Compactly Supported Solutions of Two–Scale Difference Equations*

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ABSTRACT

We consider Lebesgue–integrable, compactly supported solutions of two–scale difference equations and investigate the relations between translates of these solutions. A detailed study of corresponding invariant subspaces leads to new observations concerning the factorization of the refinement mask and certain spectral properties of corresponding coefficient matrices. In particular, new necessary conditions for the existence of integrable, compactly supported solutions are derived.

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1. INTRODUCTION

A two–scale difference equation is a functional equation of the form

$$\varphi(t/2) = \sum_{\nu=0}^{n} c_\nu \, \varphi(t-\nu), \tag{1.1}$$

where c_ν are given real or complex constants with $c_0\, c_n \neq 0$ and $n \geq 1$. A function φ satisfying (1.1) for all real t, is called *refinable*.

Functional equations of type (1.1) arise in many contextes, in the construction of wavelets as well as in interpolating subdivision schemes. There are a lot of papers studying these equations extensively (see e.g. Micchelli

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& Prautzsch [19], Daubechies & Lagarias [5, 6], Cavaretta, Dahmen & Micchelli [1], Dyn, Gregory & Levin [8], Colella & Heil [2], Jia [12, 13], Lau & Wang [17], Griepenberg [10]). In particular, several special cases of solutions are investigated. For subdivision schemes for instance, compactly supported solutions $\varphi \in C(\mathbb{R})$ (or more generally, $\varphi \in L^p(\mathbb{R})$) with L^2–stable (or L^p–stable) integer translates are considered ([8, 12]).

In what follows, we are interested in (1.1) as a functional equation, and consider solutions with the following property.

Definition 1.1. A refinable function φ is called E–solution (essential solution) of (1.1), if it is a not identically vanishing, Lebesgue–integrable and compactly supported function. Two E–solutions φ_1 and φ_2 are not considered as different, if there exists a constant c, such that $\varphi_1 = c\,\varphi_2$ almost everywhere.

As shown in [5], the assumptions in the definition yield that supp $\varphi \subseteq [0, n]$.

By Fourier transform of (1.1), we obtain

$$\hat{\varphi}(2u) = P(e^{-iu})\,\hat{\varphi}(u) \tag{1.2}$$

with $\hat{\varphi}(u) := \int_{-\infty}^{\infty} \varphi(t)\, e^{-iut}\, \mathrm{d}t$, and with the *refinement mask* (or the *two–scale symbol*)

$$P(z) := \frac{1}{2} \sum_{\nu=0}^{n} c_\nu\, z^\nu . \tag{1.3}$$

Assuming that (1.1) is given with real coefficients c_ν, in [5] it is proved that:

(i) if $|P(1)| < 1$ or $P(1) = -1$, then (1.1) has no E–solution;

(ii) if $P(1) = 1$, then (1.1) has at most one E–solution;

(iii) if $|P(1)| > 1$, and if an E–solution φ exists, then $P(1) = 2^m$ for some nonnegative integer m. If, in the last case, the coefficients c_ν ($\nu = 0, \ldots, n$) are replaced by $2^{-m} c_\nu$ in (1.1) then the new two-scale difference equation possesses a continuous solution g, and φ is the m-th derivative of g,

$$\varphi(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} g(x),$$

almost everywhere. Hence, we make the

Assumption (A1): *Throughout the paper, we assume that* $P(1) = 1$.

Then, for the Fourier transform $\hat{\varphi}(u)$ of a refinable function φ, we obtain by repeated application of (1.2),

$$\hat{\varphi}(u) = \prod_{j=1}^{\infty} P(e^{-iu/2^j}),$$

where we have assumed that $\hat{\varphi}(0) = \int_0^n \varphi(t)\,\mathrm{d}t = 1$ (see [5, 20]).

For $t = 0$ and $t = n$, (1.1) simplifies to $\varphi(0) = c_0\,\varphi(0)$ and $\varphi(n) = c_n\,\varphi(n)$, respectively.

Assumption (A2): *Throughout the paper (disregarding the exceptional case of step functions), we assume that* $c_0 \neq 1$, $c_n \neq 1$.

This implies together with the foregoing equations that the E–solution φ satisfies $\varphi(0) = \varphi(n) = 0$.

For a refinable function φ, we introduce the vector

$$\psi(t) := (\varphi(t),\, \varphi(t+1), \ldots, \varphi(t+n-1))^T. \tag{1.4}$$

Since supp $\varphi \subseteq [0, n]$, it sufficies to consider $\psi(t)$ for $0 \leq t \leq 1$. Further, in view of $\varphi(0) = \varphi(n) = 0$, there is an exact equivalence between φ and the vector ψ (cf. [2], Proposition 1).

As in [2, 5, 6, 8, 12, 19], we introduce the 1×2 block Toeplitz matrices $A_0 := (c_{2j-k})$, $j, k = 0, \ldots, n-1$ and $A_1 := (c_{2j-k+1})$, $j, k = 0, \ldots, n-1$, where $c_j = 0$ for $j < 0$ and $j > n$, respectively, i.e.,

$$A_0 := \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & c_{n-1} & c_{n-2} & c_{n-3} \\ 0 & \cdots & 0 & c_n & c_{n-1} \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & c_n & c_{n-1} & c_{n-2} \\ 0 & \cdots & 0 & 0 & c_n \end{pmatrix}.$$

Observe that the both matrices A_0 and A_1 contain $M := (c_{2j-k+1})_{j,k=0}^{n-2}$ as a submatrix with the following peculiarity: If M possesses the spectrum spec(M), then spec$(A_0) =$ spec$(M) \cup \{c_0\}$ and spec$(A_1) =$ spec$(M) \cup \{c_n\}$.

With the notations above, (1.1) can be written in the vector form

$$\psi\left(\frac{t}{2}\right) = A_0\,\psi(t), \qquad \psi\left(\frac{t+1}{2}\right) = A_1\,\psi(t) \tag{1.5}$$

for $0 \leq t \leq 1$.

The equations in (1.5) imply that

$$\psi\left(\frac{t}{2^k}\right) = A_0^k\,\psi(t), \qquad \psi\left(1 - \frac{1-t}{2^k}\right) = A_1^k\,\psi(t), \tag{1.6}$$

and for $t = 0$ and $t = 1$, respectively,

$$\psi(0) = A_0\,\psi(0), \qquad \psi(1) = A_1\,\psi(1). \tag{1.7}$$

Introducing the vector $\tilde{\psi}(t) := (\varphi(t), \ldots, \varphi(t+n-2))^T$, relations (1.7) reduce to the single relation $\tilde{\psi}(1) = M\,\tilde{\psi}(1)$. In the case $\tilde{\psi}(1) \neq 0$, which is valid for a nontrivial continuous solution (cf. [6], Proposition 2.1), $\tilde{\psi}(1)$

is necessarily a right eigenvector of M corresponding to the eigenvalue 1, and, of course, 1 is also an eigenvalue of the both matrices A_0 and A_1. Starting with an eigenvector $\tilde{\psi}(1)$ of M corresponding to 1, we can recursively compute values of φ at dyadic rationals by means of (1.5). This dyadic interpolation method is extensively explained in [6] and [19]. It also applies if φ has linearly dependent integer translates, while the subdivision algorithm usually does not work in this case (cf. e.g. [1]).

The matrix M may have the eigenvalue 1 with a multiplicity greater than 1. In this case, only one particular linear combination of corresponding eigenvectors can lead to an E–solution φ (see Example 1 later on).

In the following, it is also convenient to introduce the infinite matrix $\boldsymbol{A} := (c_{2j-k})_{j,k\geq 0}$ and the infinite column vector $\boldsymbol{\psi}(t) := (\varphi(t+j))_{j\geq 0}$ of a refinable function φ, so that (1.1) can be written in the form

$$\boldsymbol{\psi}\left(\frac{t}{2}\right) = \boldsymbol{A}\,\boldsymbol{\psi}(t) \tag{1.8}$$

for $-\infty < t \leq 1$.

Micchelli & Prautzsch [19] and Colella & Heil [2] succeeded in establishing necessary and sufficient conditions for the existence of continuous solutions of (1.1). They extensively studied the space $W \subset \mathbb{C}^n$,

$$W := \{\psi(t) - \psi(0) : t \in [0,\, 1]\}.$$

Conditions could be expressed in terms of the joint spectral radius $\rho(A_0|_W, A_1|_W)$ of the two matrices A_0 and A_1 restricted to the subspaces W (cf. [2]). As shown in [2], Proposition 3, W is the smallest subspace of $\mathbb{C}^n$ invariant under both A_0 and A_1, which contains the vector $\psi(1)-\psi(0)$. So, if $\psi(0)$, $\psi(1)$ are determined by (1.7), then W can be constructed without knowing ψ explicitly.

The space W is uniquely determined by its orthogonal complement $L_0 \subset \mathbb{C}^n$,

$$L_0 := \operatorname{span}\{w \in \mathbb{C}^n \,:\, w^T\,\psi(t) = w^T\,\psi(0),\, t \in [0,1]\,\}. \tag{1.9}$$

In Daubechies & Lagarias [6], the special case $L_0 = \{(1,\ldots,1)^T\}$ has been considered. Further, let

$$\mathcal{L}_0 := \operatorname{span}\{\boldsymbol{w} = (w_j)_{j\geq 0} : \boldsymbol{w}^T\,\boldsymbol{\psi}(t) = \boldsymbol{w}^T\,\boldsymbol{\psi}(0),\, t \in (-\infty,\, 1]\,\} \tag{1.10}$$

with $\boldsymbol{\psi}(t) := (\varphi(t+j))_{j\geq 0}$ satisfying (1.8). In particular, for $\boldsymbol{w} \in \mathcal{L}_0$ we have by definition

$$\sum_{j=0}^{\infty} w_j\,\varphi(t+j) = c \tag{1.11}$$

for all $t \leq 1$ with a fixed constant c. Since $\boldsymbol{\psi}(t)$ has only finitely many components different from zero, there arise no convergence problems.

Concerning E–solutions, we extend the spaces L_0 and $\mathcal{L}_0$, respectively, to

$$L := \text{span}\,\{w \in \mathbb{C}^n \,:\, w^T\,\psi(t) = c,\, t \in [0,1],\, a.e.\},\tag{1.12}$$

$$\mathcal{L} := \text{span}\,\{\boldsymbol{w} = (w_j)_{j\geq 0} \,:\, \boldsymbol{w}^T\,\boldsymbol{\psi}(t) = c,\, t \in (-\infty,1],\, a.e.\}\tag{1.13}$$

with certain constants c which depend on w (or $\boldsymbol{w}$), but not on t. For simplicity, we restrict ourselves to these extended spaces, though several of the following results are even valid with respect to L_0, $\mathcal{L}_0$, and we usually drop the restriction "almost everywhere" tacitly. Let us emphazise that (1.1) shall be satisfied in any case for all real t.

In Section 2, the properties of L and $\mathcal{L}$ are extensively studied. A general characterization of elements of $\mathcal{L}$ will be presented. Similar results can be found in [1].

