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# Periodic Interpolation and Wavelets on Sparse Grids

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*Dedicated to Professor Gerhard Maeß on the occasion of his 60th birthday*

Nested spaces of multivariate periodic functions forming a non-stationary multiresolution analysis are investigated. The scaling functions of these spaces are fundamental polynomials of Lagrange interpolation on a sparse grid. The approach based on Boolean sums leads to sample and wavelet spaces of significantly lower dimension and good approximation order. The algorithms for complete decomposition and reconstruction are of simple structure and low complexity.

**Keywords:** Wavelets, Multivariate periodic interpolation, Boolean sums, Sparse grids

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## 1. Introduction

Periodic processes in numerical analysis and signal processing are nowadays often examined in terms of an analysis of periodic wavelets. We are interested here in decomposing multivariate periodic functions into wavelet parts.

Therefore, we consider special bivariate sample and wavelet spaces of periodic functions. Starting with given univariate periodic interpolatory scaling functions and wavelets, we obtain new bivariate periodic interpolatory scaling functions and wavelets by investigating Boolean sums of interpolation operators. In this way, we overcome the problem of a very rapid growth of the dimensions which we have in the ordinary tensor product case. This means that we obtain sample spaces with a dimension of $\mathcal{O}(j\,2^j)$ while the tensor products yield a dimension of $\mathcal{O}(2^{2j})$ for the level $j$. The concept of Boolean sums and sparse grids is often applied in interpolation and approximation (see e.g. [7,20]) as well as in finite element methods (see e.g. [24]). A related field is the hyberbolic approximation (see e.g. [8,9,23]). We will use essentially the interpolatory properties of the underlying periodic univariate scaling functions and wavelets. The general multivariate case can be handled analogously.

Applying the Boolean sum method, we are able to construct multivariate scaling functions and wavelets from univariate periodic interpolatory scaling functions and wavelets. We obtain the bivariate sample spaces

$$V_j^{\mathrm{B}} = \sum_{r=0}^{j} V_r \otimes V_{j-r},$$

and the bivariate wavelet spaces

$$W_{j+1}^{\mathrm{B}} = \bigoplus_{r=0}^{j+1}{}^{\perp}\, W_r \otimes W_{j+1-r},$$

both introduced in Section 4, from the underlying univariate sample spaces $V_r$ and wavelet spaces $W_r$, respectively. The univariate sample spaces $V_r$ are images of interpolation operators $L_r$ on equidistant grids. Analogously, the bivariate sample spaces $V_j^{\mathrm{B}}$ are images of $j$-th order blending interpolation operators $B_j$ defined by Boolean sums in Section 4. The obtained spaces can be used to define stable subspace splittings which are needed for the construction of fast elliptic solvers (see e.g. [10,11]). Similar hyperbolic sums of wavelet spaces have been considered in [8] for the cardinal case. There the authors were interested in the characterization of the best approximation in such wavelet spaces but not in characterizing suited scaling functions or in the reconstruction and decomposition relations between scaling functions and wavelets.

In this paper, we present the algorithms for complete decomposition and reconstruction which are based on the algorithms for the univariate case. These procedures have a simple structure and a low complexity of $\mathcal{O}(j^2\, 2^j)$ essential operations for the level $j$.

There are many possibilities for the particular choice of univariate periodic wavelets. One can periodize a wavelet on the real line (see e.g. [3,5]) or build scaling functions and wavelets directly by using periodic functions like trigonometric polynomials (see e.g. [4,16,17]). More general and unified introductions to periodic univariate wavelets based on calculations with Fourier coefficients can be found in [12–14].

For our purposes, we use a univariate non-stationary multiresolution analysis with interpolatory scaling functions and interpolatory wavelets. The de la Vallée Poussin means of the Fourier sum as in [17] and certain interpolatory splines may serve as examples of univariate interpolatory scaling functions and wavelets.

## 2. Univariate Interpolation and Periodic Sample and Wavelet Spaces

Let $d \in \mathbb{N}$ be fixed. Put $d_j := 2^j d$, $j \in \mathbb{N}_0$. We consider continuous $2\pi$-periodic scaling functions $\phi_{j,0} \in L_2(\mathbb{T})$ defined as fundamental interpolants on the equidistant interpolation grid

$$\mathcal{T}_j := \left\{ t_{j,k} := k\,\tfrac{2\pi}{d_j} \;:\; k = 0, \ldots, d_j - 1 \right\},$$

i.e.,

$$\phi_{j,0}(t_{j,k}) = \delta_{k,0}, \quad k = 0, 1, \ldots, d_j - 1. \tag{1}$$

Let us denote the translates of the function $\phi_{j,0}$ by $\phi_{j,k}(x) := \phi_{j,0}(x - t_{j,k})$, $k = 0, 1, \ldots, d_j - 1$. Then we define corresponding sample spaces by

$$V_j := \mathrm{span}\ \{\phi_{j,k} \;:\; k = 0, 1, \ldots, d_j - 1\}. \tag{2}$$

From the interpolatory conditions on the functions $\phi_{j,0}$, we immediately obtain $\dim V_j = d_j$. Furthermore, we assume the spaces $V_j$ to form a chain of subspaces of the Hilbert space $L_2(\mathbb{T})$ with

$$V_j \subset V_{j+1},\ j \in \mathbb{N}_0 \quad \text{and} \quad \mathrm{clos}_{L_2(\mathbb{T})} \bigcup_{j \in \mathbb{N}_0} V_j = L_2(\mathbb{T}).$$

The interpolation operator $L_j$ projecting onto $V_j$ is given as

$$L_j f := \sum_{k=0}^{d_j - 1} f(t_{j,k})\, \phi_{j,k}\,.$$

Now let us consider the univariate wavelet spaces $W_{j+1}$ defined by

$$V_{j+1} = V_j \oplus^{\perp} W_{j+1},$$

where $\oplus^{\perp}$ denotes the orthogonal sum with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathbb{T}}$ of $L_2(\mathbb{T})$. Then assume the wavelet spaces $W_{j+1}$ to be generated by the translates of a single function $\psi_{j+1,1}$

$$W_{j+1} := \mathrm{span}\ \{\psi_{j+1,2k+1} \;:\; k = 0, 1, \ldots, d_j - 1\} \tag{3}$$

with $\psi_{j+1,2k+1}(x) := \psi_{j+1,1}(x - t_{j+1,2k})$, $k = 0, 1, \ldots, d_j - 1$, satisfying also interpolatory conditions

$$\psi_{j+1,2m+1}\left(t_{j+1,2\ell+1}\right) = \delta_{\ell,m}, \qquad \ell = 0, 1, \ldots, d_j - 1. \tag{4}$$

Next, we want to establish reconstruction and decomposition relations in matrix form. Therefore, we define the function vectors $\boldsymbol{\phi}_{j-1} := (\phi_{j-1,k})_{k=0}^{d_{j-1}-1}$, $\boldsymbol{\psi}_j := (\psi_{j,2k+1})_{k=0}^{d_{j-1}-1}$, and $\boldsymbol{P}_j\, \boldsymbol{\phi}_j := (\phi_{j,0}, \phi_{j,2}, \ldots, \phi_{j,d_j-2}, \phi_{j,1}, \phi_{j,3}, \ldots, \phi_{j,d_j-1})^{\mathrm{T}}$ and the following circulant matrices. We need the three inner product matrices

$$\boldsymbol{G}_{j-1} := \left(\langle \phi_{j-1,k},\, \phi_{j-1,\ell} \rangle_{\mathbb{T}}\right)_{k,\ell=0}^{d_{j-1}-1}, \qquad \boldsymbol{G}^{e}_{j,j-1} := \left(\langle \phi_{j,2k},\, \phi_{j-1,\ell} \rangle_{\mathbb{T}}\right)_{k,\ell=0}^{d_{j-1}-1},$$

$$\boldsymbol{G}^{o}_{j,j-1} := \left(\langle \phi_{j,2k+1},\, \phi_{j-1,\ell} \rangle_{\mathbb{T}}\right)_{k,\ell=0}^{d_{j-1}-1},$$

the two knot evaluation matrices

$$\boldsymbol{K}_{j-1} := \left(\phi_{j-1,k}(t_{j,2\ell+1})\right)_{k,\ell=0}^{d_{j-1}-1}, \qquad \boldsymbol{L}_j := \left(\psi_{j,2k+1}(t_{j,2\ell})\right)_{k,\ell=0}^{d_{j-1}-1},$$

and the Fourier matrix of order $d_j$

$$\boldsymbol{F}_j := \left(\omega_j^{k\ell}\right)_{k,\ell=0}^{d_j-1},$$

where $\omega_j := e^{-2\pi i/d_j}$ is a primitive $d_j$-th root of unity. The Kronecker product of two matrices $\boldsymbol{A} \in \mathbb{C}^{m\times n}$, $\boldsymbol{B} \in \mathbb{C}^{r\times s}$ is defined as

$$\boldsymbol{A} \otimes \boldsymbol{B} := \left(a_{k,\ell}\boldsymbol{B}\right)_{k=0,\ell=0}^{m-1,n-1}.$$

