho = E_k^2$ in contradiction to the assumption that ϱ is coherent.

  For $h \geq 3$ we even know that $\delta(\varrho) \neq \delta_\varepsilon$ for all ε without singleton class. Assume $\delta(\varrho) = \delta_\varepsilon$ and ε has at least one singleton class. But then there is some $\pi \in S_h$ with $\delta(\varrho)^{[\pi]} \neq \delta(\varrho)$ in contradiction to ϱ coherent.

  Thus $\delta(\varrho) \in \{\iota_k^h, \varrho_1, \varrho_2\}$. If $h = 4$ then $\delta(\varrho) \neq \varrho_i$ for $i \in \{1, 2\}$ because of (3), i.e., $f_i(v_1, \ldots, v_{h_i}) = (x, x, y, z) \in \delta(E_k^4) \backslash \varrho_i$. Thus we see that $\delta(\varrho) = \iota_k^h = \delta(E_k^h)$. But then $\varrho = \sigma(\varrho) \cup \delta(\varrho) = E_k^h$ contradicting ϱ coherent.

Thus we can conclude that $C \not\subseteq M$ for all $M \in p\mathscr{M}_k$ and by Theorem 7 we get $[C]_{\mathrm{P}} = \widetilde{P}_k$. ☐

5 Conclusion

We have given the number of maximal partial clones up to $k = 6$ which forced us to think about different ways for checking for completeness of sets of partial functions. It might be interesting to investigate this further and find different approaches. There are still some open problems such as the following:

- What are good candidate sets T for Theorem 11?

- Is there a formula for the number of maximal partial clones like the one for the number of maximal total clones? See [9].

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k	$\lvert p\mathscr{M}_k \rvert$	$\lvert \mathcal{S} \rvert$				$\lvert \mathcal{L} \rvert$		$P_k \cup C_\emptyset$
		3	4	5	6			
2	8					1	1	1
3	58	1				1	1	1
4	1102	15	1			4	2	1
5	325722	1023	31	1		46	16	1
6	5242621816	1048575	32767	63	1	4141	786	1

Table 1. Number of maximal partial clones I

k	$\lvert \mathcal{Q} \cup \mathcal{A} \rvert$					
	1	2	3	4	5	6
2	2	3				
3	6	30	18			
4	14	416	505	144		
5	30	16457	295080	11945	1092	
6	62	1934514	5008589703	230676900	319722	14581

Table 2. Number of maximal partial clones II

k	$\lvert \mathscr{M}_k \rvert$	$\lvert p\mathscr{M}_k \rvert$
2	5	8
3	18	58
4	82	1 102
5	643	325 722
6	15 182	5 242 621 816
7	7 848 984	?
8	549 761 933 169	?

Table 3. Number of maximal (partial) clones

k	$\lvert p\mathscr{M}_k \rvert$	$\lvert p\mathscr{M}_k^{\mathrm{C}} \rvert$	$\frac{\lvert p\mathscr{M}_k \rvert}{\lvert p\mathscr{M}_k^{\mathrm{C}} \rvert}$	$k!$	$\frac{\lvert p\mathscr{M}_k \rvert}{\lvert p\mathscr{M}_k^{\mathrm{C}} \rvert \cdot k!}$
2	8	7	1.14	2	0.57
3	58	26	2.23	6	0.37
4	1102	138	7.99	24	0.33
5	325722	3287	99.1	120	0.82
6	5242621816	7322017	716	720	0.99

Table 4. Size of $p\mathscr{M}_k^{\mathrm{C}}$ in comparison to $p\mathscr{M}_k$