Knowing the structure of vectors contained in $\mathcal{L}$, we are able to derive new consequences on eigenvectors of our matrices A_0, A_1 and $\boldsymbol{A}$, and on zeros of the refinement mask $P(z)$ from (1.3) in Section 3. Usually, papers dealing with refinement equations are restricted to the case $P(-1) = 0$, i.e., that $P(z)$ possesses the factor $z+1$. If the subdivision scheme associated with $\{c_\nu\}_{\nu=0}^n$ converges uniformly (or in $L^p(\mathbb{R})$), then $P(-1) = 0$ is necessarily satisfied (see [8, 12]). Here, we drop this assumption and consider also refinement masks with $P(-1) \neq 0$; however, we show that the refinement mask $P(z)$ of an E–solution of (1.1) always possesses a factor $p(z)$, which is a certain modification of $(z+1)$. This factor $p(z)$ can be considered as the refinement mask of a piecewise step function. Moreover, it follows that each E-solution can be represented as a finite linear combination of integer translates of a simpler E-solution of (1.1) with a refinement mask containing the factor $(z+1)$. The arguments can be pushed a little further, also allowing assertions on multiple zeros of $P(z)$.

Finally, in Section 4, the structure of $\mathcal{L}$ implies consequences on eigenvectors of the coefficient matrices A_0, A_1 and $\boldsymbol{A}$. In particular, it will be shown that the matrices A_0, A_1 have the eigenvalue 1 also in the general case; however, in case of continuous E–solutions, they cannot possess root vectors belonging to the eigenvalue 1. As corollaries, we obtain new statements on the non–existence of E–solutions of (1.1). The results will be explained by examples.

2. INVARIANT SPACES

We consider the functional equation (1.1) satisfying the Assumptions (A1) and (A2) and possessing an E–solution φ. In this section, we want to derive some properties of the corresponding spaces L and $\mathcal{L}$ in (1.12) and (1.13), respectively. We start with the following basic theorem.

THEOREM 2.1. *Let φ be an E–solution of* (1.1), *and let* (A1), (A2) *be satisfied. Then we have*

$$\sum_{\nu\in\mathbb{Z}} \varphi(t+\nu) = \int_0^n \varphi(s)\ \mathrm{d}s = \hat{\varphi}(0) \tag{2.1}$$

almost everywhere for $t \in \mathbb{R}$.

Proof. Let $\phi(t) := \sum_{\nu=0}^{n-1} \varphi(t+\nu)$. Then by (1.1) it follows for $t \in [0,1]$ that

$$\phi\left(\frac{t}{2}\right) + \phi\left(\frac{t+1}{2}\right) = 2\phi(t), \tag{2.2}$$

since the left hand–side is equal to

$$\begin{aligned}
\phi\left(\frac{t}{2}\right) + \phi\left(\frac{t+1}{2}\right) &= \sum_{\nu=0}^{n-1}\left[\varphi\left(\frac{t}{2}+\nu\right) + \varphi\left(\frac{t+1}{2}+\nu\right)\right] \\
&= \sum_{\nu=0}^{n-1}\sum_{k=0}^{n} c_k \left[\varphi(t+2\nu-k) + \varphi(t+1+2\nu-k)\right] \\
&= \sum_{k=0}^{n} c_k \sum_{\nu=0}^{2n-1} \varphi(t+\nu-k) = 2\phi(t).
\end{aligned}$$

With regard to Klemmt [16], it follows that the only Lebesque-integrable solution $\phi(t)$ of (2.2) is

$$\phi(t) = \int_0^1 \phi(s)\ \mathrm{d}s$$

almost everywhere for $t \in [0,1]$. According to the definition of $\phi(t)$, this is our assertion. ■

REMARKS: 1. In papers dealing with the construction of wavelets, (2.1) is usually assumed to be true (see e.g. [4, 5]). If $\hat{\varphi}(0) \neq 0$, then (2.1) implies that φ satisfies the moment condition of order 1, i.e., constants can be reproduced by integer translates of φ.

2. For continuous solutions, it was already shown by Fichtenholz, cf. [9], p. 789–790, that (2.2) has constant solutions only. The proof of Fichtenholz also works for Riemann–integrable functions, where (2.1) is valid for all $t \in \mathbb{R}$.

3. Let us mention that according to Gupta, cf. [15], p. 420, there exist nonintegrable solutions $\phi(t)$ satisfying (2.2), for instance $\phi(t) = \cot(\pi t)$

$(0 < t < 1)$. In fact, there exist infinitely many nonintegrable solutions, namely, given an arbitrary function $\phi(t)$ for $\frac{1}{2} < t \le 1$, it can be continued such that (2.2) holds.
4. For $n = 1$, we have $\varphi(t) = \phi(t) = c$, and (1.1) implies either $c = 0$ or $c_0 = c_1 = 1$, which contradicts Assumption (A2). Hence, in (1.1) we have in fact $n \ge 2$.

For the next considerations, we repeat the following notation from linear algebra. A vector v is called a left root vector of height $k \ge 2$ belonging to the eigenvalue λ of a quadratic matrix M if

$$v^T (M - \lambda I)^k = 0^T, \qquad v^T (M - \lambda I)^{k-1} \ne 0^T.$$

The left eigenvector of v^T of M belonging to λ is the (improper) left root vector of height 1. In the following, if we speak about root vectors, then we mean a proper root vector (of height $k \ge 2$).

THEOREM 2.2. *Let φ be an E–solution of* (1.1) *with* (A1), (A2). *Then for the space L in* (1.12) *the following assertions are satisfied.*
(i) *The vector $e := (1, \ldots, 1)^T$ is contained in L.*
(ii) *If $w \in L$, then $(w^T A_0)^T \in L$ and $(w^T A_1)^T \in L$.*
(iii) *Let the solution φ be bounded in neighbourhoods of the points k ($k = 0, \ldots, n$). If $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is an eigenvalue of the coefficient matrix A_0 (or A_1), then the left eigenvectors and left root vectors of A_0 (or A_1) corresponding to λ are contained in L.*
(iv) *Let the solution φ be continuous in the points k ($k = 0, \ldots, n$). Then the left eigenvectors and left root vectors of A_0 (or A_1) corresponding to an eigenvalue λ with $|\lambda| = 1$ are contained in L.*
(v) *If $w \in \mathbb{C}^n$ is a left eigenvector of both A_0 and A_1 corresponding to the eigenvalue 0, then $w \in L$.*
(vi) *The dimension of L is at most $n - 1$.*

Proof. Assertions (i) and (ii) immediately follow from (1.12), (1.5) and from Theorem 2.1.
(iii): Let λ with $|\lambda| > 1$ be an eigenvalue of A_0, and $w^T A_0 = \lambda\, w^T$. Then we find iteratively by (1.5) and (1.6)

$$w^T \psi(t) = \frac{1}{\lambda} w^T A_0 \psi(t) = \frac{1}{\lambda} w^T\, \psi\left(\frac{t}{2}\right) = \ldots = \frac{1}{\lambda^k} w^T\, \psi\left(\frac{t}{2^k}\right).$$

If k goes to infinity, it follows, by boundedness of ψ in the neighbourhood of 0, that $w^T\, \psi(t) = 0$, i.e., $w \in L$ with the constant $c = 0$ in (1.12).
Assuming that A_0 possesses a root vector of height 2, $\tilde{w}^T A_0 = \lambda(\tilde{w}^T + w^T)$, we can use the same argument, observing that

$$\tilde{w}^T\, \psi(t) = (\frac{1}{\lambda} \tilde{w}^T A_0 - w^T)\, \psi(t) = \frac{1}{\lambda} \tilde{w}^T\, \psi\left(\frac{t}{2}\right).$$

So, it follows that $\tilde{w} \in L$. Analogously, we can derive the assertion for all root vectors. The same arguments apply for eigenvectors and root vectors of A_1, using (1.6).
(iv): For $|\lambda| = 1$, we obtain the assertion in a similar manner, since the assumed continuity of φ implies that ψ is continuous in 0. In case of $\lambda \neq 1$, we again find $c = 0$.
(v): This assertion easily follows from (1.5).
(vi): If there were n linearly independent vectors $w^{(k)}$ $(k = 0, \ldots, n-1)$ in L, then $w^{(k)T}\,\psi(t) = c^{(k)}$ for $k = 0, \ldots, n-1$ would yield a solution $\psi(t)$ with constant components, i.e., $\varphi(t+j) = C_j$ for $j = 0, \ldots, n-1$ and $0 \leq t \leq 1$ a.e.. But (1.1) yields for $0 \leq t < 1$ that $\varphi(t/2) = c_0\varphi(t)$; hence, by Assumption (A2), it follows that $C_0 = 0$, i.e., $\varphi(t)$ vanishes identically on $[0,1)$. Applying (1.1) recursively, we find that φ vanishes identically on $[0,n]$, in contrast to our assumption. ∎

REMARKS: 1. The boundedness of φ in (iii) for $|\lambda| > 1$ can be weakened by boundedness in right neighbourhoods of k for eigenvalues of A_0, and boundedness in left neighbourhoods of k for eigenvalues of A_1.
2. Analogously, in (iv), for A_0 we need continuity of $\psi(0)$ from the right; and for A_1, continuity of $\psi(1)$ from the left is sufficient.
3. Eigenvectors of A_0 (or A_1) corresponding to eigenvalues λ with $0 < |\lambda| < 1$ can also be contained in L. For example, consider (1.1) with $c_0 = c_1 = c_3 = c_4 = 1/2$, $c_2 = 0$. Then A_0 possesses the eigenvalues $1,\ 1/2,\ 1/2,\ -1/2$ with corresponding left eigenvectors $w_1 = (1,\,1,\,1,\,1)^T$, $w_2 = (1,\,0,\,0,\,0)^T$, $w_3 = (1,\,-1,\,0,\,1)^T$ and $w_4 = (1,\,1,\,-2,\,1)^T$. Observing that $(0,\,1,\,1,\,1)^T$ is a right eigenvector of A_0 to the eigenvalue 1, we find $\psi(1) - \psi(0) = (1,\,0,\,0,\,-1)^T$. We easily check that $W = \{(1,\,0,\,0,\,-1)^T\}$, since it is already invariant under both A_0 and A_1. Hence, L is spanned by w_1, w_3 and w_4.
4. If the first five statements of Theorem 2.2 yield n linearly independent vectors of a formally constructed space L, then (1.1) has no E-solution.