By construction, it holds that

$$\boldsymbol{L}_j\, \boldsymbol{G}^e_{j,j-1} = -\boldsymbol{G}^o_{j,j-1}. \tag{5}$$

**Theorem 1.** (i) The two-scale or reconstruction relation is given by

$$\begin{pmatrix} \boldsymbol{\phi}_{j-1} \\ \boldsymbol{\psi}_j \end{pmatrix} = \boldsymbol{C}_j\, \boldsymbol{P}_j\, \boldsymbol{\phi}_j$$

with

$$\begin{aligned} \boldsymbol{C}_j &= \begin{pmatrix} \boldsymbol{I}_{d_{j-1}} & \boldsymbol{K}_{j-1} \\ \boldsymbol{L}_j & \boldsymbol{I}_{d_{j-1}} \end{pmatrix} \\ &= \frac{1}{d_{j-1}} (\boldsymbol{I}_2 \otimes \overline{\boldsymbol{F}}_{j-1})\, \boldsymbol{C}_{j,\lambda}\, (\boldsymbol{I}_2 \otimes \boldsymbol{F}_{j-1}), \end{aligned}$$

where $\boldsymbol{C}_{j,\lambda}$ consists of $2\times 2$ blocks, each of which is a diagonal matrix of dimension $d_{j-1}$.

(ii) The decomposition relation can be written as

$$\boldsymbol{P}_j\, \boldsymbol{\phi}_j = \boldsymbol{D}_j \begin{pmatrix} \boldsymbol{\phi}_{j-1} \\ \boldsymbol{\psi}_j \end{pmatrix}$$

with

$$\begin{aligned} \boldsymbol{D}_j = \boldsymbol{C}_j^{-1} &= \begin{pmatrix} \boldsymbol{A}_j & -\boldsymbol{A}_j \boldsymbol{K}_{j-1} \\ -\boldsymbol{A}_j \boldsymbol{L}_j & \boldsymbol{A}_j \end{pmatrix} \\ &= \frac{1}{d_{j-1}} (\boldsymbol{I}_2 \otimes \overline{\boldsymbol{F}}_{j-1})\, \boldsymbol{D}_{j,\lambda}\, (\boldsymbol{I}_2 \otimes \boldsymbol{F}_{j-1}), \end{aligned}$$

where $\boldsymbol{A}_j = \boldsymbol{G}_{j-1}^{-1}\, \boldsymbol{G}^e_{j,j-1}$ and $\boldsymbol{D}_{j,\lambda}$ consists of $2\times 2$ blocks, each of which is a diagonal matrix of dimension $d_{j-1}$.

*Proof.* (i) We obtain the form of the reconstruction matrix $\boldsymbol{C}_j$ immediately from the interpolatory properties (1) and (4) of the scaling functions and the wavelets.

(ii) Using the dual scaling functions and wavelets and again (1) and (4), we get the decomposition matrix $\boldsymbol{D}_j$. For more details, we refer to [16] where this assertion is proved for a special case.

The matrices $\boldsymbol{C}_j$ and $\boldsymbol{D}_j$, respectively, are matrices with $2 \times 2$ circulant blocks. These circulant blocks can be diagonalized by the Fourier matrices (see e.g. [6]). That gives the factorized forms of the matrices $\boldsymbol{C}_j$ and $\boldsymbol{D}_j$. $\square$

An interesting special case of such periodic interpolating scaling functions and wavelets are the trigonometric ones described in [16–18]. We briefly recall the definitions of these trigonometric scaling functions. For $n, m \in \mathbb{N}, n > m$, the de la Vallée Poussin means $\varphi_n^m$ are given as

$$\varphi_n^m(x) := \frac{1}{4mn} \sum_{\ell=n-m}^{n+m-1} \sum_{k=-\ell}^{\ell} e^{ikx} .$$

For $j \in \mathbb{N}_0$, the univariate trigonometric scaling function $\phi_{j,0}$ can be defined as a special de la Vallée Poussin mean

$$\phi_{j,0}^t := \begin{cases} 1 & \text{for } j = 0, d = 1, \\ \varphi_{n_j}^{m_j} & \text{otherwise,} \end{cases}$$

where

$$n_j := d\, 2^{j-1} \quad \text{and} \quad m_j := \begin{cases} 2^{j-1-\mu} & \text{for } j > \mu, \\ 1 & \text{for } j \le \mu, \end{cases} \quad \mu \in \mathbb{N}, \mu \ge 2.$$

For $d \in 2\mathbb{N} \cup \{1\}$ and with the definition of the univariate trigonometric wavelet function

$$\psi_{j+1,1}^t(x) := 2\phi_{j+1,1}^t(x) - \phi_{j,0}^t\left(x - t_{j+1,1}\right) ,$$

one obtains an example for the univariate periodic sample and wavelet spaces as defined in (2) and (3). The diagonalized matrices for reconstruction and decomposition are

$$\boldsymbol{C}_{j,\lambda}^t = \begin{pmatrix} \boldsymbol{I}_{d_{j-1}} & \operatorname{diag}\left(\kappa_{j-1,r}\right)_{r=0}^{d_{j-1}-1} \\ \operatorname{diag}\left(-\overline{\kappa}_{j-1,r}\right)_{r=0}^{d_{j-1}-1} & \boldsymbol{I}_{d_{j-1}} \end{pmatrix},$$

$$\boldsymbol{D}_{j,\lambda}^t = \begin{pmatrix} \operatorname{diag}\left(\frac{1}{\gamma_{j-1,r}}\right)_{r=0}^{d_{j-1}-1} & \operatorname{diag}\left(-\frac{\kappa_{j-1,r}}{\gamma_{j-1,r}}\right)_{r=0}^{d_{j-1}-1} \\ \operatorname{diag}\left(\frac{\overline{\kappa}_{j-1,r}}{\gamma_{j-1,r}}\right)_{r=0}^{d_{j-1}-1} & \operatorname{diag}\left(\frac{1}{\gamma_{j-1,r}}\right)_{r=0}^{d_{j-1}-1} \end{pmatrix},$$

where

$$\kappa_{j,r} = \omega_{j+1}^r \begin{cases} 1 & \text{if } 0 \le r \le n_j - m_j, \\ \frac{n_j - r}{m_j} & \text{if } n_j - m_j \le r \le n_j + m_j, \\ -1 & \text{if } n_j + m_j \le r \le 2n_j - 1, \end{cases}$$

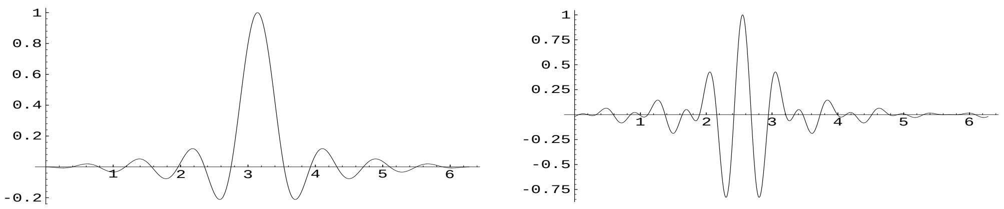


Figure 1. The trigonometric scaling function $\phi^t_{4,8}$ and wavelet $\psi^t_{5,13}$ for $d = 1, \mu = \infty$.

$$\gamma_{j,r} = \frac{1}{m_j^2} \begin{cases} (m_j^2 + (n_j - r)^2) & \text{for } n_j - m_j < r < n_j + m_j, \\ 2m_j^2 & \text{otherwise.} \end{cases}$$

