ON ORTHOGONAL DOUBLE COVERS BY TREES

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ABSTRACT. A collection $\mathcal{P}$ of n spanning subgraphs of the complete graph K_n is said to be an Orthogonal Double Cover (ODC) if every edge of K_n belongs to exactly two members of $\mathcal{P}$ and every two elements of $\mathcal{P}$ share exactly one edge. We consider the case when all graphs in $\mathcal{P}$ are isomorphic to some tree G and improve former results on the existence of ODCs, especially for trees G of short diameter and for trees G on few vertices.

1. INTRODUCTION

Let K_n be the (undirected) complete graph on n vertices. Throughout we assume that its set of vertices is $[n] := \{0, 1, \dots, n-1\}$. We say that two graphs on $[n]$ are *orthogonal* if they have exactly one edge in common. Let $\mathcal{P} = \{P_0, P_1, \dots, P_{n-1}\}$ be a collection of spanning subgraphs of K_n, called *pages* of $\mathcal{P}$. We call $\mathcal{P}$ an *Orthogonal Double Cover (ODC)* of K_n if the following conditions are satisfied:

(1) Every edge of K_n is contained in exactly two pages of $\mathcal{P}$.

(2) The pages of $\mathcal{P}$ are pairwise orthogonal.

An (almost immediate) consequence of this definition is that every page of an ODC $\mathcal{P}$ of K_n consists of exactly $n-1$ edges. If all pages of $\mathcal{P}$ are isomorphic to some graph G, then we say that $\mathcal{P}$ is an ODC of K_n by G.

The investigation of the existence of ODCs by special graphs G has its origin in a paper by Demetrovics, Füredi and Katona [4] who were searching for minimal databases satisfying a certain closure condition. They conjectured that there is always an ODC by G if G consists of disjoint triangles and an isolated vertex only. This was verified (with one exception) by Ganter and Gronau [5]. Later results concentrate on graphs of maximum degree 2 [2, 9], graphs consisting of disjoint cycles [12, 6, 7] or directed cycles [10], clique graphs [1, 5], trees [8] and especially paths [11, 13]. The existence of an ODC by G with at most 9 vertices was decided in [14].

Especially valuable are *cyclic* ODCs since they often are suitable starting points for recursive constructions. A cyclic ODC of K_n by some graph $G = (V, E)$ with $|V| = n$ and $|E| = n-1$ is an ODC of the

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form $\mathcal{P} = \{G_0, G_1, \ldots, G_{n-1}\}$, where $G_0 = ([n], E_0)$ is isomorphic to G and (computed modulo n) the set of edges of G_i $(i = 1, \ldots, n-1)$ is

$$E_i \;=\; \{\, \{x+i, y+i\} \mid \{x, y\} \in E_0\} \,.$$

Every element of a cyclic ODC $\mathcal{P}$ is called *starter* of $\mathcal{P}$.

Arranging the elements of $[n]$ equidistantly and in monotone order on a circle line, a cyclic ODC of K_n can be obtained by taking a suitable starter G_0 and all graphs arising from rotation of G_0. For $x, y \in [n]$, the *Lee–distance* $\ell(x, y)$ is the length of the shorter one of the two paths from x to y on the circle line, i.e.

$$\ell(x, y) := \; \min\{\, |x-y|,\, n - |x-y| \,\}.$$

If there is an edge connecting x and y, we say that $\ell(x, y)$ is the *length* of the edge $\{x, y\}$. The *rotation–index* $r(e_1, e_2)$ of two edges $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ with $\ell(e_1) = \ell(e_2)$ is the length of the shorter one of the two rotations mapping one edge onto the other, i.e.

$$r(e_1, e_2) := \begin{cases} \ell(x_1, x_2), & \text{if } \; \ell(x_1, x_2) = \ell(y_1, y_2) \\ \ell(x_1, y_2), & \text{if } \; \ell(x_1, y_2) = \ell(x_2, y_1). \end{cases}$$

If for some graph $H = ([n], E)$ the set E contains exactly two edges e_1, e_2 of length k, we denote $r(e_1, e_2)$ by $r_H(k)$, or simply by $r(k)$ if there is no danger of ambiguity.