Analogous considerations yield the following

COROLLARY 2.1. *Let φ be an E–solution of* (1.1) *with* (A1), (A2). *Then for the space $\mathcal{L}$ in* (1.13) *we have:*
(i) *The vector $\mathbf{e} := (1,\,1,\ldots)^T$ is contained in $\mathcal{L}$.*
(ii) *If $\boldsymbol{w} \in \mathcal{L}$, then $(\boldsymbol{w}^T\boldsymbol{A})^T \in \mathcal{L}$.*
(iii) *Let the solution φ be bounded in neighbourhoods of the points k $(k = 0, \ldots, n)$. If $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is an eigenvalue of $\boldsymbol{A}$, then the left eigenvectors and left root vectors of $\boldsymbol{A}$ corresponding to λ are contained in $\mathcal{L}$.*
(iv) *Let the solution φ be continuous in the points k $(k = 0, \ldots, n)$. Then the left eigenvectors and left root vectors of $\boldsymbol{A}$ corresponding to an eigenvalue λ with $|\lambda| = 1$ are contained in $\mathcal{L}$.*
(v) *If $\boldsymbol{w}$ is a left eigenvector of both $\boldsymbol{A}$ and $\boldsymbol{A}' := (c_{2j-k+1})_{j,k=0}^{\infty}$ corre-*

sponding to the eigenvalue 0*, then* $\boldsymbol{w} \in \mathcal{L}$*.*
(vi) *If* $\boldsymbol{w} \in \mathcal{L}$ *then* $w \in L$ *for the corresponding restriction.*

Moreover, we find

THEOREM 2.3. *Let* φ *be an E–solution of* (1.1) *with* (A1), (A2). *Then for the spaces* L, $\mathcal{L}$ *in* (1.12) *and* (1.13)*, respectively, the following assertions are satisfied.*
(i) *If* $w = (w_j)_{j=0}^{n-1}$ *is a left eigenvector or left root vector of* A_0 *to the eigenvalue* λ *with* $\lambda \neq c_n$*, then* w *is uniquely extendable to a left eigenvector or left root vector* $\boldsymbol{w} = (w_j)_{j\geq 0}$ *of* $\boldsymbol{A}$*.*
(i') *If* $w = (w_j)_{j=0}^{n-1}$ *is a left eigenvector or left root vector of* A_1 *to the eigenvalue* λ*, then* w *is uniquely extendable to a left eigenvector or left root vector* $\boldsymbol{w} = (w_j)_{j\geq 0}$ *of* $\boldsymbol{A}' := (c_{2j-k+1})_{j,k=0}^{\infty}$*.*
(ii) *If* $(w_j)_{j\geq 0} \in \mathcal{L}$*, then we have* $(w_{j+k})_{j\geq 0} \in \mathcal{L}$ *for every* $k \in \mathbb{N}$*.*
(iii) *Let* $w := (w_j)_{j=0}^{n-1} \in L$ *and let its extension* $\boldsymbol{w} := (w_j)_{j\geq 0}$ *be contained in* $\mathcal{L}$*. Then this extension* $\boldsymbol{w}$ *is uniquely determined by* w*.*
(iv) *If* $(w_j)_{j\geq 0} \in \mathcal{L}$*, then* $(w_{j+k})_{j=0}^{n-1} \in L$ *for every* $k \in \mathbb{N}_0$*.*
(v) $\dim \mathcal{L} \leq \dim L \leq n-1$.

Proof. (i): If w is a left eigenvector of A_0 corresponding to $\lambda \neq 0$ and $\lambda \neq c_n$, then we have

$$\sum_{j=0}^{n-1} w_j c_{2j-k} = \lambda\, w_k$$

for $k = 0, \ldots, n-1$. From this equation, w can successively be extended to a vector $\boldsymbol{w} = (w_j)_{j\geq 0}$ such that $\sum_{j=0}^{\infty} w_j c_{2j-k} = \lambda w_k$ is also satisfied for $k \geq n$, since j runs in fact up to $\lfloor \frac{n+k}{2} \rfloor \leq k$. This means, that $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{A}$ to λ.
For $\lambda = 0$, w can also be extended to $\boldsymbol{w} = (w_j)_{j\geq 0}$, since the components w_k $(k = n, n+1, \ldots)$ can be found successively from $\sum_{j=0}^{k} w_j\, c_{2j-2k+n} = 0$ in view of $c_n \neq 0$. Actually, there are two equations determining w_k by w_l $(0 \leq l \leq k-1)$, namely

$$\sum_{j=0}^{k} w_j\, c_{2j-2k+n} = 0, \qquad \sum_{j=0}^{k} w_j\, c_{2j-2k+n-1} = 0.$$

But the second equation is a consequence of the former ones, since both equations are contained in the system $(w_j)_{j=k-n+1}^{k} A_0 = 0$ with $\det A_0 = 0$. Same ideas apply for root vectors.
(i'): For left eigenvectors of A_1 we can prove the assertion analogously as (i), considering $\boldsymbol{A}' := (c_{2i-j+1})_{i,j=0}^{\infty}$ instead of $\boldsymbol{A}$.

(ii): Replacing t by $t-k$ in (1.11) with an arbitrary $k \in \mathbb{N}$, we obtain

$$\sum_{j=-k}^{\infty} w_{j+k}\, \varphi(t+j) = c$$

for $t \le k+1$ and, in view of $\varphi(t) = 0$ for $t < 0$,

$$\sum_{j=0}^{\infty} w_{j+k}\, \varphi(t+j) = c \tag{2.3}$$

for $t \le 1$. Hence, (ii) holds.
(iii): First, it can easily be seen that an E-solution $\varphi(t)$ can neither vanish identically in $[0,1]$ nor in $[n-1,n]$, otherwise it would vanish for all $t \in [0,n]$. By (2.3), for a certain $t_0 \in [0,1]$ with $\varphi(t_0+n-1) \neq 0$, we find

$$\sum_{j=0}^{n-1} w_{j+k}\, \varphi(t_0+j) = c.$$

This recursion formula shows that w_ν for $\nu \ge n$ is uniquely determined by the initial values $w_1, \ldots, w_{n-1}$, hence there can be at most one extension from w to $\boldsymbol{w}$.
Assertion (iv) follows from (ii) and from Corollary 2.1 (vi).
Finally, (v) is a consequence of (iii) and the definition of L. ∎

Remarks: 1. Using Theorem 2.3 (i) and Corollary 2.1 (iii), (iv) we find: If φ is bounded in neighbourhoods of the points k $(k = 0, \ldots, n)$, and if $w \in L$ is a left eigenvector or left root vector of A_0 (or A_1) to the eigenvalue λ $(|\lambda| > 1)$, then w is uniquely extendable to $\boldsymbol{w} \in L$. If we replace the boundedness condition by continuity in all points k $(k = 0, \ldots, n)$, then this assertion also holds for eigenvalues λ with $|\lambda| = 1$.
2. If $(w_j)^T_{j\ge 0}\, \boldsymbol{A} = \lambda\, (w_j)^T_{j\ge 0}$ then we also have $(w_{j+1})^T_{j\ge 0}\, \boldsymbol{A}' = \lambda\, (w_{j+1})^T_{j\ge 0}$.
3. For $\lambda = c_n$, it can happen that an eigenvector of A_0 corresponding to λ cannot be extended to an eigenvector of $\boldsymbol{A}$, but then it can always be extended to a root vector of $\boldsymbol{A}$ corresponding to λ.
For example, consider (1.1) with $c_0 = c_2 = 1/2$, $c_1 = 3/4$ and $c_3 = 1/4$. Then $(-2,1,-2)^T$ is a left eigenvector of A_0 to $1/4$. This eigenvector cannot be extended to an eigenvector of $\boldsymbol{A}$, since there is no x, such that $(1,-2,x)\, A_1 = \frac{1}{4}(1,-2,x)$. But, since $(1,-2,x)^T$ is a root vector of A_1, it can be extended to a root vector of $\boldsymbol{A}'$; hence $(-2,1,-2)^T$ can be extended to a root vector of $\boldsymbol{A}$.
4. If a formally constructed space $\mathcal{L}$ has dimension n, then (1.1) has no E-solution.

Considering solutions of (1.1) with linearly independent integer translates, we have $\dim W = n-1$. Hence, $\dim L = 1$, and by Theorem 2.1 it follows that $L = \operatorname{span}\{(1,\dots,1)^T\}$ and $\mathcal{L} = \operatorname{span}\{\boldsymbol{e}\} = \operatorname{span}\{(1,1,1,\dots)^T\}$. By Theorem 2.2 (ii),

$$(1,\dots,1)\,A_0 = (\sum_\nu c_{2\nu}, \sum_\nu c_{2\nu+1},\dots) = \lambda(1,\dots,1).$$

Taking into account that $2P(1) = \sum_\nu c_\nu = 2$ according to Assumption (A1), it follows that the first sum rule, cf. [6],

$$\sum_\nu c_{2\nu} = \sum_\nu c_{2\nu+1} \tag{2.4}$$

is satisfied, and λ must be 1.

In the following, we are interested in the structure of elements of L and $\mathcal{L}$, and in consequences for the refinement mask.

THEOREM 2.4. *Let φ be an E–solution of* (1.1) *with* (A1), (A2)*, let $\mathcal{L}$ be defined as in* (1.13)*, and let $\boldsymbol{w} = (w_j)_{j\geq 0} \in \mathcal{L}$. Then there are $\zeta_k \in \mathbb{C}\setminus\{0\}$ $(k = 1,\dots,l)$ such that the jth element of $\boldsymbol{w}$ has the form*

$$w_j = \sum_{k=1}^{l} d_k(j)\,\zeta_k^j \qquad (j \geq 0) \tag{2.5}$$

with $l < n$, and where $d_k(j)$ are polynomials of degree ν_k in j. In case of $\nu_k \geq 0$, the vectors $(j^\nu \zeta_k^j)_{j\geq 0}$ $(0 \leq \nu \leq \nu_k)$ belong to $\mathcal{L}$.

Proof. 1. According to Theorem 2.3 (ii), by $\boldsymbol{w} \in \mathcal{L}$, the vectors $(w_{j+m})_{m\geq 0}$ $(j \in \mathbb{N}_0)$ are also contained in $\mathcal{L}$, i.e.,

$$\sum_{m=0}^{\infty} w_{j+m}\,\varphi(t+m) = c \qquad (t \leq 1). \tag{2.6}$$

Observe that $\varphi(t)$ is not a constant function for $t \in [0,1]$, since $c_0 \neq 1$ by Assumption (A2). Hence, we can choose $t_0,\, t_1 \in [0,1]$, such that (2.6) is satisfied and $\varphi(t_0) \neq \varphi(t_1)$. Considering (2.6) for these t_0 and t_1, and putting $b_m := \varphi(t_0+m) - \varphi(t_1+m)$, we obtain

$$\sum_{m=0}^{n-1} b_m\, w_{j+m} = 0 \tag{2.7}$$

for all $j \in \mathbb{N}_0$.

This is a difference equation with constant coefficients, where in particular, $b_0 \neq 0$. Hence, the solution is of the form (2.5) for $j \in \mathbb{N}_0$, and the numbers ζ_k $(k = 1, \ldots, l)$ are the pairwise different zeros of the characteristic polynomial of the difference equation, i.e.,

$$\sum_{m=0}^{n-1} b_m\, z^m = b_p \prod_{k=1}^{l} (z - \zeta_k)^{\mu_k + 1},$$

where the index p of the coefficient b_p is determined by $p = \max\{j : b_j \neq 0\}$. In view of Corollary 2.1 (i), both equations (2.6) and (2.7) are satisfied at least for $w_j = 1$ $(j \in \mathbb{N}_0)$, and hence, $p \geq 1$. Further, $b_0 \neq 0$ implies that $\zeta_k \neq 0$ for all k. The polynomials d_k in (2.5) are of degree $\nu_k \leq \mu_k$, where $\mu_k + 1$ is the multiplicity of the zero ζ_k in the characteristic polynomial. (The zero polynomial $d_k(j) \equiv 0$ is included with degree $\nu_k = -1$.)