The periodized linear splines (compare [3,14]) and spline wavelets may serve as another example in this field. For $d = 2$, $j \in \mathbb{N}_0$, the periodic hat function $\phi^s_{j,0}$ is defined

$$\phi^s_{j,0}(x) := \begin{cases} 1 - \frac{d_j}{2\pi}x & \text{for } x \in [0, t_{j,1}], \\ 0 & \text{for } x \in [t_{j,1}, t_{j,d_j-1}], \\ \frac{d_j}{2\pi}(x - t_{j,d_j-1}) & \text{for } x \in [t_{j,d_j-1}, 2\pi] \end{cases}$$

in $[0, 2\pi]$ and extended $2\pi$-periodically. The univariate periodic interpolatory linear spline wavelet is given by

$$\psi^s_{j+1,1} := \phi^s_{j+1,1} - \frac{6}{d_j} \sum_{k=0}^{d_j-1} \beta_k \phi^s_{j+1,2k},$$

where the coefficients

$$\beta_k := \sum_{\ell=0}^{d_j-1} \frac{\omega_j^{-k\ell} + \omega_j^{-(k-1)\ell}}{\omega_j^{k\ell} + 10 + \omega_j^{-k\ell}}$$

are computed according to Theorem 1 and relation (5). Here, the diagonalized matrices for the changes of basis are given by

$$\boldsymbol{C}^s_{j,\lambda} = \begin{pmatrix} \boldsymbol{I}_{d_{j-1}} & \frac{1}{2}\,\mathrm{diag}\left(1 + \omega^r_{j-1}\right)_{r=0}^{d_{j-1}-1} \\ -6\,\mathrm{diag}\left(\frac{1+\omega^r_{j-1}}{\omega^r_{j-1}+10+\omega^{-r}_{j-1}}\right)_{r=0}^{d_{j-1}-1} & \boldsymbol{I}_{d_{j-1}} \end{pmatrix},$$

$$\boldsymbol{D}_{j,\lambda}^{s} = \begin{pmatrix} \frac{1}{2}\operatorname{diag}\left(\frac{\omega_{j-1}^{r}+10+\omega_{j-1}^{-r}}{\omega_{j-1}^{r}+4+\omega_{j-1}^{-r}}\right)_{r=0}^{d_{j-1}-1} & -\frac{1}{4}\operatorname{diag}\left(\frac{(1+\omega_{j-1}^{r})(\omega_{j-1}^{r}+10+\omega_{j-1}^{-r})}{\omega_{j-1}^{r}+4+\omega_{j-1}^{-r}}\right)_{r=0}^{d_{j-1}-1} \\ 3\operatorname{diag}\left(\frac{1+\omega_{j-1}^{r}}{\omega_{j-1}^{r}+4+\omega_{j-1}^{-r}}\right)_{r=0}^{d_{j-1}-1} & \frac{1}{2}\operatorname{diag}\left(\frac{\omega_{j-1}^{r}+10+\omega_{j-1}^{-r}}{\omega_{j-1}^{r}+4+\omega_{j-1}^{-r}}\right)_{r=0}^{d_{j-1}-1} \end{pmatrix}.$$

## 3. Univariate Decomposition and Reconstruction

In this section, we want to give the relations for complete decomposition and reconstruction of univariate periodic functions. Assume, we have a univariate continuous (or Riemann integrable) $2\pi$-periodic function $f$. We obtain the projection $f_j$ of this function into the sample space $V_j$ simply by interpolation

$$f_j := L_j f = \sum_{r=0}^{d_j-1} f(t_{j,r})\,\phi_{j,r}\,.$$

This means, $f_j = \boldsymbol{a}_j^{\mathrm{T}}\,\boldsymbol{\phi}_j$ with the coefficient vector $\boldsymbol{a}_j$ containing the sampled function values $a_{j,r} = f(t_{j,r})$, $r = 0,\dots,d_j-1$, and the function vector $\boldsymbol{\phi}_j$ as defined in Section 2. On the other hand, we can write $f_j$ as the sum of its wavelet parts

$$f_j = f_0 + \sum_{k=1}^{j} g_k = \sum_{r=0}^{d_0-1} a_{0,r}\phi_{0,r} + \sum_{k=1}^{j}\sum_{r=0}^{d_{k-1}-1} c_{k,r}\psi_{k,2r+1}$$

where $f_0 \in V_0$ and $g_k \in W_k$, respectively. Using vector notation, we have $f_j = \boldsymbol{a}_0^{\mathrm{T}}\,\boldsymbol{\phi}_0 + \sum_{k=1}^{j}\boldsymbol{c}_k^{\mathrm{T}}\,\boldsymbol{\psi}_k$, with the function vectors $\boldsymbol{\psi}_k$ as defined in Section 2. We collect the coefficient vectors and the function vectors in two vectors

$$\boldsymbol{b}_j^{\mathrm{T}} := \left(\boldsymbol{a}_0^{\mathrm{T}}, \boldsymbol{c}_1^{\mathrm{T}}, \dots, \boldsymbol{c}_j^{\mathrm{T}}\right), \qquad (\boldsymbol{\psi}_j^{a})^{\mathrm{T}} := \left(\boldsymbol{\phi}_0^{\mathrm{T}}, \boldsymbol{\psi}_1^{\mathrm{T}}, \dots, \boldsymbol{\psi}_j^{\mathrm{T}}\right),$$

both having length $d_j$. The function $f_j$ can be written as

$$f_j = \boldsymbol{a}_j^{\mathrm{T}}\,\boldsymbol{\phi}_j = \boldsymbol{b}_j^{\mathrm{T}}\,\boldsymbol{\psi}_j^{a}.$$

This means that the functions $f_j \in V_j$ can be represented in the highest order scaling function basis as well as in the basis of the wavelets contained in this sample space.

We want to establish relations between these bases of a univariate sample space $V_j$. The complete decomposition of the scaling functions into wavelets parts describes one possible change of basis. There have to exist coefficients $z_{k,\ell}^{j}$ such that

$$\phi_{j,k} = \sum_{\nu=0}^{d_0-1} z_{k,2^j\nu}^{j}\,\phi_{0,\nu} + \sum_{\mu=0}^{j-1}\sum_{\nu=0}^{d_\mu-1} z_{k,2^{j-\mu-1}(2\nu+1)}^{j}\,\psi_{\mu+1,2\nu+1}$$

where the second index $\ell$ of the entries of $\boldsymbol{Z}_j := (z^j_{k,\ell})_{k,\ell=0}^{d_j-1}$ corresponds to that point $t_{j,\ell}$ in which the function $\phi_{0,\nu}$ or $\psi_{\mu+1,2\nu+1}$, respectively, satisfies the interpolatory condition (1) or (4) with the function value 1. On the other hand, there have to be coefficients $r^j_{k,\ell}$ such that the complete reconstruction of the wavelets into terms of scaling functions of a higher level in the univariate case reads as

$$\begin{aligned} \phi_{0,\nu} &= \sum_{\ell=0}^{d_j-1} r^j_{2^j\nu,\ell}\,\phi_{j,\ell} = \sum_{\ell=0}^{d_j-1} \phi_{0,\nu}(t_{j,\ell})\,\phi_{j,\ell}, \\ \psi_{\mu+1,2\nu+1} &= \sum_{\ell=0}^{d_j-1} r^j_{2^{j-\mu-1}(2\nu+1),\ell}\,\phi_{j,\ell} = \sum_{\ell=0}^{d_j-1} \psi_{\mu+1,2\nu+1}(t_{j,\ell})\,\phi_{j,\ell}, \end{aligned} \tag{6}$$

where the first index $k$ of the entries of $\boldsymbol{R}_j := (r^j_{k,\ell})_{k,\ell=0}^{d_j-1}$ corresponds to that point $t_{j,k}$ of the fine grid in which the function $\phi_{0,\nu}$ or $\psi_{\mu+1,2\nu+1}$, respectively, fulfils (1) or (4) with the function value 1.

Of course, one can describe the matrices $\boldsymbol{Z}_j$ of complete decomposition and $\boldsymbol{R}_j$ of complete reconstruction by a stepwise application of the matrices $\boldsymbol{D}_r$ and $\boldsymbol{C}_r$ from Theorem 1, respectively. We want to use discrete Fourier transform in our decomposition and reconstruction algorithms. Therefore, we need to introduce some notations. We restrict ourselves to a special case. Assume here and in the sequel that $d = d_0$ is a power of 2, i.e. $d = 2^{t_d}$. Otherwise things can be done in a similar way (see [22]) but will become somewhat more complicated. Let $k \in \{0, \dots, d_j - 1\}$ have the representation

$$k = \sum_{s=0}^{t} k_s\, 2^s\,, \qquad k_s \in \{0,1\},\ t < t_d + j. \tag{7}$$

Then, the binary representation mirrored at $j + t_d - 1$ (i.e., in bit reversed order) is given by

$$M_j(k) := \sum_{s=0}^{t} k_s\, 2^{j+t_d-1-s}. \tag{8}$$

The following lemma collects some assertions on such mirrored represention. It can be proved by some simple calculations.

**Lemma 2.** Let $k \in \mathbb{N}_0$ have the representation (7). Then

(i) $M_j(M_j(k)) = k$.
(ii) $M_J(2^{J-j}k) = M_j(k)$, for $2^{J-j}k \in \mathbb{N}_0,\ k \in \{0, \dots, d_j - 1\}$.