In order to formulate an equivalent condition for the existence of a cyclic ODC of the K_n by $G = (V, E)$, we need the notion of an *orthogonal labelling.* A labelling of $G = (V, E)$ with $|V| = n$ is simply a bijection $\phi\colon V \to [n]$. A labelling ϕ of $G = (V, E)$ with $|V| = n$ and $|E| = n - 1$ is called orthogonal if (with respect to ϕ) the following conditions hold:

(a) For every $k \in \{1, \ldots, \lfloor\frac{n-1}{2}\rfloor\}$, the graph G contains exactly two edges of length k.

(b) The rotation–indices form a permutation of $\{1, \ldots, \lfloor\frac{n-1}{2}\rfloor\}$, that means $\{r_G(k) \mid k = 1, \ldots, \lfloor\frac{n-1}{2}\rfloor\} = \{1, \ldots, \lfloor\frac{n-1}{2}\rfloor\}$.

The next theorem is due to Gronau, Mullin and Rosa.

Theorem 1. [8] *Let $G = (V, E)$ be some graph with $|V| = n$ and $|E| = n - 1$. There exists a cyclic ODC of K_n by G iff there is an orthogonal labelling of G.*

As a simple example we mention that every labelling of the n–star (the tree of diameter 2 with n vertices) is orthogonal.

2. Recursive Constructions

In this section we describe two recursive constructions of ODCs, the PBD-construction and the adding–construction. The first one is using pairwised balanced designs while the second "adds" two ODCs in such a way that a new one turns out.

To formulate the PBD-construction we need some definitions. A *pairwise balanced design* PBD(v, K) is a pair $(V, \mathcal{B})$ where V is a set of *elements* and $\mathcal{B}$ is a collection of subsets of V called *blocks* such that every 2-subset of V is contained in exactly one block of $\mathcal{B}$ and $|B| \in K$ for every $B \in \mathcal{B}$. For existence theorems for PBDs see [15, 3].

Let $\mathcal{G} = \{G_0, \dots, G_{n-1}\}$ be an ODC of K_n with $G_i \cong G$ for all i. We call $\mathcal{G}$ a *rooted* ODC if there exists a vertex (root) $x \in V(G)$ such that the enumerations of x in the G_i's form the set $[n]$. If we have rooted ODCs of K_k for all $k \in K$ and there is a PBD(n, K), then we can construct an ODC of K_n.

Theorem 2. [5, 8] *Let $([n], \mathcal{B})$ be a PBD(n, K) and suppose that for all $k \in K$ there exists a rooted ODC of K_k. For any block $B \in \mathcal{B}$, let $\mathcal{G}_B = \{G_B^{b_1}, \dots, G_B^{b_{|B|}}\}$ be a rooted ODC of $K_{|B|}$ on B. Furthermore, let $H_x = \bigcup_{B \in \mathcal{B}} G_B^x$ be the graph obtained by amalgamating, at x, all graphs in $\bigcup_{B \in \mathcal{B}} \mathcal{G}_B$ having x as a root. Then $\mathcal{H} = \{H_0, \dots, H_{n-1}\}$ is a rooted ODC of K_n.*

Proof. It is easy to see that $\mathcal{H}$ satisfies the two conditions for an ODC. The vertex, where we amalgamate, is enumerated with i in H_i, so $\mathcal{H}$ is rooted. □

In [8] the PBD–construction is used to show that an ODC exists for (almost) every graph of some special classes of trees. In section 4 we will show the same for another class of trees.

The second construction starts with two ODCs, $\mathcal{P}_1$ on the vertex–set $\{v_1, \dots, v_n\}$ and $\mathcal{P}_2$ on $\{w_1, \dots, w_m\}$. Add the vertices $\{w_1, \dots, w_m\}$ to every page of $\mathcal{P}_1$ and $\{v_1, \dots, v_n\}$ to the pages of $\mathcal{P}_2$. Now let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. To make $\mathcal{P}$ an ODC on $n+m$ vertices we have to allot the edges $\{v_i, w_j\}$ $(i = 1, \dots, n; j = 1, \dots, m)$ to the elements of $\mathcal{P}$ in such a way that each of these edges appears in exactly one page of $\mathcal{P}_1$ and exactly one of $\mathcal{P}_2$. (This corresponds to an $n \times m$–array of these edges.) Every such allotment gives an ODC of K_n, but the problem is to find one with the pages of $\mathcal{P}$ mutually isomorphic.

In the sections 4 and 5 we use the adding–construction with cyclic ODCs $\mathcal{P}_1$ and $\mathcal{P}_2$ to prove the existence of an ODC for some classes of trees and some special trees.