2. If we assume first that the zeros ζ_k are simple, then the coefficients d_k in (2.5) are independent from j. Replacing j successively by $j+1$, $j+2, \ldots, j+l-1$ in (2.5), the arising equations can be writen in form of the system

$$\left(d_1\zeta_1^j,\, d_2\zeta_2^j, \ldots, d_l\zeta_l^j\right) \begin{pmatrix} 1 & \zeta_1 & \ldots & \zeta_1^{l-1} \\ \vdots & \vdots & & \vdots \\ 1 & \zeta_l & \ldots & \zeta_l^{l-1} \end{pmatrix} = (w_j, \ldots, w_{j+l-1})\,,$$

where the determinant $\det\,(\zeta_k^{\mu-1})_{\mu,k=1}^{l}$ is the Vandermondian of $\zeta_1, \ldots, \zeta_l$ and therefore different from zero. Hence, all $(d_k\,\zeta_k^j)_{j\geq 0}$ $(k = 1, \ldots, l)$ are linear combinations of $(w_{j+k})_{j\geq 0}$ $(k = 0, \ldots, l-1)$, i.e., in case of $d_k \neq 0$, also $(\zeta_k^j)_{j\geq 0}$ is is contained in $\mathcal{L}$.

3. In the same manner, we can conclude for variable $d_k(j)$, where instead of the Vandermondian the confluent Vandermondian appears. In this case, for $\nu_k > 0$, we can enlarge j such that all coefficients of $d_k(j)$ are different from zero. ∎

Remarks: 1. For $(\zeta^j)_{j\geq 0} \in \mathcal{L}$, we have by Theorem 2.3 (ii)

$$\sum_{j=0}^{\infty} \zeta^{j+k}\, \varphi(t+j) = c$$

for all $k \in \mathbb{N}_0$. Hence, for $k = 0$ and $k = 1$, respectively,

$$0 = \sum_{j=0}^{\infty} \zeta^j\, \varphi(t+j) - \sum_{j=0}^{\infty} \zeta^{j+1} \varphi(t+j) = (1-\zeta)\, c,$$

so that $c = 0$ for $\zeta \neq 1$. Only for $\zeta = 1$, there can be $c \neq 0$. In fact, from $\boldsymbol{e}^T\, \boldsymbol{\psi}(t) = 1$ for $\int_0^n \varphi(t)\; \mathrm{d}t = 1$ (see Theorem 2.1) it follows that

$$\mathcal{L} = \{\boldsymbol{w} : \boldsymbol{w}^T\, \psi(t) = 0,\, t \in (-\infty, 1],\, a.e.\} \cup \{\boldsymbol{e}\}.$$

2. Instead of $\mathcal{L}$, in [1], Theorem 6.4, the structure of

$$N_\varphi := \{\lambda : \sum_{\alpha\in\mathbb{Z}^s} \lambda_\alpha\, \varphi(x-\alpha) = 0,\ x \in \mathbb{R}^s\}$$

was considered, with a result similar to Theorem 2.4. Observe that for $s = 1$, $N_\varphi \subset \mathcal{L}_0$ in view of the substitution $\alpha = -j$. More precisely, we have $\mathcal{L}_0 = N_\varphi \cup \{e\}$. The investigation of N_φ was especially addressed in detail for the cube spline by Dahmen & Micchelli [3].

Example 2.1. We consider (1.1) with the coefficients $c_0 = c_6 = \frac{1}{2}$, $c_3 = 1$ and $c_1 = c_2 = c_3 = c_4 = 0$, i.e., $P(z) = \frac{1}{4}(1+z^3)^2$. The corresponding matrices A_0 and A_1 read

$$A_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

The matrix A_0 possesses the eigenvalue 1 with multiplicity 2 with corresponding right eigenvectors $v^{(0)} = (0,\ 1,\ 2,\ 0,\ 2,\ 1)^T$, and $v^{(1)} = (0,\ -1,\ -2,\ 6,\ -2,\ -1)^T$. Analogously, $(1,\ 2,\ 0,\ 2,\ 1,\ 0)^T$ and $(-1,\ -2,\ 6,\ -2,\ -1,\ 0)^T$ are right eigenvectors of A_1 corresponding to 1. But only the linear combination $\psi(0) = (0,\ 1,\ 2,\ 3,\ 2,\ 1)^T = \frac{3}{2}v^{(0)} + \frac{1}{2}v^{(1)}$ leads to an E–solution of (1.1), namely

$$\varphi(t) = \begin{cases} t & 0 \le t < 3, \\ (6-t) & 3 \le t \le 6, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that $\psi(1) - \psi(0) = (1,\ 1,\ 1,\ -1,\ -1,\ -1)$ is already invariant under both A_0 and A_1, such that $W := \operatorname{span}\{(1,\ 1,\ 1,\ -1,\ -1,\ -1)^T\}$. Hence, L is e.g. given by

$$L = \operatorname{span}\left\{ \begin{array}{l} (1,0,0,1,0,0)^T,\ (0,1,0,0,1,0,0)^T,\ (0,0,1,0,0,1)^T, \\ (0,0,0,1,-1,0)^T,\ (0,0,0,0,1,-1)^T \end{array} \right\}.$$

In particular, the left eigenvectors of A_0 and A_1 corresponding to 1 and -1, are contained in L. The vectors in L can be extended to vectors in $\mathcal{L}$, i.e.,

$$\mathcal{L} = \operatorname{span}\left\{ \begin{array}{l} (1,0,0,1,0,0,1,0,0,1,0,0,1,0,0,\ldots)^T, \\ (0,1,0,0,1,0,0,1,0,0,1,0,0,1,0,\ldots)^T, \\ (0,0,1,0,0,1,0,0,1,0,0,1,0,0,1,\ldots)^T, \\ (0,0,0,1,-1,0,2,-2,0,3,-3,0,4,-4,\ldots)^T, \\ (0,0,0,0,1,-1,0,2,-2,0,3,-3,0,4,-4,\ldots)^T \end{array} \right\}.$$

Considering the proof of Theorem 2.4, the characteristic polynomial of the difference equation (2.7) reads

$$(1+z+z^2-z^3-z^4-z^5) = (z-1)\,(z-e^{2\pi i/3})^2\,(z-e^{4\pi i/3})^2.$$

Hence, $\mathcal{L}$ is also spanned by the vectors $\boldsymbol{e} := (1,1,1,\ldots)^T$, $(e^{2\pi ij/3})_{j\geq 0}$, $(j\,e^{2\pi ij/3})_{j\geq 0}$, $(e^{4\pi ij/3})_{j\geq 0}$ and $(j\,e^{4\pi ij/3})_{j\geq 0}$.

3. FACTORIZATION OF THE MASK

As before, let φ be an E–solution of (1.1) under the assumptions (A1) and (A2), and let $\mathcal{L}$ be defined as in (1.13). Knowing the structure of vectors contained in $\mathcal{L}$, we shall derive consequences on eigenvectors of the infinite matrix $\boldsymbol{A} := (c_{2i-j})_{i,j\geq 0}$ and on zeros of the refinement mask $P(z)$ defined in (1.3). We shall show that for each E-solution φ, the refinement mask necessarily contains a polynomial factor, which is a certain modification of $(1+z)$.

Let $\sqrt{\zeta}$ denote an arbitrarily chosen, but then fixed value of the two square roots of ζ $(\zeta \neq 0)$.

THEOREM 3.1. *Let $(\zeta^j)_{j\geq 0} \in \mathcal{L}$ $(\zeta \neq 0)$ and*

$$\alpha = \alpha(\zeta) := \sum_{j=0}^{\infty} \zeta^j\, c_{2j}, \qquad \beta = \beta(\zeta) := \sum_{j=0}^{\infty} \zeta^j\, c_{2j-1}. \tag{3.1}$$

Then, one of the following 4 cases arises:

(i) *$\alpha = \beta = 0$; then $P(\pm\sqrt{\zeta}) = 0$, and $(\zeta^j)_{j\geq 0}$ is a left eigenvector of $\boldsymbol{A}$ to the eigenvalue 0.*

(ii) *$\beta = \alpha\sqrt{\zeta}$ $(\alpha \neq 0)$; then $P(-\sqrt{\zeta}) = 0$, and $((\sqrt{\zeta})^j)_{j\geq 0} \in \mathcal{L}$.*

(iii) *$\beta = -\alpha\sqrt{\zeta}$ $(\alpha \neq 0)$; then $P(\sqrt{\zeta}) = 0$, and $((-\sqrt{\zeta})^j)_{j\geq 0} \in \mathcal{L}$.*

(iv) *$\beta^2 \neq \alpha^2\zeta$; then both $((\sqrt{\zeta})^j)_{j\geq 0}$ and $((-\sqrt{\zeta})^j)_{j\geq 0}$ belong to $\mathcal{L}$.*

Proof. If $(\zeta^j)_{j\geq 0} \in \mathcal{L}$, then

$$\begin{aligned}
(\zeta^i)^T_{i\geq 0}\,\boldsymbol{A} &= \left(\sum_{i=0}^{\infty} \zeta^i\, c_{2i-j}\right)^T_{j\geq 0} = (\alpha,\,\beta,\,\zeta\alpha,\,\zeta\beta,\,\zeta^2\alpha,\zeta^2\beta,\ldots) \\
&= \frac{1}{2}(\alpha+\frac{\beta}{\sqrt{\zeta}})((\sqrt{\zeta})^j)^T_{j\geq 0} + \frac{1}{2}(\alpha-\frac{\beta}{\sqrt{\zeta}})((-\sqrt{\zeta})^j)^T_{j\geq 0} \qquad (3.2) \\
&= P(\sqrt{\zeta})((\sqrt{\zeta})^j)^T_{j\geq 0} + P(-\sqrt{\zeta})((-\sqrt{\zeta})^j)^T_{j\geq 0}
\end{aligned}$$

also belongs to $\mathcal{L}$, according to Corollary 2.1.
Case (i): From $\alpha = \beta = 0$ it obviously follows that $(\zeta^j)_{j\geq 0}$ is a left zero vector of $\boldsymbol{A}$. Moreover, we find $\alpha \pm \beta/\sqrt{\zeta} = P(\pm\sqrt{\zeta}) = 0$.
Case (ii): For $\beta = \alpha\sqrt{\zeta}$ $(\alpha \neq 0)$, the vector $(\zeta^j)_{j\geq 0}^T \boldsymbol{A}$ is equal to $(\alpha(\sqrt{\zeta})^j)_{j\geq 0}$ and $P(\sqrt{\zeta}) \neq 0$. Hence, $((\sqrt{\zeta})^j)_{j\geq 0} \in \mathcal{L}$, and by $\alpha - \beta/\sqrt{\zeta} = 0$ we obtain that $P(-\sqrt{\zeta}) = 0$. Analogously, we can conclude in the case (iii).
Case (iv): By Theorem 2.3 (ii), the shifted vector

$$(\zeta^{j+1})_{j\geq 0}^T \boldsymbol{A} = (\beta, \zeta\alpha,\, \zeta\beta,\, \zeta^2\alpha, \zeta^2\beta,\, \ldots)^T$$

also belongs to $\mathcal{L}$. Multiplying the matrix consisting of the two row vectors $(\zeta^j)_{j\geq 0}^T\boldsymbol{A}$ and $(\zeta^{j+1})_{j\geq 0}^T\boldsymbol{A}$ by

$$\begin{pmatrix} \alpha & \beta \\ \beta & \zeta\alpha \end{pmatrix}^{-1} = \frac{1}{\zeta\alpha^2 - \beta^2}\begin{pmatrix} \zeta\alpha & -\beta \\ -\beta & \alpha \end{pmatrix},$$

it follows that the vectors

$$\begin{aligned} &(1,\, 0,\, \zeta,\, 0, \zeta^2,\, 0, \ldots)^T, \\ &(0,\, 1,\, 0,\, \zeta,\, 0,\, \zeta^2, \ldots)^T, \end{aligned}$$

and therefore also the both vectors $((\pm\sqrt{\zeta})^j)_{j\geq 0}$ belong to $\mathcal{L}$. ■

Considering the more general case if $(j^\nu \zeta^j)_{j\geq 0} \in \mathcal{L}$ with $\nu > 0$, analogous assertions as in Theorem 3.1 can be derived.