Now, we want to give factorizations of the matrices for complete univariate decomposition $\boldsymbol{Z}_j$ and reconstruction $\boldsymbol{R}_j$. We use the notation $\boldsymbol{A} \boxplus \boldsymbol{B} := \operatorname{diag}(\boldsymbol{A}, \boldsymbol{B})$ for the direct sum of matrices.

**Theorem 3.** Let $j \in \mathbb{N}$ and $\boldsymbol{Q}_s$, $\boldsymbol{F}_{s,\lambda}$ be the matrices

$$\boldsymbol{Q}_s := \left(\delta_{k,M_s(\ell)}\right)_{k,\ell=0}^{d_s-1}, \qquad \boldsymbol{F}_{s,\lambda} := \begin{pmatrix} \boldsymbol{I}_{d_{s-1}} & \boldsymbol{I}_{d_{s-1}} \\ \operatorname{diag}\left(\omega_s^k\right)_{k=0}^{d_{s-1}-1} & -\operatorname{diag}\left(\omega_s^k\right)_{k=0}^{d_{s-1}-1} \end{pmatrix}.$$

(i) The matrix $\boldsymbol{R}_j$ for complete univariate reconstruction can be factorized as

$$\boldsymbol{Q}_j \boldsymbol{R}_j \boldsymbol{Q}_j^{\mathrm{T}} = \frac{1}{d_{j-1}} \left( \boldsymbol{Q}_0 \overline{\boldsymbol{F}}_0 \boxplus \mathop{\boxplus}_{r=0}^{j-1} \boldsymbol{Q}_r \overline{\boldsymbol{F}}_r \right) \boldsymbol{R}_{j,\lambda} \left( \boldsymbol{I}_2 \otimes \boldsymbol{F}_{j-1} \boldsymbol{Q}_{j-1}^{\mathrm{T}} \right)$$

with the inner sparse matrix

$$\boldsymbol{R}_{j,\lambda} := \left( \prod_{r=1}^{j-1} (\boldsymbol{C}_{r,\lambda} \boldsymbol{F}_{r,\lambda}) \boxplus \boldsymbol{I}_{d_j - d_r} \right) \boldsymbol{C}_{j,\lambda}\,.$$

(ii) The matrix $\boldsymbol{Z}_j$ for complete univariate decomposition can be factorized as

$$\boldsymbol{Q}_j \boldsymbol{Z}_j \boldsymbol{Q}_j^{\mathrm{T}} = \frac{1}{d_{j-1}} \left( \boldsymbol{I}_2 \otimes \boldsymbol{Q}_{j-1} \overline{\boldsymbol{F}}_{j-1} \right) \boldsymbol{Z}_{j,\lambda} \left( \boldsymbol{F}_0 \boldsymbol{Q}_0^{\mathrm{T}} \boxplus \mathop{\boxplus}_{r=0}^{j-1} \boldsymbol{F}_r \boldsymbol{Q}_r^{\mathrm{T}} \right)$$

with the inner sparse matrix

$$\boldsymbol{Z}_{j,\lambda} := \boldsymbol{D}_{j,\lambda} \left( \prod_{r=1}^{j-1} (\overline{\boldsymbol{F}}_{j-r,\lambda}^{\mathrm{T}} \boldsymbol{D}_{j-r,\lambda}) \boxplus \boldsymbol{I}_{d_j - d_{j-r}} \right).$$

All occurring matrix products are taken from left to right.

*Proof.* By construction, it holds that,

$$\boldsymbol{Q}_j \boldsymbol{R}_j \boldsymbol{Q}_j^{\mathrm{T}} = \prod_{r=1}^{j} \left( \left( (\boldsymbol{I}_2 \otimes \boldsymbol{Q}_{r-1}) \boldsymbol{C}_r \boldsymbol{P}_r \boldsymbol{Q}_r^{\mathrm{T}} \right) \boxplus \boldsymbol{I}_{d_j - d_r} \right).$$

With some calculations, assertion (i) follows from Theorem 1 (i), the orthogonality of the matrices $\boldsymbol{Q}_r$ and the relation

$$\frac{1}{d_{s-1}} (\boldsymbol{I}_2 \otimes \boldsymbol{F}_{s-1})\, \boldsymbol{P}_s \overline{\boldsymbol{F}}_s = \boldsymbol{F}_{s,\lambda}.$$

The proof of (ii) can be done analogously. □

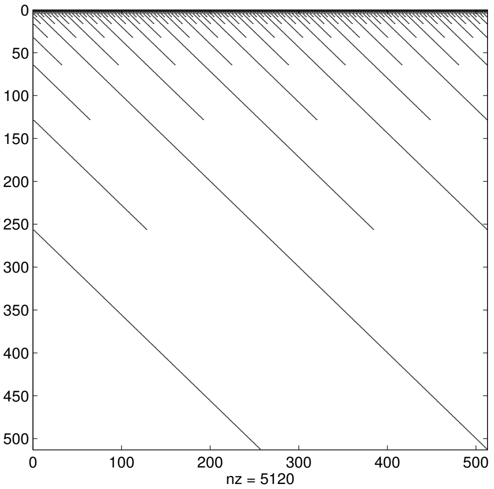


Figure 2. Non-zero entries of the matrix $\boldsymbol{R}_9^\lambda$.

We return to our function $f_j \in V_j$ which we wanted to represent in the basis of scaling functions as well as in the basis of wavelets. With the decomposition and reconstruction relations for the basis functions, we obtain the decomposition and reconstruction relations for the coefficient vectors

$$\boldsymbol{b}_j = \left(\boldsymbol{Q}_0^{\mathrm{T}} \boxplus \mathop{\boxplus}_{r=0}^{j-1} \boldsymbol{Q}_r^{\mathrm{T}}\right) \left(\boldsymbol{Q}_j \boldsymbol{Z}_j^{\mathrm{T}} \boldsymbol{Q}_j^{\mathrm{T}}\right) \boldsymbol{Q}_j \boldsymbol{a}_j,$$

$$\boldsymbol{a}_j = \boldsymbol{Q}_j^{\mathrm{T}} \left(\boldsymbol{Q}_j \boldsymbol{R}_j^{\mathrm{T}} \boldsymbol{Q}_j^{\mathrm{T}}\right) \left(\boldsymbol{Q}_0 \boxplus \mathop{\boxplus}_{r=0}^{j-1} \boldsymbol{Q}_r\right) \boldsymbol{b}_j.$$

Using Theorem 3, this simplifies to

$$\boldsymbol{b}_j = \left(\boldsymbol{F}_0 \boxplus \mathop{\boxplus}_{r=0}^{j-1} \boldsymbol{F}_r\right) \boldsymbol{Z}_{j,\lambda}^{\mathrm{T}} \left(\boldsymbol{I}_2 \otimes \overline{\boldsymbol{F}}_{j-1}\right) \boldsymbol{P}_j \boldsymbol{a}_j,$$

$$\boldsymbol{a}_j = \boldsymbol{P}_j^{\mathrm{T}} \left(\boldsymbol{I}_2 \otimes \boldsymbol{F}_{j-1}\right) \boldsymbol{R}_{j,\lambda}^{\mathrm{T}} \left(\overline{\boldsymbol{F}}_0 \boxplus \mathop{\boxplus}_{r=0}^{j-1} \overline{\boldsymbol{F}}_r\right) \boldsymbol{b}_j.$$

So the univariate complete decomposition and the univariate complete reconstruction, respectively, for the level $j$ cost some discrete Fourier transforms of length at most $d_{j-1}$ and a multiplication with a matrix with at most $jd_j$ non-zero entries, i.e., altogether $\mathcal{O}(j2^j)$ essential operations in the general periodic case. If we use the interpolatory spline wavelets from our second example then there are direct algorithms for complete decomposition and complete reconstruction, respectively, working with $\mathcal{O}(2^j)$ essential operations (see [2]).

## 4. Sample and Wavelet Spaces for Sparse Grids

Starting from the univariate interpolation, we want to investigate bivariate interpolation on a sparse grid. Therefore, we recall that by the definition of the interpolation projectors $L_j$ and the univariate sample spaces $V_j$, it holds that

$$\text{Im } L_j \subset \text{Im } L_{j+1} \quad \text{and} \quad \mathcal{T}_j \subset \mathcal{T}_{j+1},$$

thus, $L_j \le L_{j+1}$, $j \in \mathbb{N}_0$, is a chain of projectors, where the order relation $A \le B$ holds if and only if $AB = BA = A$.