3. Trees of Diameter 3

There are ODCs by all trees of diameter 3, except P_4 (the path with 4 vertices). A proof is contained in [8] or can easily be found using the adding–construction from the previous section, starting with two cyclic ODCs by stars.

If we restrict ourselves to cyclic ODCs, the situation is another. Consider a tree T of diameter 3. Then T has exactly two vertices x and y of degree at least 2, and x and y are adjacent. Let x and y be adjacent to exactly p and q vertices of degree 1, respectively. We call T

a $(p,q;3)$–*tree*. The authors of [8] showed that there are no cyclic ODCs by the $(1,q;3)$–trees with $q \not\equiv 0 (\mathrm{mod}\ 3)$ and the $(2,q;3)$–trees with q odd. By the following theorem we give a complete characterization of those trees of diameter 3 that generate cyclic solutions.

As usual, for natural numbers a and b we denote by (a,b) the largest integer dividing both a and b.

Theorem 3. *Let G be a $(p,q;3)$–tree on $n = p+q+2$ vertices. There is a cyclic ODC of K_n by G iff $(n,pq) \geq 2$.*

Proof. $(\Longrightarrow)$ Let T be a $(p,q;3)$–tree such that there is a cyclic ODC of K_n by T. According to theorem 1, there exists an orthogonal labelling ϕ of T, so let us assume that the vertices of G are labelled with respect to ϕ. W.l.o.g. we may suppose that the two vertices of degree at least 2 are 0 and i, where $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Trivially, $\ell(0,i) = i$. Since all vertices of degree 1 are adjacent either to 0 or to i, we have

$$r(i) = i. \tag{1}$$

Let $x \in [n] - \{n-i\}$ be some vertex of degree 1 which is adjacent to 0. If $x+i$ (taken modulo n) was adjacent to i, then the edges $\{0,x\}$ and $\{i, x+i\}$ had the same Lee–distance. Their rotation–index would be i which is impossible by (1) and condition (b) in the definition of an orthogonal labelling. Consequently, $x+i$ must be adjacent to 0, too. Let $g := (n,i)$ and

$$G_j := \{x \in [n] \,:\, x \equiv j (\mathrm{mod}\ g),\, x \neq 0,\, x \neq i\}$$

for $j = 0,1,\ldots,g-1$. Iterating the last argument, we obtain that for fixed $j \in \{0,1,\ldots,g-1\}$ either all elements of G_j are adjacent to 0 or all elements of G_j are adjacent to i. Since 0 and i both are divisible by g, we can write one of the numbers p, q as $\frac{n}{g}a$ with some natural number a. Therefore $\frac{n}{g}$ is a common divisor of pq and n, where $\frac{n}{g} > 1$ by the definition of g.

$(\Longleftarrow)$ Let $(n,pq) \geq 2$, w.l.o.g we assume $(n,p) \geq 2$. From the proof of the other direction it is quite clear how to construct a $(p,q;3)$–tree with an orthogonal labelling. Let $g := \frac{n}{(n,p)}$ and $a := \frac{p}{(n,p)}$. Choose 0 and g as the two (adjacent) vertices of degree greater than 1.

We again define the cosets

$$G_j := \{x \in [n] \,:\, x \equiv j (\mathrm{mod}\ g),\, x \neq 0,\, x \neq g\}$$

for $j = 0,1,\ldots,g-1$. Let 0 be adjacent to all elements of exactly a of the sets $G_1,\ldots,G_{g-1}$, and let g be adjacent to all other elements of $[n] - \{0\} - \{g\}$. Then 0 and g are adjacent to exactly p and q vertices of degree 1, respectively.

We now are going to show that the labelling given above is orthogonal. For $j = 0,1,\ldots,g-1$ let E_j be the set of edges containing a vertex from G_j. Obviously, the Lee–distances of the elements of E_j form the set

$$P_j = \{x \in [n] : 1 \le x < \tfrac{n}{2},\ x \equiv j (\mathrm{mod}\ g) \text{ or } x \equiv -j (\mathrm{mod}\ g)\},$$

where for $j \notin \{0, \frac{g}{2}\}$ every $x \in P_j$ is the length of exactly one edge from E_j. Since $P_j = P_{g-j}$ for $j \notin \{0, \frac{g}{2}\}$ every possible Lee–distance $\equiv j(\mathrm{mod}\ g)$ is appearing exactly twice. Every element of P_0 is the length of exactly two edges from E_0. If g is even, then every element from $P_{\frac{g}{2}}$ is the Lee–distance of exactly two edges from $E_{\frac{g}{2}}$. Therefore the first condition for an orthogonal labelling is satisfied.