Let, for $\nu \in \mathbb{N}_0$,

$$N^{(\nu)} := \{z \in \mathbb{C} :\ \mathrm{D}^\mu P(z) = 0,\ \mu = 0, \ldots, \nu - 1\} \tag{3.3}$$

be the set of ν-fold zeros of $P(z)$, where $P(z)$ is the refinement mask in (1.3), and where D denotes the usual differential operator $\mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}z}$.

We state the following generalization of Theorem 3.1:

Theorem 3.2. *Assume that* $(j^\nu \zeta^j)_{j\geq 0} \in \mathcal{L}$ $(\nu \in \mathbb{N}_0,\ \zeta \neq 0)$. *Then there exist integers* r, $\tilde{r}$ *with* $0 \leq r,\, \tilde{r} \leq \nu + 1$, *such that*
(1) $(j^\mu\, (\sqrt{\zeta})^j)_{j\geq 0} \in \mathcal{L}$ *for* $\mu = 0, \ldots, r - 1$ *and* $\sqrt{\zeta} \in N^{(\nu - r + 1)}$;
(2) $(j^\mu\, (-\sqrt{\zeta})^j)_{j\geq 0} \in \mathcal{L}$ *for* $\mu = 0, \ldots, \tilde{r} - 1$ *and* $-\sqrt{\zeta} \in N^{(\nu - \tilde{r} + 1)}$.

Example 3.1. Let us apply Theorem 3.1 to the Example 1. For $\zeta_1 = 1$, we obtain $\alpha(1) = \beta(1) = 1$. Hence, putting $\sqrt{\zeta_1} = 1$, it follows by case (ii) that $P(-1) = 0$. For $\zeta_2 = e^{2\pi i/3}$ and $\sqrt{\zeta_2} = e^{\pi i/3}$, we find $\alpha(e^{2\pi i/3}) = 1$, $\beta(e^{2\pi i/3}) = e^{4\pi i/3}$. Thus, we have to apply case (iii) yielding that $P(e^{\pi i/3}) = 0$ and that $(-e^{\pi i j/3})_{j\geq 0} = (e^{4\pi i j/3})_{j\geq 0} \in \mathcal{L}$. Finally, from $\zeta_3 =$

$e^{4\pi i/3}$ with $\sqrt{\zeta_3} = e^{2\pi i/3}$ it follows that $\alpha(e^{4\pi i/3}) = 1$, $\beta(e^{4\pi i/3}) = e^{2\pi i/3}$. Hence, by case (ii), we obtain $P(e^{5\pi ij/3}) = 0$ and $(e^{2\pi ij/3})_{j\geq 0} \in \mathcal{L}$.

In some papers considering compactly supported solutions of refinement equations (see e.g. Daubechies & Lagarias [6]), the first sume rule (2.4) is assumed, yielding that $e := (1,\ldots,1)^T$ is a left eigenvector of both A_0 and A_1. In particular, it follows that the corresponding refinement mask $P(z)$ possesses the factor $(z+1)$. For integrable solutions of (1.1) with compact support in $[0,n]$ and with linearly independent integer translates, the first sum rule (2.4) is satisfied everytimes (see Section 2). Now, we show that a refinement mask $P(z)$ yielding an E–solution φ, necessarily possesses a factor $p(z)$ which is a certain modification of $(z+1)$.

THEOREM 3.3. *Assume that* (1.1) *with* (A1), (A2) *possesses an E–solution φ. Then there is a number $k \in \mathbb{N}_0$, such that $p(z) := p_k(z)$ is a factor of the refinement mask $P(z)$, where $p_k(z)$ is obtained iteratively in the following manner:*
1) $p_0(z) := (z+1)/2$;
2) $p_l(z)$ is obtained by replacing z by z^2 in $p_{l-1}(z)$ or in a polynomial factor of $p_{l-1}(z)$ $(l = 1,2,\ldots,k)$.
The resulting factor $p_k(z)$ of $P(z)$ is a polynomial satisfying $p_k(1) = 1$, and all its zeros are roots of -1 of order 2^r with $r \in \{0,\ldots,k\}$.

Proof. With the notation $\boldsymbol{e} := (1,1,1,\ldots)^T$, we have that $\boldsymbol{e} \in \mathcal{L}$ in view of Corollary 2.1 (i). Hence, we can apply Theorem 3.1 to $\zeta = 1$. By $2P(1) = \alpha + \beta = 2$ (with α, β defined in (3.1)) we obtain

$$\boldsymbol{e}^T \boldsymbol{A} = (\alpha,\, \beta,\, \alpha,\, \beta, \ldots) = \boldsymbol{e}^T + (\alpha - 1)((-1)^j)^T_{j\geq 0}.$$

Let us first consider the case that $\alpha = \beta = 1$; i.e., the case (ii) of Theorem 3.1. Then, $P(-1) = 0$, i.e., $p_0(z) = (z+1)/2$ is a factor of the mask $P(z)$, and the last statement of (ii) (in Theorem 3.1) gives no new relation in view of $\zeta = 1$.

For $\alpha \neq 1$, we have $\beta^2 = (2-\alpha)^2 \neq \alpha^2$, and case (iv) of Theorem 3.1 is applicable. Hence, $((-1)^j)_{j\geq 0} \in \mathcal{L}$. Now the procedure can be repeated with $\zeta = -1$.

We compute $\alpha(-1)$, $\beta(-1)$ and check which of the four cases of Theorem 3.1 can be applied. For $\alpha(-1) = \beta(-1) = 0$, it follows that $P(\pm i) = 0$, i.e., $p_1(z) = (1+z^2)/2$. If one of the last three cases occurs, then, again (at least one) new element of $\mathcal{L}$ is found, namely $(i^j)_{j\geq 0}$ or $((-i)^j)_{j\geq 0}$, or both of them. The procedure must then be applied to the remaining roots $\zeta = i$ and $\zeta = -i$, respectively, or to both of them, and so on. Observe, that this algorithm only produces numbers ζ, which are roots of unity, namely roots of -1 of order 2^r in the r-th step. Moreover, all elements $\boldsymbol{e}$, $((-1)^j)_{j\geq 0}, \ldots$

of $\mathcal{L}$, found iteratively by this procedure, are linearly independent. Since the dimension of $\mathcal{L}$ is finite, we must arrive at case (i) (of Theorem 3.1) after finitely many steps, and the procedure stops. The algorithm is completely described by determining what to do when arriving at one of the cases (i) – (iv), after replacing the factor $z-\zeta$ of $p_{l-1}(z)$ by $z^2-\zeta = (z-\sqrt{\zeta})(z+\sqrt{\zeta})$.
Case (i) means: $(z^2-\zeta)$ is a factor of $P(z)$ and the procedure stops for this ζ.
Case (ii) means: $(z+\sqrt{\zeta})$ is a factor of $P(z)$ and we continue with the factor $z-\sqrt{\zeta}$ applying Theorem 3.1 to $((\sqrt{\zeta})^j)_{j\geq 0}$.
Case (iii) means: $(z-\sqrt{\zeta})$ is a factor of $P(z)$ and we continue with the factor $z+\sqrt{\zeta}$ applying Theorem 3.1 to $((-\sqrt{\zeta})^j)_{j\geq 0}$.
Case (iv) means: We continue with both factors $(z+\sqrt{\zeta})(z-\sqrt{\zeta})$ applying Theorem 3.1 to both $(\sqrt{\zeta}^j)_{j\geq 0}$ and $((-\sqrt{\zeta})^j)_{j\geq 0}$.
Since some factors $z-\zeta$ can be gathered up, we have exactly the procedure as described in the Theorem leading to a factor $p(z) = p_k(z)$ with the mentioned properties. ■

REMARKS: 1. If the refinement mask of an E-solution φ does not possess the factor $p_0(z) = (1+z)/2$, then in view of the foregoing case (i), it has symmetric zeros on the unit circle. Hence, the condition that $P(z)$ has no symmetric zeros on the unit circle, is necessary for linear independence as well as for Riesz stability of integer translates of φ (see e.g. [13], Theorem 3.3).
2. The simplest zero set corresponding to $p_0(z)$ is $R = \{1\}$. Other examples for zero sets are $R = \{1/2, 3/2\}$ corresponding to $p_1(z) = (z^2+1)/2$ and $R = \{1/8, 9/8, 5/4, 3/2\}$ corresponding to $p_3(z) = (z^2+e^{-5i\pi/4})(z-e^{-5i\pi/4})(z+i)$.
3. Characterising the roots $e^{i\pi p/q}$ of -1 by p/q, we can interpret the result of Theorem 3.3 by means of the tree graph in Figure 1, in order to get the zeros of $p(z)$. Geometrically, the endpoints of a certain finite subtree of the graph with invariant root 1 form the set R of zeros of a possible factor $p(z) = p_k(z)$ in the Theorem.