For bivariate functions, we will consider the $j$-th order blending operator defined by the $j$-th order Boolean sum

$$B_j := \bigoplus_{r=0}^{j} L_r \otimes L_{j-r},$$

where $A \oplus B := A + B - AB$. The representation of $B_j$ in terms of ordinary sums is known (see [1,7]) to be

$$B_j = \sum_{r=0}^{j} L_r \otimes L_{j-r} - \sum_{r=0}^{j-1} L_r \otimes L_{j-r-1}. \tag{9}$$

The Boolean sums also form a chain [7]. They have the range

$$\text{Im } B_j = \sum_{r=0}^{j} \text{Im } L_r \otimes \text{Im } L_{j-r} = \sum_{r=0}^{j} V_r \otimes V_{j-r}$$

and the interpolate on the grid

$$\mathcal{T}_j^{\text{B}} := \bigcup_{r=0}^{j} \mathcal{T}_r \times \mathcal{T}_{j-r}\,.$$

In this manner, the Boolean sum operator $B_j$ is an interpolation projector on a sparse grid. Furthermore, as in the univariate case, it holds that

$$\text{Im } B_j \subset \text{Im } B_{j+1} \quad \text{and} \quad \mathcal{T}_j^{\text{B}} \subset \mathcal{T}_{j+1}^{\text{B}}. \tag{10}$$

Taking into account the properties of the $j$-th order blending interpolation operator $B_j$, we are now able to define new sample spaces of bivariate functions by

$$V_j^{\text{B}} := \text{Im } B_j = \sum_{r=0}^{j} V_r \otimes V_{j-r}. \tag{11}$$

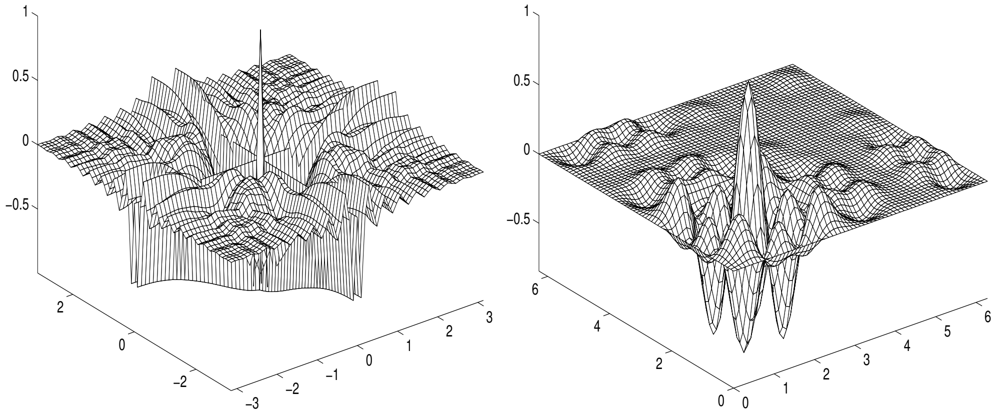


Figure 3. The trigonometric scaling functions $\Phi^t_{6,0,0}$ (left) and $\Phi^t_{6,16,16}$ (right).

From this definition and (10), we obtain

$$V_j^{\mathrm{B}} \subset V_{j+1}^{\mathrm{B}}, \qquad \operatorname{clos}_{L_2(\mathbb{T}^2)} \bigcup_{j\in\mathbb{N}_0} V_j^{\mathrm{B}} = L_2(\mathbb{T}^2)$$

and

$$V_j^{\mathrm{B}} = \operatorname{span} \bigcup_{r=0}^{j} \{\, \phi_{r,k} \otimes \phi_{j-r,\ell} \;:\; k = 0,1,\dots,d_r - 1; \quad \ell = 0,1,\dots,d_{j-r} - 1 \,\}. \tag{12}$$

Hence, we immediately get a generating system, but unfortunately this set of functions does not form a basis. In order to characterize the bivariate scaling functions again by their interpolatory properties, we investigate the relation $\mathcal{T}_j^{\mathrm{B}} \subset \mathcal{T}_j \times \mathcal{T}_j$ between the sparse and the regular interpolation grid. Thus, we can define an index set $\mathcal{J}_j$ by

$$\mathcal{J}_j := \left\{ (k,\ell) \in \mathbb{Z}^2 \;:\; (t_{j,k}, t_{j,\ell}) \in \mathcal{T}_j^{\mathrm{B}} \right\}$$

and search scaling functions $\Phi_{j,k,\ell}$ as fundamental interpolants in $V_j^{\mathrm{B}}$ on the sparse grid $\mathcal{T}_j^{\mathrm{B}}$, i.e. with the property,

$$\Phi_{j,k,\ell}(t_{j,m}, t_{j,n}) = \delta_{k,m}\delta_{\ell,n}, \qquad (k,\ell),(m,n) \in \mathcal{J}_j.$$

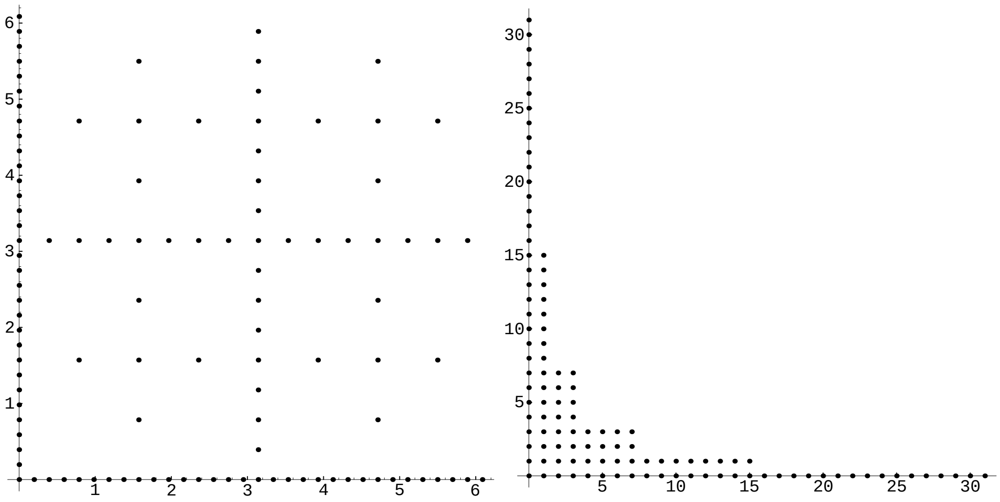


Figure 4. Sparse grid $\mathcal{T}_5^{\mathrm{B}}$ and the index set $M_5(\mathcal{J}_5)$ for $d = 1$.

Such functions can easily be found using tensor products of the univariate interpolating scaling functions on the level $j$. The application of the blending interpolation projector $B_j$ leads to

$$\begin{aligned}\Phi_{j,k,\ell} &= B_j(\phi_{j,k} \otimes \phi_{j,\ell})\\ &= \sum_{s=0}^{j} \phi_{s,2^{s-j}k} \otimes \phi_{j-s,2^{-s}\ell} - \sum_{s=0}^{j-1} \phi_{s,2^{s-j}k} \otimes \phi_{j-s-1,2^{-s-1}\ell}\end{aligned}$$

where $(k,\ell) \in \mathcal{J}_j$ and $\phi_{s,t} \equiv 0$ for $t \notin \mathbb{Z}$.

For further investigations, we take a finer splitting of the index set $\mathcal{J}_j$

$$\mathcal{J}_j = 2\,\mathcal{J}_{j-1} \cup (\mathcal{J}_j \backslash 2\,\mathcal{J}_{j-1}), \qquad \mathcal{J}_j \backslash 2\,\mathcal{J}_{j-1} = \bigcup_{p=0}^{j} \mathcal{K}_{j,p}$$

with

$$\mathcal{K}_{0,0} = \mathcal{J}_0$$

and for $j \geq 1$,

$$\begin{aligned}\mathcal{K}_{j,p} &= \{(M_j(\eta), M_j(\kappa)) \,:\, \eta = d_{p-1}, \dots, d_p - 1;\\ &\qquad \kappa = d_{j-p-1}, \dots, d_{j-p} - 1\}, \quad p = 1, \dots, j-1,\\ \mathcal{K}_{j,0} &= \{(M_j(\eta), M_j(\kappa)) \,:\, \eta = 0, \dots, d_0 - 1;\ \kappa = d_{j-1}, \dots, d_j - 1\},\\ \mathcal{K}_{j,j} &= \{(M_j(\eta), M_j(\kappa)) \,:\, \eta = d_{j-1}, \dots, d_j - 1;\ \kappa = 0, \dots, d_0 - 1\}.\end{aligned}$$

Hence,

$$\mathcal{J}_j = \bigcup_{t=0}^{j} 2^{j-t} \bigcup_{r=0}^{t} \mathcal{K}_{t,r} \tag{13}$$

and

$$M_j(\mathcal{J}_j) = \bigcup_{r=0}^{j} \bigcup_{s=0}^{r} M_r(\mathcal{K}_{r,s}) \,. \tag{14}$$

With the finer splitting of the index set $\mathcal{J}_j$, we can more precisely describe the fundamental interpolants in $V_j^{\mathrm{B}}$.