In order to prove that the second condition holds, we show that the rotation–indices of the elements of $E_j \cup E_{g-j}$ form a permutation of P_j for $j = 0, 1, \ldots, g-1$. We distinguish between two cases.

Case 1. All edges from $E_j \cup E_{g-j}$ contain 0, or all edges from $E_j \cup E_{g-j}$ contain g. In this case it is easy to see that $r(k) = k$ for all $k \in P_j$.

Case 2. All edges from E_j contain 0, and all edges from E_{g-j} contain g. Consider a pair of edges $e_1 \in E_j$ and $e_2 \in E_{g-j}$ of the same length $k \in P_j$. Then we have $r(k) = \min\{g+k,\ n-(g+k)\}$, and again the set of rotation–indices forms a permutation of P_j. □

Theorem 3 yields the results on $(1, q; 3)$– and $(2, q; 3)$–trees mentioned above as simple corollaries. As another consequence one obtains that the $(\frac{n-2}{2}, \frac{n-2}{2}; 3)$–tree admits a cyclic solution iff $n \equiv 2(\mathrm{mod}\ 4)$. The most interesting corollary perhaps is the following.

Corollary 1. *Let n be a prime and T a tree of diameter 3 on n vertices. There is no cyclic ODC of K_n by T.*

4. Trees of Diameter 4

We now investigate some special trees of diameter 4, namely those consisting of a path (x, y, z) and vertices adjacent to x, y or z, respectively. Let p, q and r be the numbers of such vertices, then we call the tree a *$(p, q, r; 4)$–tree.* The path with 4 edges for instance is a $(1, 0, 1; 4)$–tree.

Finally we will show the existence of an ODC for another class of trees of diameter 4 by the PBD-construction as well.

Lemma 1. *Let $n \ge 5$ and let T be a $(n-4, 0, 1; 4)$–tree. Then there exists a cyclic ODC of K_n by T.*

Proof. It is easy to check that the labelling generating the edges $\{0, 3\}, \ldots, \{0, n-1\}, \{n-1, 1\}, \{1, 2\}$ is orthogonal. □

Lemma 2. *Let $n \ge 5$ be odd and let T be a $(\frac{n-3}{2}, 0, \frac{n-3}{2}; 4)$–tree. Then there exists a cyclic ODC of K_n by T.*

Proof. The following labelling is easily verfied to be orthogonal:
$\{1, 0\}, \{0, 3\}, \{3, 2\},$

$\{0, a\}$ for $a = 5, 7, 9, \ldots, n-2$,
$\{2, b\}$ for $b = 4, 6, 8, \ldots, n-1$. □

By the help of lemma 1 we are able to prove the following theorem.

Theorem 4. *Let T be a $(p, 0, q; 4)$–tree on $n = p+q+3$ vertices. Then there exists an ODC of K_n by T.*

Proof. We use the adding–construction. Let $q > 5$ and let $\mathcal{P}$ and $\mathcal{Q}$ be cyclic ODCs by $(p, 0, 1; 4)$ and $(1, 0, q-5; 4)$, respectively. These ODCs exist because of lemma 1. Let in $P_i \in \mathcal{P}$ the vertex of degree 2, who is adjacent to a vertex of degree 1, be labelled with i_P. We add the edges $\{i_p, j_q\}, j_Q = 0, \ldots, q-2$ to every P_i. Let in $Q_j \in \mathcal{Q}$ the vertex of degree $q-4$ be labelled with j_Q and let the path of length 3 starting with j_Q be (j_Q, a, b, c). We add the edges $\{j_Q, 0_P\}, \ldots, \{j_Q, 4_P\}$ and $\{b, 5_P\}, \ldots, \{b, (p+3)_P\}$ to every Q_j. Then P_i and Q_j have exactly the edge $\{i_P, j_Q\}$ for $i = 0, \ldots, 4$ and $\{i_P, b\}$ for $i = 5, \ldots, p+3$ in common. Since $\mathcal{P}$ and $\mathcal{Q}$ are cyclic, all edges of the form $\{x_P, y_Q\}$ are covered.