The factor $p(z) (= p_k(z))$ found in Theorem 3.3 can also be characterized as follows:

THEOREM 3.4. *The polynomial $p(z)$ can be found by the iteration process in Theorem 3.3 if and only if it is of the form*

$$p(z) = \frac{q(z^2)}{2\,q(z)}, \tag{3.4}$$

where $q(z)$ is a polynomial of the same degree and possessing a set Q of zeros with the following property: Q contains roots of unity with powers of

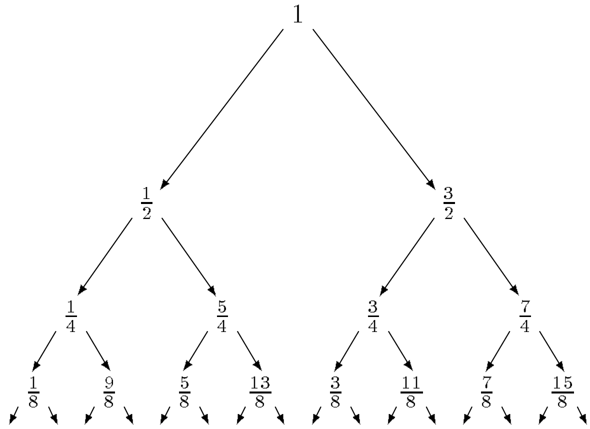

Figure 1

2 as root exponent, and it is closed regarding to the operation $z \to z^2$ (i.e., for $z \in Q$ it follows that $z^2 \in Q$). Moreover, denoting the zero set of $p(z)$ by R and introducing the set $\tilde{Q}$ of all square roots of elements of Q,

$$\tilde{Q} := \{z : z^2 \in Q\},$$

we have the relations:

$$\begin{aligned} R &= \tilde{Q} \setminus Q, \\ Q &= \{z^{2^j} : j \in \mathbb{N},\, z \in R\}. \end{aligned}$$

Proof. For $m = 1$, the representation is valid with $q(z) = z - 1$. For $m > 1$, the construction of $p(z)$ can be described in the following way:

$$\begin{aligned} z^2 + 1 &= q_1(z)r_1(z) \to q_1(z)r_1(z^2) = q_2(z)r_2(z) \to \dots \\ &= q_k(z)r_k(z) \to q_k(z)r_k(z^2) = 2p(z) \end{aligned}$$

with polynomials $q_i(z)$, $r_i(z)$ $(i = 1, \dots, k)$ and $q_i(z)r_i(z^2) = q_{i+1}(z)r_{i+1}(z)$ $(i = 1, \dots, k-1)$. We find recursively

$$2p(z) = (z^2 + 1) \prod_{i=1}^{k} \frac{r_i(z^2)}{r_i(z)},$$

so that (3.4) is obtained with $q(z) = (z^2 - 1) \prod_{i=1}^{k} r_i(z)$. Denoting the set of zeros of $p(z)$ by $R := \{\xi_1, \dots, \xi_m\}$, and the zero set of $q(z)$ by $Q := \{\sigma_1, \dots, \sigma_m\}$, equation (3.4) shows that

$$\{\xi_\nu,\, \sigma_\nu : \nu = 1, \dots, m\} = \{\sqrt{\sigma_\nu},\, -\sqrt{\sigma_\nu} : \nu = 1, \dots, m\},$$

i.e., $R \cup Q = \tilde{Q}$ with $\tilde{Q} := \{\sqrt{\sigma_\nu}, -\sqrt{\sigma_\nu} : \nu = 1, \ldots, m\}$. Thus, for every ν, there exist numbers a_ν, $b_\nu \in \{1, \ldots, m\}$ with $\xi_\nu^2 = \sigma_{a_\nu}$, $\sigma_\nu^2 = \sigma_{b_\nu}$. Hence, the set $\{\sigma_1, \ldots, \sigma_m\}$ is the closure of the set $\{\xi_1^2, \ldots, \xi_m^2\}$ by the operation $z \to z^2$, i.e., $Q = \{z^{2^j} : j \in \mathbb{N}, z \in R\}$. Since all ξ_l are roots of unity with a power of 2 as root exponent, the σ_l have the same property. Finally, since the $2m$ elements of $\tilde{Q}$ are different, it follows that $R = \tilde{Q} \setminus Q$.

The second direction of the proof follows conversely. ■

Remarks: 1. In [1], Lemma 6.6, it was already shown that a finite set R of complex numbers with the property $R \subseteq R^2 := \{z^2 : z \in R\}$ and cardinality n can only consist of elements which are roots of unitary or zero. Further, for any element $z \in R$ there is a k, $1 \le k \le n$, such that $z^{2^k} = z$.
2. In the special case $2p(z) = (z^{2^k} + 1)$, (3.4) is satisfied with $q(z) = z^{2^k} - 1$. Of course, $P(z)$ can have several factors of the type $p(z)$, such that the zeros ξ_ν can also appear with a higher multiplicity.
3. If we assume that $p(z)$ has real coefficients only, then $p(z)$ can be decomposed into factors of the form $z^2 + a_k z + 1$. The coefficients a_k can be found iteratively from $a_0 := 0$, $a_l := \pm\sqrt{2 - a_{l-1}}$, since

$$(z^2 + \sqrt{2 - a_k}\, z + 1)\,(z^2 - \sqrt{2 - a_k}\, z + 1) = z^4 + a_k z^2 + 1.$$

The factor $p_k(z)$ of $P(z)$, found iteratively from $(z+1)/2$ as described in Theorem 3.3, can be considered as the refinement mask of a special compactly supported step function. This is the exeptional case, where Assumption (A2) is not satisfied.

Let

$$\varphi(t) = \sum_{\nu=0}^{k} b_\nu\, \chi(t - \nu) \qquad (t \in \mathbb{R}) \tag{3.5}$$

with $b_0 b_k \neq 0$ $(k \ge 1)$, and where $\chi(t)$ is the characteristic function of the interval $[0, 1)$, i.e.,

$$\chi(t) = \begin{cases} 1 & t \in [0,1), \\ 0 & t \notin [0,1). \end{cases} .$$

Then according to Lawton, Lee & Shen [18] we have:

Proposition 3.1. *The function φ of the form (3.5) is refinable, if and only if the corresponding refinement mask $P(z)$ is of the form $2P(z) = Q(z^2)/Q(z)$, where $Q(z) := (z-1)(\sum_{\nu=0}^{k} b_\nu z^\nu)$ with the coefficients b_ν in (3.5), and where the zero set $\{z \in \mathbb{C} : Q(z) = 0\}$ is closed regarding to the operation $z \to z^2$.*

Observe that the zeros of $Q(z)$ in Proposition 3.1 can be arbitrary roots of unity. The factorization $P(z) = p(z)Q(z)$ with $p(z)$ in (3.4), given by Theorem 3.4, allows to simplify the E-solution of (1.1). In order to show this, we need the

THEOREM 3.5. *Let $P(z)$ and $\tilde{P}(z)$ be polynomials of the form*

$$P(z) = P_1(z)\,P_2(z^2), \qquad \tilde{P}(z) = P_1(z)\,P_2(z),$$

where $P_2(z) = \sum_{\nu=0}^{k} r_\nu\, z^\nu$ with $r_0 \neq 0$ and with $P(1) = \tilde{P}(1) = 1$. Then we have: Equation (1.1) *with the refinement mask $P(z)$ possesses an E-solution φ if and only if* (1.1) *with $\tilde{P}(z)$ possesses an E-solution $\tilde{\varphi}$, and we have, for $t \in \mathbb{R}$,*

$$\varphi(t) = \sum_{\nu=0}^{k} r_\nu\, \tilde{\varphi}(t-\nu). \tag{3.6}$$

Proof. Since $P(1) = \tilde{P}(1) = 1$, there exist unique compactly supported distributions φ, $\tilde{\varphi}$ satisfying (1.1) with the refinement masks $P(z)$, $\tilde{P}(z)$, respectively (see e.g. [20], Theorem 1.1, [5, 7]). The Fourier transforms of φ, $\tilde{\varphi}$ are given by

$$\begin{aligned}
\hat{\varphi}(u) &= \prod_{j=1}^{\infty} P_1(e^{-iu/2^j})\,P_2(e^{-2iu/2^j}) \\
&= P_2(e^{-iu}) \prod_{j=1}^{\infty} P_1(e^{-iu/2^j})\,P_2(e^{-iu/2^j}), \\
\hat{\tilde{\varphi}}(u) &= \prod_{j=1}^{\infty} P_1(e^{-iu/2^j})\,P_2(e^{-iu/2^j}),
\end{aligned}$$

where the infinite products converge uniformly on every compact subset of $\mathbb{C}$ (cf. [20]). Hence,

$$\hat{\varphi}(u) = P_2(e^{-iu})\,\hat{\tilde{\varphi}}(u).$$

Inverse Fourier transform yields that

$$\varphi(t) = \sum_{\nu=0}^{k} r_\nu\, \tilde{\varphi}(t-\nu).$$

This equation shows: If $\tilde{\varphi}$ is an E-solution, then so is φ. Vice versa, if φ is an E-solution, then, by $r_0 \neq 0$, the function $\tilde{\varphi}$ can recursively be constructed by means of (3.6) as a nonvanishing locally Lebesgue-integrable function. Since $\tilde{\varphi}$ is compactly supported, $\tilde{\varphi}$ is an E-solution. ∎

We easily conclude that, if $\tilde{\varphi}(t)$ is continuous, then also $\varphi(t)$ is continuous. Observe, that $P_1(z)$ in Theorem 3.5 is not necessarily a polynomial as in the example $P(z) = \frac{1}{2}(1+z^4)$, $\tilde{P}(z) = \frac{1}{2}(1+z)$, $P_1(z) = (1+z^2)^{-1}$.

By Theorem 3.4, each E-solution φ of (1.1) has a refinement mask of the form $P(z) = p(z)\,Q(z)$ with $2p(z) = q(z^2)/q(z)$. The polynomial $q(z)$ possesses the simple zero 1, i.e., $q(z) = (z-1)\,r(z)$ with $r(1) \neq 0$ (see the proof of Theorem 3.4). We can apply Theorem 3.5 as follows:

Proposition 3.2. *Let φ be an E-solution of* (1.1) *and let* (A1), (A2) *be satisfied. Assume that the corresponding refinement mask $P(z)$ has the representation $P(z) = p(z)\,Q(z)$. Further, let $p(z)$ be a polynomial factor of the form*

$$p(z) = \frac{(1+z)\,r(z^2)}{2\,r(z)}$$

with $p(1) = 1$, and with $r(z) := \sum_{\nu=0}^{k} r_\nu z^\nu$ and $r_0 \neq 0$. Then (1.1) *corresponding to the refinement mask $\tilde{P}(z) := \frac{z+1}{2}\,Q(z)$ possesses an E-solution $\tilde{\varphi}$ and we have*

$$\varphi(t) = \sum_{\nu=0}^{k} r_\nu\,\tilde{\varphi}(t-\nu).$$

Proof. We apply Theorem 3.5 as follows: Putting

$$P_1(z) = \frac{(z+1)\,Q(z)}{2\,r(z)}, \qquad P_2(z) = r(z),$$

we have $P(z) = P_1(z)\,P_2(z^2)$. Hence, the refinement mask $\tilde{P}(z) = P_1(z)\,P_2(z)$ also provides an E-solution $\tilde{\varphi}$ of (1.1), and the assertion follows. ∎

Remarks: 1. By Proposition 3.2, each E-solution φ of (1.1) can be represented as a finite linear combination of integer translates of an E-solution $\tilde{\varphi}$ with a refinement mask $\tilde{P}(z)$ containing the factor $p_0(z) = (z+1)/2$. This argument can even be pushed a little further, showing that each E-solution can be given as a finite linear combination of integer translates of a refinable function $\tilde{\varphi}$ with linearly independent integer shifts (see [12], Theorem 5.3).

2. If $P(z) = P_1(z)\,P_2(z^4)$ then it can also be represented as

$$P(z) = \frac{P_1(z)}{P_2(z^2)}\,P_2(z^2)P_2(z^4),$$

and we can apply Theorem 3.5 with $P_2(z)\,P_2(z^2)$ instead of $P_2(z)$. Analogously, $P(z) = P_1(z)\,P_2(z^{2^k})$ can be reduced to the original case replacing $P_2(z)$ by $P_2(z)\,P_2(z^2)\ldots P_2(z^{2^{k-1}})$.