**Theorem 4.** For indices $(k,\ell) \in 2^{j-t}\mathcal{K}_{t,p}, 0 \le p \le t \le j$, the fundamental interpolant $\Phi_{j,k,\ell}$ has the form

$$\begin{aligned}\Phi_{j,k,\ell} &= \sum_{s=p}^{j+p-t} \phi_{s,2^{s-j}k} \otimes \phi_{j-s,2^{-s}\ell} - \sum_{s=p}^{j+p-t-1} \phi_{s,2^{s-j}k} \otimes \phi_{j-s-1,2^{-s-1}\ell} \\ &= \sum_{s=p}^{j+p-t} \phi_{s,M_s(M_j(k))} \otimes \phi_{j-s,M_{j-s}(M_j(\ell))} \\ &\quad - \sum_{s=p}^{j+p-t-1} \phi_{s,M_s(M_j(k))} \otimes \phi_{j-s-1,M_{j-s-1}(M_j(\ell))} \,.\end{aligned} \tag{15}$$

*Proof.* The assertion follows from the definition of $\mathcal{K}_{t,p}$ and Lemma 2. □

As usual we define the wavelet space $W_{j+1}^{\mathrm{B}}$ as the orthogonal complement of the sample space on the lower level $j$ relative to the sample space on the higher level $j+1$, i.e.,

$$V_{j+1}^{\mathrm{B}} = V_j^{\mathrm{B}} \oplus^{\perp} W_{j+1}^{\mathrm{B}} \tag{16}$$

with respect to the inner product in $L_2(\mathbb{T}^2)$. The wavelet spaces can be characterized in terms of tensor products of univariate wavelet spaces.

**Theorem 5.** For $j \in \mathbb{N}_0$, it holds that

$$W_{j+1}^{\mathrm{B}} = \bigoplus_{r=0}^{j+1}{}^{\perp}\, W_r \otimes W_{j+1-r} \tag{17}$$

with the notation $W_0 := V_0$.

The proof can be found in [21]. For $(m,n) \in \mathcal{K}_{j+1,p}$, we define the wavelet function

$$\Psi_{j+1,m,n} := \psi_{p,2^{p-j-1}m} \otimes \psi_{j+1-p,2^{-p}n} \tag{18}$$

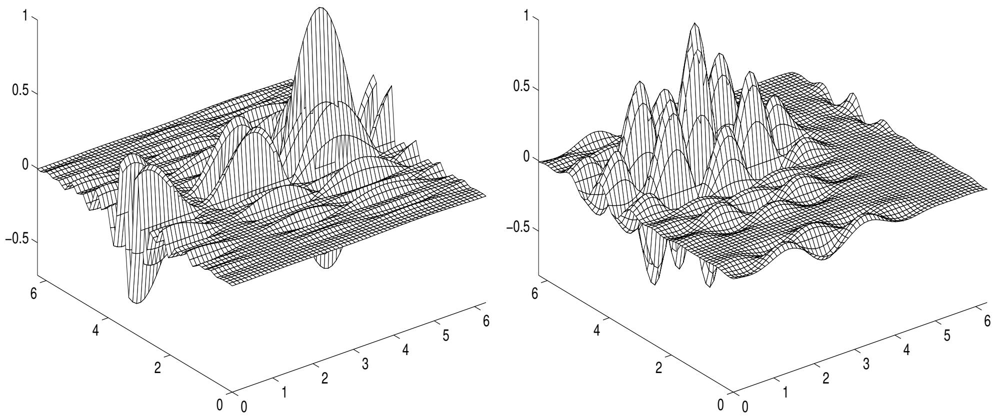

Figure 5. The trigonometric wavelet functions $\Psi^t_{7,68,96}$ (left) and $\Psi^t_{7,88,48}$ (right).

with $\psi_{0,r} := \phi_{0,r}$. Using the representation (17) of the wavelet space, we immediately obtain bases of the sample and the wavelet spaces.

**Corollary 6.** Let $j \in \mathbb{N}_0$.

(i) The basis and the dimension of the sample space are given by

$$\begin{aligned}\mathcal{B}(V_j^{\mathrm{B}}) &= \{\Phi_{j,k,\ell} \;:\; (k,\ell) \in \mathcal{J}_j\},\\ \dim V_j^{\mathrm{B}} &= d^2(j2^{j-1} + 2^j).\end{aligned}$$

(ii) The basis and the dimension of the wavelet space are given by

$$\begin{aligned}\mathcal{B}(W_{j+1}^{\mathrm{B}}) &= \{\Psi_{j+1,m,n} \;:\; (m,n) \in \mathcal{K}_{j+1,p}\ ,\ p = 0,\ldots,j+1\},\\ \dim W_{j+1}^{\mathrm{B}} &= d^2(j2^{j-1} + 2^{j+1}).\end{aligned}$$

*Proof.* Part (ii) of the assertion follows from Theorem 5. Denoting $W_0^{\mathrm{B}} := V_0^{\mathrm{B}}$, we can write the sample space in terms of orthogonal sums of wavelet spaces

$$V_j^{\mathrm{B}} = \bigoplus_{r=0}^{j}{}^{\perp}\ W_r^{\mathrm{B}} = \bigoplus_{r=0}^{j}{}^{\perp}\ \bigoplus_{k=0}^{r}{}^{\perp}\ (W_k \otimes W_{r-k})\,,$$

and obtain the dimension of the sample space as above. The functions $\Phi_{j,k,\ell}$ are fundamental interpolants in $V_j^{\mathrm{B}}$. Hence, they are linearly independent. By (13) their number equals the just computed dimension of the sample space. That proves part (i). $\square$

This means, we have the dimension of the constructed bivariate sample space growing as $\mathcal{O}(j2^j)$, while the dimension of ordinary tensor products of univariate sample spaces grows as $\mathcal{O}(2^{2j})$.

## 5. Complete Decomposition and Reconstruction

We want to establish the relation for the complete decomposition of the fundamental interpolants on the sparse grid, i.e., the scaling functions in the bivariate case, into wavelet parts in matrix form

$$\boldsymbol{\Phi}_j = \boldsymbol{Z}_j^{\mathrm{B}}\, \boldsymbol{\Psi}_j$$

with

$$\boldsymbol{\Phi}_j = (\Phi_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j} \qquad \text{and} \qquad \boldsymbol{\Psi}_j = (\Psi_{u,m,n})_{u=0,\ldots,j;\; q=0,\ldots,u;\; (m,n)\in\mathcal{K}_{u,q}}$$

where the indices in the vector $\boldsymbol{\Phi}_j$ follow the ordering

$$2^j\mathcal{K}_{0,0},\ 2^{j-1}\mathcal{K}_{1,0},\ 2^{j-1}\mathcal{K}_{1,1},\ \ldots,\ \mathcal{K}_{j,j}. \tag{19}$$

For the numbering of rows and columns of the matrices involved we use notations like $(a_{k,\ell,m,m})_{(k,\ell)\in\mathcal{J}_j,(m,n)\in M_j(\mathcal{J}_j)}$. In this example the numbering of rows follows the ordering (19). The numbering of columns uses the splitting (14). The subsets and the tuples in each subset are ordered lexicographically. We want to describe the matrix $\boldsymbol{Z}_j^{\mathrm{B}}$ in terms of the matrices $\boldsymbol{Z}_s$ for the univariate decompositions.