This construction works for all $(p, 0, q; 4)$–trees with $p > 5$ or $q > 5$. By a computer search we found for all $(p, 0, q; 4)$–trees with $p \leq 5$ and $q \leq 5$ (even cyclic) ODCs. □

Let us now consider trees that arise from amalgamating n–stars at vertices of degree 1. In [8] the existence of an ODC for trees consisting of amalgamated 4–stars is proven. We show the existence for trees consisting of 5–stars.

Theorem 5. *Let T_k be a tree on $4k+1$ vertices that arises from amalgamating 5–stars at a vertex of degree 1 (see figure 1). Then there exists an ODC of K_{4k+1} by T_k for all $k \geq 1$ with the possible exceptions $k = 7$ or 8.*

Proof. We use the PBD–construction. The PBD–closure of $\{5, 9, 13\}$ includes already all integers $n \equiv 1 (\mathrm{mod}\ 4)$ with the exceptions $n = 17, 29, 33$ (see [3]). The assertion is proven by giving cyclic ODCs for T_1, T_2, T_3 and T_4 since by theorem 2, we can construct ODCs for T_k, $k \neq 4, 7, 8$ from solutions for T_1 with a vertex of degree 1 as a root, for T_2 with a vertex of degree 2 as a root and for T_3 with a vertex of degree 3 as a root.

Since T_1 is a star, a solution is trivial. A cyclic ODC by T_2 is given by lemma 2. Figure 1 shows starters for T_3 and T_4 (one of the open cases). □

5. Trees on at most 13 Vertices

Gronau, Mullin and Rosa [8] investigated all trees on up to 10 vertices. They came to the result that for every tree T on n vertices, $2 \leq n \leq 10$, there exists an ODC of K_n by T iff T is not P_4. They conjecture that this statement can be extended to all $n \geq 2$.

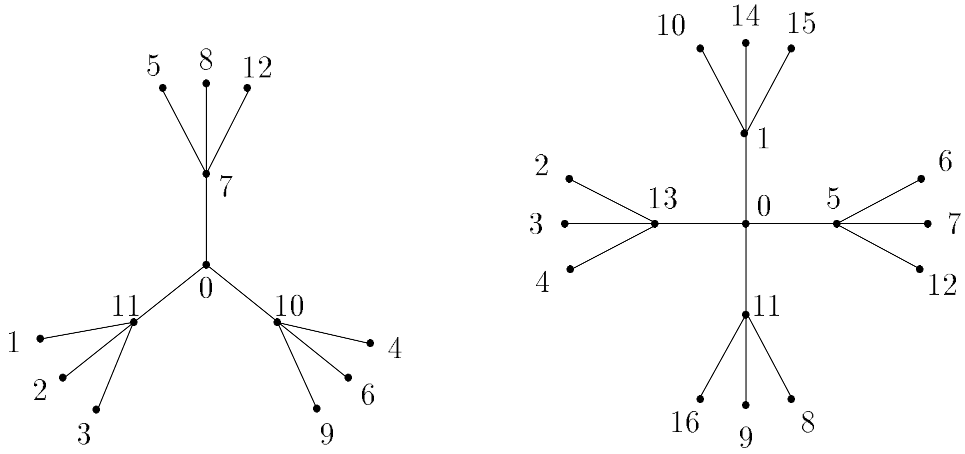


Figure 1. Starters for cyclic ODCs by T_3 and T_4

We investigated all trees with at most 13 vertices. We used the program `MAKEG` by Brendan McKay to generate all trees with up to 13 vertices. By a computer program we found out those having an orthogonal labelling (or equivalently, a cyclic ODC). A complete list of all these labellings is obtainable at WWW page `ftp://zeus.math.uni-rostock.de/WWW/vleck.html`. It turned out that all but 13 of the 2087 trees on $11, 12$ and 13 vertices admit an orthogonal labelling (see table 1).

vertices	trees	ODCs	
		cyclic	not cyclic
2	1	1	-
3	1	1	-
4	2	1	-
5	3	2	1
6	6	6	-
7	11	8	3
8	23	20	3
9	47	45	2
10	106	103	3
11	235	230	5
12	551	550	1
13	1301	1294	7

Table 1. ODCs for trees

From the remaining trees are 10 of diameter 3, that means they have an ODC (see section 3). The last open cases are $(3,2,3;4)$, $(3,4,3;4)$ and $(2,4,4;4)$.