Example 3.2. (to Theorem 3.5) Let $P(z) = \frac{1}{2}(z^2 + \sqrt{2}z + 1)\,(z^4 - \sqrt{2}z^2 + 1)$. A corresponding solution φ of (1.1) is

$$\varphi(t) = \begin{cases} 1 & t \in [0,1) \cup [5,6); \\ 1 - \sqrt{2} & t \in [1,2) \cup [4,5); \\ 2 - \sqrt{2} & t \in [2,4); \\ 0 & \text{otherwise.} \end{cases}$$

Considering the simplified mask

$$\tilde{P}(z) = \frac{1}{2}(z^2 + \sqrt{2}z + 1)\,(z^2 - \sqrt{2}z + 1) = \frac{1}{2}(z^4 + 1)$$

providing the solution $\tilde{\varphi} = \chi_{[0,4)}$ (where χ denotes the characteristic function), it follows that

$$\varphi(t) = \tilde{\varphi}(t) - \sqrt{2}\tilde{\varphi}(t-1) + \tilde{\varphi}(t-2).$$

Analogously as in the foregoing theorem, we find:

Theorem 3.6. *Let (1.1) possess an E–solution $\varphi(t)$ and assume that the corresponding refinement mask $P(z)$ factorizes*

$$P(z) = (z^l + 1)\,Q(z) \quad (l \in \mathbb{N}).$$

Then (1.1) with the refinement mask $(z^{ml} + 1)\,Q(z)$ $(m \in \mathbb{N})$ also provides an E–solution $\Phi(t)$, namely

$$\Phi(t) = \sum_{\nu=0}^{m-1} \varphi(t - l\nu). \tag{3.7}$$

The case $m = 2^k$, $l = 1$ can be treated with both Theorem 3.5 and Theorem 3.6 in view of

$$(1+z)(1+z^2)\ldots(1+z^{2^{k-1}}) = \sum_{\nu=0}^{2^k-1} z^\nu.$$

4. EIGENVECTORS OF THE COEFFICIENT MATRIX

In the considerations above, the structure of the spaces L and $\mathcal{L}$ has been used for deriving assertions on zeros of the refinement mask corresponding to E–solutions. As seen in Section 2, eigenvectors and root vectors of A_0, A_1 and $\boldsymbol{A}$ can be special elements of L and $\mathcal{L}$, respectively. Now, we want to investigate conversely, if given zeros of the refinement mask imply consequences for the eigenvalues and eigenvectors of the matrices A_0, A_1 and $\boldsymbol{A}$.

First, we show that there are no root vectors of A_0 and A_1 corresponding to the eigenvalue 1 in case of continuous E-solutions.

THEOREM 4.1. *Let φ be a continuous E–solution of* (1.1). *Then the Jordan block in the Jordan decomposition of A_0 (or A_1) belonging to the eigenvalue* 1 *is the identity matrix, i.e., there are no root vectors of A_0 (or A_1) belonging to the eigenvalue* 1. *Moreover, if w is a left eigenvector of A_0 (or A_1) to the eigenvalue* 1, *then all coefficients d_k in its representation* (2.5) *are constants.*

Proof. We only show the assertion for A_0.
1. Observe that by (1.7), A_0 possesses the eigenvalue 1. Let us assume that there are vectors $w := (w_j)_{j=0}^{n-1}$ and $\tilde{w} := (\tilde{w})_{j=0}^{n-1}$ with

$$w^T A_0 = w^T, \qquad \tilde{w}^T A_0 = \tilde{w}^T + w^T, \tag{4.1}$$

so that w is an eigenvector and $\tilde{w}$ is a root vector of height 2 to the eigenvalue 1. We show that this assumption leads to a contradiction.

By Theorem 2.2 (iv), w and $\tilde{w}$ are contained in L. Further, the both vectors can be extended to $\boldsymbol{w} = (w_j)_{j\geq 0} \in \mathcal{L}$, $\tilde{\boldsymbol{w}} = (\tilde{w}_j)_{j\geq 0} \in \mathcal{L}$ by Theorem 2.3 (i) and Corollary 2.1 (iv). As we have shown in Theorem 2.4, $\boldsymbol{w}$ and $\tilde{\boldsymbol{w}}$ can be represented in the form

$$w_j = \sum_{k=1}^{l} d_k(j)\, \zeta_k^j, \qquad \tilde{w}_j = \sum_{k=1}^{l} \tilde{d}_k(j)\, \zeta_k^j, \tag{4.2}$$

and also all $(j^\nu\, \zeta_k^j)_{j\geq 0}$, appearing in (4.2) are contained in $\mathcal{L}$. We can suppose that, for each k, $d_k(j)$ and $\tilde{d}_k(j)$ do not both vanish identically.
2. First, let us assume that all d_k, $\tilde{d}_k$ in (4.2) are independent of j. According to (3.2), we have

$$(\zeta_k^j)_{j\geq 0}^T\, \boldsymbol{A} = \gamma_k\, \left((\sqrt{\zeta_k})^j\right)_{j\geq 0}^T + \delta_k\, \left((-\sqrt{\zeta_k})^j\right)_{j\geq 0}^T \tag{4.3}$$

with

$$\gamma_k := P(\sqrt{\zeta_k}) = \frac{1}{2}\sum_{j=0}^{\infty} c_j\, (\sqrt{\zeta_k})^j, \quad \delta_k := P(-\sqrt{\zeta_k}) = \frac{1}{2}\sum_{j=0}^{\infty} c_j\, (-\sqrt{\zeta_k})^j. \tag{4.4}$$

Now, (4.1)–(4.3) yield for $j \in \mathbb{N}_0$

$$\left(\sum_{k=1}^{l} d_k \left(\gamma_k \left(\sqrt{\zeta_k}\right)^j + \delta_k \left(-\sqrt{\zeta_k}\right)^j\right)\right) = \sum_{k=1}^{l} d_k \zeta_k^j \tag{4.5}$$

and

$$\left(\sum_{k=1}^{l} \tilde{d}_k \left(\gamma_k \left(\sqrt{\zeta_k}\right)^j + \delta_k \left(-\sqrt{\zeta_k}\right)^j\right)\right) = \sum_{k=1}^{l} (d_k + \tilde{d}_k) \zeta_k^j. \tag{4.6}$$

Since $((\sqrt{\zeta_k})^j)_{j\geq 0}$ and $((-\sqrt{\zeta_k})^j)_{j\geq 0}$ $(k = 1, \ldots, l)$ on the left–hand side and $(\zeta_k^j)_{j\geq 0}$ $(k = 1, \ldots, l)$ on the right–hand side are linearly independent vectors, it follows that some coefficients of the set $\{\gamma_k, \delta_k : k = 1, \ldots, l\}$ must vanish. On the other side, for each $k \in \{1, \ldots, l\}$, there exist numbers $p_k \in \{0, 1\}$, $q_k \in \{1, \ldots, l\}$ such that

$$\zeta_k = (-1)^{p_k} \sqrt{\zeta_{q_k}}, \tag{4.7}$$

where the corresponding coefficients

$$\epsilon_k := \begin{cases} \gamma_{q_k} & \text{for } p_k = 0, \\ \delta_{q_k} & \text{for } p_k = 1 \end{cases} \tag{4.8}$$

do not vanish. Comparing the coefficients in (4.5) and (4.6), we find $d_k = \epsilon_k d_{q_k}$ and $d_k + \tilde{d}_k = \epsilon_k \tilde{d}_{q_k}$. Iterating the first equation, we must arive at a cycle, say with m steps $(1 \leq m \leq l)$. Then, after changing the notation of the indices, we find (disregarding the preperiod)

$$d_1 = \epsilon_1 d_2, \quad d_2 = \epsilon_2 d_3, \quad \ldots \quad, d_m = \epsilon_m d_1,$$

which implies either $d_1 = \ldots = d_m = 0$ or $\epsilon_1 \epsilon_2 \ldots \epsilon_m = 1$. In the last case, from

$$d_1 + \tilde{d}_1 = \epsilon_1 \tilde{d}_2, \quad d_2 + \tilde{d}_2 = \epsilon_2 \tilde{d}_3, \quad \ldots \quad, d_m + \tilde{d}_m = \epsilon_m \tilde{d}_1$$

we obtain by elimination

$$\begin{aligned} &\tilde{d}_1 = \epsilon_1(\tilde{d}_2 - d_2), \quad \tilde{d}_2 - d_2 = \epsilon_2(\tilde{d}_3 - 2d_3), \quad \ldots, \\ &\tilde{d}_m - (m-1)d_m = \epsilon_m(\tilde{d}_1 - md_1), \end{aligned}$$

and so

$$\tilde{d}_1 = \epsilon_1 \epsilon_2 \ldots \epsilon_m (\tilde{d}_1 - md_1) = \tilde{d}_1 - md_1.$$

It follows as before that $d_1 = 0$ and hence, by $\epsilon_i \neq 0$, that $d_2 = d_3 = \ldots = d_m = 0$. If $m < l$, we can conclude in any case that also the other d_μ must vanish. But this contradicts the fact that the eigenvector w is nonvanishing.

3. Next, we consider the case that $d_k(j)$ and $\tilde{d}_k(j)$ are linear in j, i.e., $d_k(j) = a_k + jb_k$, $\tilde{d}_k(j) = \tilde{a}_k + j\tilde{b}_k$ with a_k, b_k, $\tilde{a}_k$, $\tilde{b}_k$ independent of j. Equation (4.3) implies by differentiation with respect to ζ_k (which, in this connection, can be considered as a variable) and multiplication with ζ_k,

$$(j\zeta_k^j)_{j\geq 0}^T \boldsymbol{A} = \left((\gamma_k'\,\zeta_k + \frac{j}{2}\gamma_k)(\sqrt{\zeta_k})^j + (\delta_k'\,\zeta_k + \frac{j}{2}\,\delta_k)(-\sqrt{\zeta_k})^j\right)_{j\geq 0}^T .$$

So (4.1)–(4.3) yield for $j \in \mathbb{N}_0$,

$$\begin{aligned} & \sum_{k=1}^{l} a_k \left(\gamma_k(\sqrt{\zeta_k})^j + \delta_k(-\sqrt{\zeta_k})^j\right) \\ + \quad & b_k \left((\gamma_k'\zeta_k + \frac{j}{2}\gamma_k)(\sqrt{\zeta_k})^j + (\delta_k'\zeta_k + \frac{j}{2}\delta_k)(-\sqrt{\zeta_k})^j\right) \\ = \quad & \sum_{k=1}^{l}(a_k + jb_k)\,\zeta_k^j, \end{aligned}$$

and an analogous equation with $\tilde{a}_k$, $\tilde{b}_k$ instead of a_k, b_k on the left-hand side and $\sum_{k=1}^{l}(a_k + jb_k + \tilde{a}_k + j\tilde{b}_k)$ on the right hand-side. With the same arguments as before, we can derive that ζ_k is of the form (4.7) and a comparison of the coefficients of $j\zeta_k^j$ yields

$$b_k = \frac{1}{2}\epsilon_k\, b_{q_k}, \quad b_k + \tilde{b}_k = \frac{1}{2}\epsilon_k\,\tilde{b}_{q_k}$$

with ϵ_k as in (4.8). Now, we can conclude as before (with $\frac{\epsilon_k}{2}$ instead of ϵ_k) that all b_k vanish. Hence, we have reduced the case of linear $d_k(j)$ to the case of constant d_k, which already was seen to be impossible.