**Theorem 7.** With the permutation matrices

$$\begin{aligned}
\boldsymbol{S}_{k,s} &:= \left(\delta_{\eta,\lfloor d_{k-s}^{-1} r\rfloor}\, \delta_{\kappa,r \bmod d_{k-s}}\right)_{(\eta,\kappa)\in M_j(\mathcal{J}_j),\, r=0,\ldots,d\,d_k-1}, \\
\boldsymbol{T}_j &:= \left(\delta_{M_j(k),m}\, \delta_{M_j(\ell),n}\right)_{(k,\ell)\in\mathcal{J}_j,\,(m,n)\in M_j(\mathcal{J}_j)},
\end{aligned}$$

and $\boldsymbol{Q}_s$ as in Theorem 3, the matrix $\boldsymbol{Z}_j^{\mathrm{B}}$ for complete decomposition has the form

$$\begin{aligned}
\boldsymbol{Z}_j^{\mathrm{B}} = \boldsymbol{T}_j \Bigg( &\sum_{s=0}^{j} \boldsymbol{S}_{j,s} \left( (\boldsymbol{Q}_s \boldsymbol{Z}_s \boldsymbol{Q}_s^{\mathrm{T}}) \otimes (\boldsymbol{Q}_{j-s} \boldsymbol{Z}_{j-s} \boldsymbol{Q}_{j-s}^{\mathrm{T}}) \right) \boldsymbol{S}_{j,s}^{\mathrm{T}} \\
&- \sum_{s=0}^{j-1} \boldsymbol{S}_{j-1,s} \left( (\boldsymbol{Q}_s \boldsymbol{Z}_s \boldsymbol{Q}_s^{\mathrm{T}}) \otimes (\boldsymbol{Q}_{j-s-1} \boldsymbol{Z}_{j-s-1} \boldsymbol{Q}_{j-s-1}^{\mathrm{T}}) \right) \boldsymbol{S}_{j-1,s}^{\mathrm{T}} \Bigg) \boldsymbol{T}_j^{\mathrm{T}}.
\end{aligned}$$

*Proof.* Decomposing the scaling function $\Phi_{j,k,\ell}$, we compute the entries of $\boldsymbol{Z}_j^{\mathrm{B}}$ for $(k,\ell)\in 2^{j-t}\mathcal{K}_{t,p}$, $(m,n)\in 2^{j-u}\mathcal{K}_{u,q}$ as

$$z^{\mathrm{B}}_{j,k,\ell,m,n} = \sum_{s=p}^{j+p-t} z^s_{2^{s-j}k,2^{s-j}m}\, z^{j-s}_{2^{-s}\ell,2^{-s}n} - \sum_{s=p}^{j+p-t-1} z^s_{2^{s-j}k,2^{s-j}m}\, z^{j-s-1}_{2^{-s-1}\ell,2^{-s-1}n}.$$

From Lemma 2 we obtain

$$2^{s-j}k = M_s(M_j(k)), \quad 2^{-s}\ell = M_{j-s}(M_j(\ell)), 2^{-s-1}\ell = M_{j-s-1}(M_j(\ell)),$$

$$2^{s-j}m = M_s(M_j(m)), \quad 2^{-s}n = M_{j-s}(M_j(n)), \quad 2^{-s-1}n = M_{j-s-1}(M_j(n)).$$

for $(k,\ell) \in 2^{j-t}\mathcal{K}_{t,p}$ and $(m,n) \in 2^{j-u}\mathcal{K}_{u,q}$. Then, we can rewrite the formula for $z^{\mathrm{B}}_{j,k,\ell,m,n}$ as

$$\begin{aligned} z^{\mathrm{B}}_{j,k,\ell,m,n} = &\sum_{s=p}^{j+p-t} z^{s}_{M_s(M_j(k)),M_s(M_j(m))}\, z^{j-s}_{M_{j-s}(M_j(\ell)),M_{j-s}(M_j(n))} \\ &- \sum_{s=p}^{j+p-t-1} z^{s}_{M_s(M_j(k)),M_s(M_j(m))}\, z^{j-s-1}_{M_{j-s-1}(M_j(\ell)),M_{j-s-1}(M_j(n))}, \end{aligned}$$

which proves the assertion. □

We also want to establish the relation for complete reconstruction in the bivariate case in matrix form

$$\boldsymbol{\Psi}_j = \boldsymbol{R}^{\mathrm{B}}_j\, \boldsymbol{\Phi}_j$$

with $\boldsymbol{\Psi}_j$ and $\boldsymbol{\Phi}_j$ as before. We describe the matrix $\boldsymbol{R}^{\mathrm{B}}_j$ in terms of the matrices $\boldsymbol{R}_s$ for the univariate reconstructions.

**Theorem 8.** With the permutation matrices $\boldsymbol{Q}_s$, $\boldsymbol{S}_{k,s}$ and $\boldsymbol{T}_j$ as in Theorem 7, the matrix $\boldsymbol{R}^{\mathrm{B}}_j$ for complete reconstruction has the form

$$\begin{aligned} \boldsymbol{R}^{\mathrm{B}}_j = \boldsymbol{T}_j \Bigg( &\sum_{s=0}^{j} \boldsymbol{S}_{j,s} \left( (\boldsymbol{Q}_s \boldsymbol{R}_s \boldsymbol{Q}_s^{\mathrm{T}}) \otimes (\boldsymbol{Q}_{j-s} \boldsymbol{R}_{j-s} \boldsymbol{Q}_{j-s}^{\mathrm{T}}) \right) \boldsymbol{S}^{\mathrm{T}}_{j,s} \\ &- \sum_{s=0}^{j-1} \boldsymbol{S}_{j-1,s} \left( (\boldsymbol{Q}_s \boldsymbol{R}_s \boldsymbol{Q}_s^{\mathrm{T}}) \otimes (\boldsymbol{Q}_{j-s-1} \boldsymbol{R}_{j-s-1} \boldsymbol{Q}_{j-s-1}^{\mathrm{T}}) \right) \boldsymbol{S}^{\mathrm{T}}_{j-1,s} \Bigg) \boldsymbol{T}_j^{\mathrm{T}} . \end{aligned}$$

*Proof.* Let the matrix on the right-hand side be $\boldsymbol{A}_j = (a_{j,k,\ell,m,n})_{(k,\ell),(m,n)\in\mathcal{J}_j}$, then for $(k,\ell) \in 2^{j-t}\mathcal{K}_{t,p}$, $(m,n) \in 2^{j-u}\mathcal{K}_{u,q}$,

$$\begin{aligned} a_{j,k,\ell,m,n} = &\sum_{s=p}^{j+p-t} r^{s}_{M_s(M_j(k)),M_s(M_j(m))}\, r^{j-s}_{M_{j-s}(M_j(\ell)),M_{j-s}(M_j(n))} \\ &- \sum_{s=p}^{j+p-t-1} r^{s}_{M_s(M_j(k)),M_s(M_j(m))}\, r^{j-s-1}_{M_{j-s-1}(M_j(\ell)),M_{j-s-1}(M_j(n))}. \end{aligned}$$

From Lemma 2, we obtain

$$a_{j,k,\ell,m,n} = \sum_{s=p}^{j+p-t} r^s_{2^{s-j}k,2^{s-j}m}\, r^{j-s}_{2^{-s}\ell,2^{-s}n} - \sum_{s=p}^{j+p-t-1} r^s_{2^{s-j}k,2^{s-j}m}\, r^{j-s-1}_{2^{-s-1}\ell,2^{-s-1}n}\,.$$

Because we are working on nested grids, we have $r^s_{2^{s-j}k,2^{s-j}m} = r^j_{k,m}$ for $s \le j$. This leads to

$$a_{j,k,\ell,m,n} = r^j_{k,m}\, r^j_{\ell,n} = \Psi_{t,2^{t-j}k,2^{t-j}\ell}(t_{j,m}, t_{j,n}).$$

This proves the assertion. $\square$

## 6. Algorithms

In this section, we want to present the algorithms for complete decomposition and reconstruction, respectively, for our Boolean sum approach. At first, we project a bivariate continuous (or more general: Riemann integrable) $2\pi$-periodic function $f$ into the sample space $V_j^{\mathrm{B}}$. We obtain the projection $f_j$ of $f$ by interpolation on the sparse grid

$$f_j := B_j f = \sum_{(k,\ell)\in\mathcal{J}_j} f(t_{j,k}, t_{j,\ell})\, \Phi_{j,k,\ell}\,.$$

Writing this as vectors, we have $f_j = \boldsymbol{v}_j^{\mathrm{T}} \boldsymbol{\Phi}_j$, with the coefficient vector $\boldsymbol{v}_j$ consisting of the function values $v_{j,k,\ell} = f(t_{j,k}, t_{j,\ell})$, $(k,\ell) \in \mathcal{J}_j$, sampled from the sparse grid. The function vector $\boldsymbol{\Phi}_j$ was introduced in Section 5. In terms of the wavelet basis, we get another representation of $f_j$ as $f_j = \boldsymbol{w}_j^{\mathrm{T}} \boldsymbol{\Psi}_j$, with a coefficient vector $\boldsymbol{w}_j$ and the function vector $\boldsymbol{\Psi}_j$ as in the previous section. Then we use the relations of complete decomposition and reconstruction for the basis vectors

$$\boldsymbol{\Phi}_j = \boldsymbol{Z}_j^{\mathrm{B}}\, \boldsymbol{\Psi}_j \qquad \text{and} \qquad \boldsymbol{\Psi}_j = \boldsymbol{R}_j^{\mathrm{B}}\, \boldsymbol{\Phi}_j\,,$$

and obtain the relations of complete decomposition and reconstruction, respectively, for the coefficient vectors

$$\boldsymbol{w}_j = (\boldsymbol{Z}_j^{\mathrm{B}})^{\mathrm{T}}\, \boldsymbol{v}_j \qquad \text{and} \qquad \boldsymbol{v}_j = (\boldsymbol{R}_j^{\mathrm{B}})^{\mathrm{T}}\, \boldsymbol{w}_j \tag{20}$$

in the same way as before. Here, the multiplication with the corresponding matrix reduces by Theorems 7 and 8 to the stepwise application of the algorithms in the univariate case.