Lemma 3. *Let T be a*

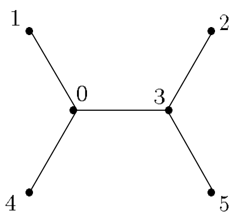
(i) $(3,2,3;4)$*–tree,* (ii) $(3,4,3;4)$*–tree* *or* (iii) $(2,4,4;4)$*–tree.*

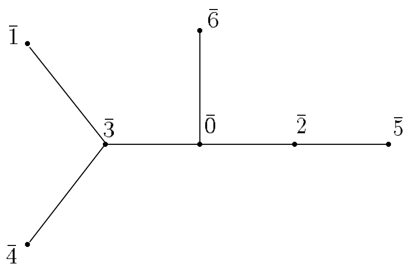
Then there exists an ODC by T of K_{11} or K_{13}, respectively.

Proof. (i) We use the adding–construction with cyclic ODCs $\mathcal{P}$ and $\mathcal{Q}$ by the $(3,1;3)$–tree and the 5–star, respectively. Let in every $P_i \in \mathcal{P}$ the vertex of degree 2 be labelled with i_P and let in every $Q_j \in \mathcal{Q}$ the vertex with degree 4 be labelled with j_Q. Add the edges $\{i_P, 0_Q\}, \{i_P, 1_Q\}$, $\{i_P + 2(\text{mod } 6),\ 2_Q\}$, $\{i_P + 2(\text{mod } 6),\ 3_Q\}$, $\{i_P + 2(\text{mod } 6),\ 4_Q\}$ to every P_i and the edges $\{j_Q + 1(\text{mod } 5),\ 0_P\}$, $\{j_Q + 1(\text{mod } 5),\ 2_P\}$, $\{j_Q + 1(\text{mod } 5),\ 4_P\}$, $\{j_Q - 1(\text{mod } 5),\ 1_P\}$, $\{j_Q - 1(\text{mod } 5),\ 3_P\}$, $\{j_Q - 1(\text{mod } 5),\ 5_P\}$ to every Q_j. This gives an ODC by $(3,2,3;4)$.

(ii) We use the same construction as above, but with $\mathcal{Q}$ an ODC by the 7–star. We add the edges $\{i_P, 0_Q\}, \{i_P, 1_Q\}, \{i_P, 2_Q\}, \{i_P, 3_Q\}$, $\{i_P + 2(\text{mod } 6),\ 2_Q\}$, $\{i_P + 2(\text{mod } 6),\ 3_Q\}$, $\{i_P + 2(\text{mod } 6),\ 4_Q\}$ to every P_i and the same edge as above to every Q_j, but this time modulo 7 instead of modulo 5.

(iii) We again use the adding–construction with the cyclic ODCs given by the following starters:





Then an ODC by $(2,4,4;4)$ is described by this allotment of the edges between both ODCs:

$3\bar 2$	$5\bar 3$	$5\bar 4$	$5\bar 5$	$5\bar 6$	$3\bar 0$	$3\bar 1$
$4\bar 0$	$4\bar 1$	$4\bar 2$	$0\bar 3$	$0\bar 4$	$0\bar 5$	$0\bar 6$
$5\bar 2$	$1\bar 3$	$1\bar 4$	$1\bar 5$	$1\bar 6$	$5\bar 0$	$5\bar 1$
$0\bar 0$	$0\bar 1$	$0\bar 2$	$2\bar 3$	$2\bar 4$	$2\bar 5$	$2\bar 6$
$1\bar 2$	$3\bar 3$	$3\bar 4$	$3\bar 5$	$3\bar 6$	$1\bar 0$	$1\bar 1$
$2\bar 0$	$2\bar 1$	$2\bar 2$	$4\bar 3$	$4\bar 4$	$4\bar 5$	$4\bar 6$

In this array row i contains the edges to be added to P_{i-1} and column j the edges to be added to Q_{j-1}. The element in row $i+1$ and column $j+1$ is the edge P_i and Q_j have in common. $\square$

Altogether, we obtained the following result.

Theorem 6. *Let T be a tree on n vertices, $2 \le n \le 13$. There exists an ODC of K_n by T iff $T \ne P_4$.*

The construction in parts (i) and (ii) of the proof of lemma 3 works for all $(3,k,3;4)$–trees if we use cyclic ODCs by $(3,1;3)$ and the $(k+3)$–star (that always exist). Consequently, the next lemma holds.

Lemma 4. *Let T be a $(3,k,3;4)$–tree. Then there exists an ODC of K_{k+9} by T.*

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