For polynomials $d_k(j)$, $\tilde{d}_k(j)$ of higher degree the contradiction follows in the same manner. ■

With the help of the foregoing theorem we simply observe

Corollary 4.1. *If the matrix A_0 (or the matrix A_1) has nonsimple Jordan blocks corresponding to the eigenvalue* 1, *then* (1.1) *does not possess a continuous E–solution φ.*

This case really can happen as it is shown by

Example 4.1. We consider (1.1) with the coefficients $c_0 = -c_4 = a$, $c_1 = c_3 = 1$, $c_2 = 0$, with $a \in \mathbb{R}$, $|a| < 1$, $a \neq 0$. We simply observe that the assumptions (A1), (A2) are satisfied. However, (1.1) does not possess a continuous E–solution, since the corresponding matrix A_0 contains a nonsimple Jordan block corresponding to 1,

$$A_0 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ -a & 1 & 0 & 1 \\ 0 & 0 & -a & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{a}{1-a} & a & 1-a & -a \\ -1 & 0 & 1 & 1 \\ -\frac{a}{1-a} & -a & a & a \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{a}{1-a} & 0 & 1 & -\frac{1}{a} \\ 0 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Next, assuming that the refinement mask $P(z)$ possesses a factor $p_k(z)$ (as described in Theorem 3.3), we are interested in consequences for eigenvectors and root vectors of the coefficient matrix $\boldsymbol{A}$.

THEOREM 4.2. *Let* (1.1) *have an E–solution with* (A1), (A2). *Then there exists an integer $k \geq 0$ such that $(\boldsymbol{e}^T \boldsymbol{A}^k)^T$ is a left eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue* 1, *and for $k \geq 1$, the vectors $(\boldsymbol{e}^T \boldsymbol{A}^{\nu+1} - \boldsymbol{e}^T \boldsymbol{A}^{\nu})^T$ $(\nu = 0, \ldots, k-1)$ are left root vectors of height $k - \nu$ of $\boldsymbol{A}$ corresponding to the eigenvalue* 0. *Here again, $\boldsymbol{e} := (1, 1, \ldots)^T$.*

Proof. By Corollary 2.1, we have $(\boldsymbol{e}^T \boldsymbol{A} - \boldsymbol{e}^T)^T \in \mathcal{L}$. Let $P(z)$ possess the factor $p(z) = p_k(z)$ found after k iterations as described in Theorem 3.3. Recalling the proof of Theorem 3.3, we observe that, by iterative application of (4.3), $(\boldsymbol{e}^T \boldsymbol{A} - \boldsymbol{e}^T)\, \boldsymbol{A}^k = \boldsymbol{0}^T$, for $k \geq 0$, where $\boldsymbol{0} := (0, 0, \ldots)^T$, whereas $(\boldsymbol{e}^T \boldsymbol{A} - \boldsymbol{e}^T)\boldsymbol{A}^{k-1} \neq \boldsymbol{0}^T$ for $k \geq 1$. Hence, $(\boldsymbol{e}^T \boldsymbol{A}^k)^T$ is a left eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue 1, and by $(\boldsymbol{e}^T \boldsymbol{A}^{\nu+1} - \boldsymbol{e}^T \boldsymbol{A}^{\nu})\boldsymbol{A}^{k-\nu} = \boldsymbol{0}^T$, we obtain the assertion concerning the root vectors of $\boldsymbol{A}$ belonging to 0. ∎

REMARKS: 1. Obviously, we have the identity

$$\boldsymbol{e}^T \boldsymbol{A}^k = \boldsymbol{e}^T + \sum_{\nu=0}^{k-1} (\boldsymbol{e}^T \boldsymbol{A}^{\nu+1} - \boldsymbol{e}^T \boldsymbol{A}^{\nu})$$

between the eigenvector $\boldsymbol{e}^T \boldsymbol{A}^k$ of $\boldsymbol{A}$ corresponding to 1, $\boldsymbol{e}$, and the root vectors of $\boldsymbol{A}$ corresponding to 0. It can easily be seen (by repeated application of (4.3)) that $\boldsymbol{e}^T \boldsymbol{A}^{\nu}$ is a 2^{ν}–periodic vector. Writing $\boldsymbol{e}^{(\nu)\,T} = \boldsymbol{e}^T \boldsymbol{A}^{\nu+1} - \boldsymbol{e}^T \boldsymbol{A}^{\nu}$ and $\boldsymbol{e}^{(\nu)} = (e_j^{(\nu)})_{j \geq 0}$ for $\nu = 0, \ldots, k-1$, it follows that $(e_j^{(\nu)})_{j=0}^{n-1}, \ldots, (e_{j+2^{\nu}-1}^{(\nu)})_{j=0}^{n-1}$ also belong to L. Moreover, they must be root vectors of height $k - \nu$ of A_0 corresponding to 0.

2. In the case $p(z) = z^{2^k} + 1$, we observe that

$$\boldsymbol{e}^{(\nu)} = (2^{\nu}, \underbrace{0, \ldots, 0}_{2^{\nu}-1}, -2^{\nu}, \underbrace{0, \ldots, 0}_{2^{\nu}-1}, 2^{\nu}, \underbrace{0, \ldots, 0}_{2^{\nu}-1}, -2^{\nu}, 0 \ldots)^T.$$

In particular, the $e^{(\nu)}$ $(\nu = 0, \ldots, k-1)$ are linearly independent in this case.

Now, let $\boldsymbol{w} \in \mathcal{L}$, i.e., the elements of $\boldsymbol{w}$ possess the representation (2.5). Then we have:

PROPOSITION 4.1. *If $\boldsymbol{w} \in \mathcal{L}$ with the representation* (2.5) *is an eigenvector (or root vector) of $\boldsymbol{A}$ corresponding to an eigenvalue $\lambda \neq 0$. Then the ζ_k occuring in the representation* (2.5)*of $\boldsymbol{w}$ are roots of unity and the set $\{\zeta_k : k = 1, \ldots, l\}$ is closed regarding to the operation $z \to z^2$.*

Proof. Let $\boldsymbol{w}^T \boldsymbol{A} = \lambda \boldsymbol{w}^T$ with $\lambda \neq 0$. Assume first that the d_k in (2.5) are independent of j. Then we obtain, analogously as in the proof of Theorem 4.1,

$$\sum_{k=1}^{k} d_k (\gamma_k (\sqrt{\zeta})^j + \delta_k (-\sqrt{\zeta})^j) = \lambda \sum_{k=1}^{l} d_k \zeta_k^j \qquad (j \geq 0)$$

with γ_k and δ_k as in (4.4). Hence, with the same argument as in the proof of Theorem 4.1, the assertion follows. Similar ideas apply for polynomials $d_k(j)$ and for root vectors. ■

REMARKS: 1. In the case $\lambda = 0$, not only roots of unity but arbitrary ζ_k can appear.
2. Assertions on eigenvectors and root vectors of $\boldsymbol{A}$ to the eigenvalue 0 can be concluded by proving the converse of Theorem 3.1 and Theorem 3.2. If both, $\sqrt{\zeta}$ and $-\sqrt{\zeta}$, are zeros of the refinement mask $P(z)$, then $(\zeta^j)_{j \geq 0}$ is a left eigenvector of $\boldsymbol{A}$ corresponding to 0, since $\sum_{j=0}^{\infty} c_j (\sqrt{\zeta})^j = 0$ and $\sum_{j=0}^{\infty} c_j (-\sqrt{\zeta})^j = 0$ imply that by (4.3),

$$(\zeta^j)_{j \geq 0}^T \boldsymbol{A} = P(\sqrt{\zeta})\,((\sqrt{\zeta})^j)_{j \geq 0}^T + P(-\sqrt{\zeta})\,((-\sqrt{\zeta})^j)_{j \geq 0}^T = \mathbf{0}^T.$$

Analogously, if $\pm\sqrt{\zeta} \in N^{(\nu)}$ (with $N^{(\nu)}$ defined in (3.3)), then $(j^\mu \zeta^j)_{j \geq 0}$ $(\mu = 0, \ldots, \nu - 1)$ are eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvector 0.

REFERENCES

1 A. Cavaretta, W. Dahmen and C. A. Micchelli, Stationary Subdivision, *Mem. Amer. Math. Soc.* 453: 1–186 (1991).

2 D. Colella and C. Heil, Characterizations of scaling functions, I. Continuous solutions, *J. Math. Anal. Appl.* 15: 496–518 (1994).

3 W. Dahmen and C. A. Micchelli, On the solution of certain systems of partial difference equations and linear dependence of translates of box splines, *Trans. Amer. Math. Soc.* 292: 305–320 (1985).

4 I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

5 I. Daubechies and J. Lagarias, Two–scale difference equations: I. Global regularity of solutions, *SIAM J. Math. Anal.* 22: 1388–1410 (1991).

6 I. Daubechies and J. Lagarias, Two–scale difference equations: II. Local regularity, infinite products of matrices and fractals, *SIAM J. Math. Anal.* 23: 1031–1079 (1992).

7 G. Deslauriers, J. Dubois and S. Dubuc, Multidimensional iterative interpolation, *Canad. J. Math.* 43: 297–312 (1991).

8 N. Dyn, J. A. Gregory and D. Levin, Analysis of uniform binary subdivision schemes for curve design, *Constr. Approx.* 7: 127–147 (1991).

9 G. M. Fichtenholz, *Differential- und Integralrechnung II*, Verlag der Wissenschaften, Berlin, 1966.

10 G. Griepenberg, On subdivision interpolation schemes, *SIAM J. Math. Anal.* 27: 578–593 (1996).

11 C. Heil, Methods for solving dilation equations, in *Probabilistic and Stochastic Methods in Analysis, with Applications* (J. S. Byrnes et al., Eds.)., Kluwer Academic Publ., pp. 15–45 (1992).

12 R. Q. Jia, Subdivision schemes in L_p spaces, *Advances in Comp. Math.* 3: 309–341 (1995).

13 R. Q. Jia, Refinable shift–invariant spaces: From splines to wavelets, in *Approximation Theory VIII* (C.K. Chui and L.L. Schumaker, Eds.)., World Scientific Publishing Co., pp. 1–30 (1995).

14 R. Q. Jia and J. Z. Wang, Stability and linear independence associated with wavelet decompositions, *Proc. Amer. Math. Soc.* 117: 1115–1124 (1993).

15 M. Kuczma, B. Choczewski and G. Ger, *Iterative Functional Equations*, Cambridge Univ. Press, Cambridge, 1990.

16 H.-J. Klemmt, Ein maßtheoretischer Satz mit Anwendungen auf Funktionalgleichungen, *Archiv Math.* 36: 537–540 (1981).

17 K. S. Lau and J. Z. Wang, Characterization of L^p-solutions for the two–scale dilation equations, *SIAM J. Math. Anal.* 26: 1018–1046 (1995).

18 W. Lawton, S. L. Lee and Z. Shen, Characterization of compactly supported refinable splines, *Adv. Comput. Math.* 3: 137–146 (1995).

19 C. Micchelli and H. Prautzsch, Uniform refinement of curves, *Linear Algebra Appl.* 114/115: 841–870 (1989).

20 L. Villemoes, Wavelet analysis of refinement equations, *SIAM J. Math. Anal.* 25: 1433–1460 (1994).