The equations (20) are of the form $\boldsymbol{y}_j = (\boldsymbol{A}_j^{\mathrm{B}})^{\mathrm{T}}\, \boldsymbol{x}_j$ with the symbol $\boldsymbol{A}$ standing for $\boldsymbol{Z}$ in the complete decomposition relation and for $\boldsymbol{R}$ in the reconstruction relation, respectively. The application of such a matrix $(\boldsymbol{A}_j^{\mathrm{B}})^{\mathrm{T}}$ can be realized by the following procedure.

**Algorithm 9.** (Complete decomposition or reconstruction)

Input vector: $\boldsymbol{x}_j = (x_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Output vector: $\boldsymbol{y}_j = (y_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Help vector: $\boldsymbol{z}_j = (z_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Notation: $\boldsymbol{z}_{j,k} = (z_{j,k,\ell})_{\ell\in\mathcal{L}}$, $\boldsymbol{z}_{j,\ell} = (z_{j,k,\ell})_{k,\in\mathcal{K}}$, with an appropriate choice of the index sets $\mathcal{K},\mathcal{L}$ (the ranges for $k,\ell$ as given in the algorithm)

Initialization: $y_{j,k,\ell} := 0$ for all $(k,\ell)\in\mathcal{J}_j$

For all $s = 0,\ldots,j$ do

$$\begin{aligned}
&z_{j,k,\ell} := x_{j,M_j(k),M_j(\ell)} && \text{for } k=0,\ldots,d_s-1;\ \ell=0,\ldots,d_{j-s}-1\\
&\boldsymbol{z}_{j,k} := \boldsymbol{Q}_{j-s}\,\boldsymbol{A}_{j-s}^{\mathrm{T}}\,\boldsymbol{Q}_{j-s}^{\mathrm{T}}\,\boldsymbol{z}_{j,k} && \text{for } k=0,\ldots,d_s-1\\
&\boldsymbol{z}_{j,\ell} := \boldsymbol{Q}_s\,\boldsymbol{A}_s^{\mathrm{T}}\,\boldsymbol{Q}_s^{\mathrm{T}}\,\boldsymbol{z}_{j,\ell} && \text{for } \ell=0,\ldots,d_{j-s}-1\\
&y_{j,M_j(k),M_j(\ell)} := y_{j,M_j(k),M_j(\ell)} + z_{j,k,\ell} && \text{for } k=0,\ldots,d_s-1;\ \ell=0,\ldots,d_{j-s}-1
\end{aligned}$$

For all $s = 0,\ldots,j-1$ do

$$\begin{aligned}
&z_{j,k,\ell} := x_{j,M_j(k),M_j(\ell)} && \text{for } k=0,\ldots,d_s-1;\ \ell=0,\ldots,d_{j-s-1}-1\\
&\boldsymbol{z}_{j,k} := \boldsymbol{Q}_{j-s-1}\,\boldsymbol{A}_{j-s-1}^{\mathrm{T}}\,\boldsymbol{Q}_{j-s-1}^{\mathrm{T}}\,\boldsymbol{z}_{j,k} && \text{for } k=0,\ldots,d_s-1\\
&\boldsymbol{z}_{j,\ell} := \boldsymbol{Q}_s\,\boldsymbol{A}_s^{\mathrm{T}}\,\boldsymbol{Q}_s^{\mathrm{T}}\,\boldsymbol{z}_{j,\ell} && \text{for } \ell=0,\ldots,d_{j-s-1}-1\\
&y_{j,M_j(k),M_j(\ell)} := y_{j,M_j(k),M_j(\ell)} - z_{j,k,\ell} && \text{for } k=0,\ldots,d_s-1;\ \ell=0,\ldots,d_{j-s-1}-1
\end{aligned}$$

The products $\boldsymbol{Q}_s\boldsymbol{A}_s^{\mathrm{T}}\boldsymbol{Q}_s^{\mathrm{T}}$ are of the structure given in Theorem 3. That's why they can be realized by fast Fourier transforms and multiplication with a sparse matrix.

Now, we are interested in the number of essential operations for the bivariate complete decomposition and the complete reconstruction, respectively, for the level $j$ if we use the Boolean sum approach. Therefore, we count the univariate complete decompositions and the univariate complete reconstructions, respectively, for the levels occuring in the algorithm. In the end we need altogether $\mathcal{O}(j^2\,2^j)$ essential operations for the bivariate complete decomposition or reconstruction, respectively, for the level $j$. In the case of interpolating spline wavelets this reduces again to $\mathcal{O}(j\,2^j)$ essential operations.

## 7. An Example

Now we want to compare the wavelet decompositions of two test functions with different smoothness properties. Both functions are periodized splines centered in $(\pi, \pi)$. The first function $f_1 := N_1 \otimes N_1$ is the bilinear hat function, the second $f_2 := B_{1,1,1}$ is the linear 3-directional box spline.

The investigation of the decay of their Fourier coefficients leads to the order $\alpha$ of isotropic Sobolev spaces $H_2^\alpha(\mathbb{T}^2)$ and of the Sobolev spaces with dominating mixed smoothness properties $S_{2,2}^{\alpha,\alpha}H(\mathbb{T}^2)$ (cf. [19]) the test functions belong to:

| $f_k \in$ | $H_2^\alpha(\mathbb{T}^2)$ | $S_{2,2}^{\alpha,\alpha}H(\mathbb{T}^2)$ |
|---|---|---|
| $f_1$ | $\forall\ \alpha < 3/2$ | $\forall\ \alpha < 3/2$ |
| $f_2$ | $\forall\ \alpha < 3/2$ | $\forall\ \alpha < 3/4$ |

We choose the trigonometric scaling functions $\phi_{j,k}^t$ and the corresponding wavelets with the parameters $d = 2$, $\mu = 2$. The test functions are sampled on the grid of level $j$. So we obtain the coefficient vectors of the projections into the sample space of the highest level $j$. The interpolation error depends on the smoothness of the interpolated functions [15]:

| smoothness of $f$ | $\|L_j \otimes L_j f - f\|_{L_2(\mathbb{T}^2)}$ | $\|B_j f - f\|_{L_2(\mathbb{T}^2)}$ |
|---|---|---|
| $f \in H_2^\alpha(\mathbb{T}^2)$, | $\mathcal{O}(2^{-\alpha j})$ | $\mathcal{O}(j\, 2^{-\alpha j/2})$ |
| $f \in S_{2,2}^{\alpha,\alpha}H(\mathbb{T}^2)$ | $\mathcal{O}(2^{-\alpha j})$ | $\mathcal{O}(j\, 2^{-\alpha j})$ |

Now, these coefficient vectors can be decomposed by the algorithms given in Sections 3 and 6. Figure 6 shows density plots of the abolute value of the wavelet part at level $j$ of the interpolants of the test functions on the $j$-th grid. The values given below the density plots mark the used range (minimum=white, maximum=black). On the left-hand side one can see the wavelet parts of $f_1$, on the right-hand side those of $f_2$. For the figures in the first row we interpolated on the sparse grid $\mathcal{T}_8^{\mathrm{B}}$ with 5 120 grid points, for the second row we interpolated on the full grid $\mathcal{T}_8 \times \mathcal{T}_8$ with 262 144 points. The third row corresponds to the interpolation on a full grid with 9 126 points what is near the number of points used in the first row. This is now a grid of level 4, the parameters for the trigonometric wavelets are $d = 6$, $\mu = 0$. The number of essential operations behave like 4 : 132 : 1.

What all density plots have in common is that the absolute value of the wavelet parts appears to be large in regions where the underlying functions are not smooth. Now we will compare the second row with the third row. As expected we obtain better results for the finer sampling. Comparing the left-hand with the right-hand side in those rows, one finds the absolute values of the wavelet parts to have nearly the same range. The interpolated functions belong to the same isotropic Sobolev spaces. This is why the error of interpolation on the full grid is of the same order for both functions. This is reflected by the shown wavelet part

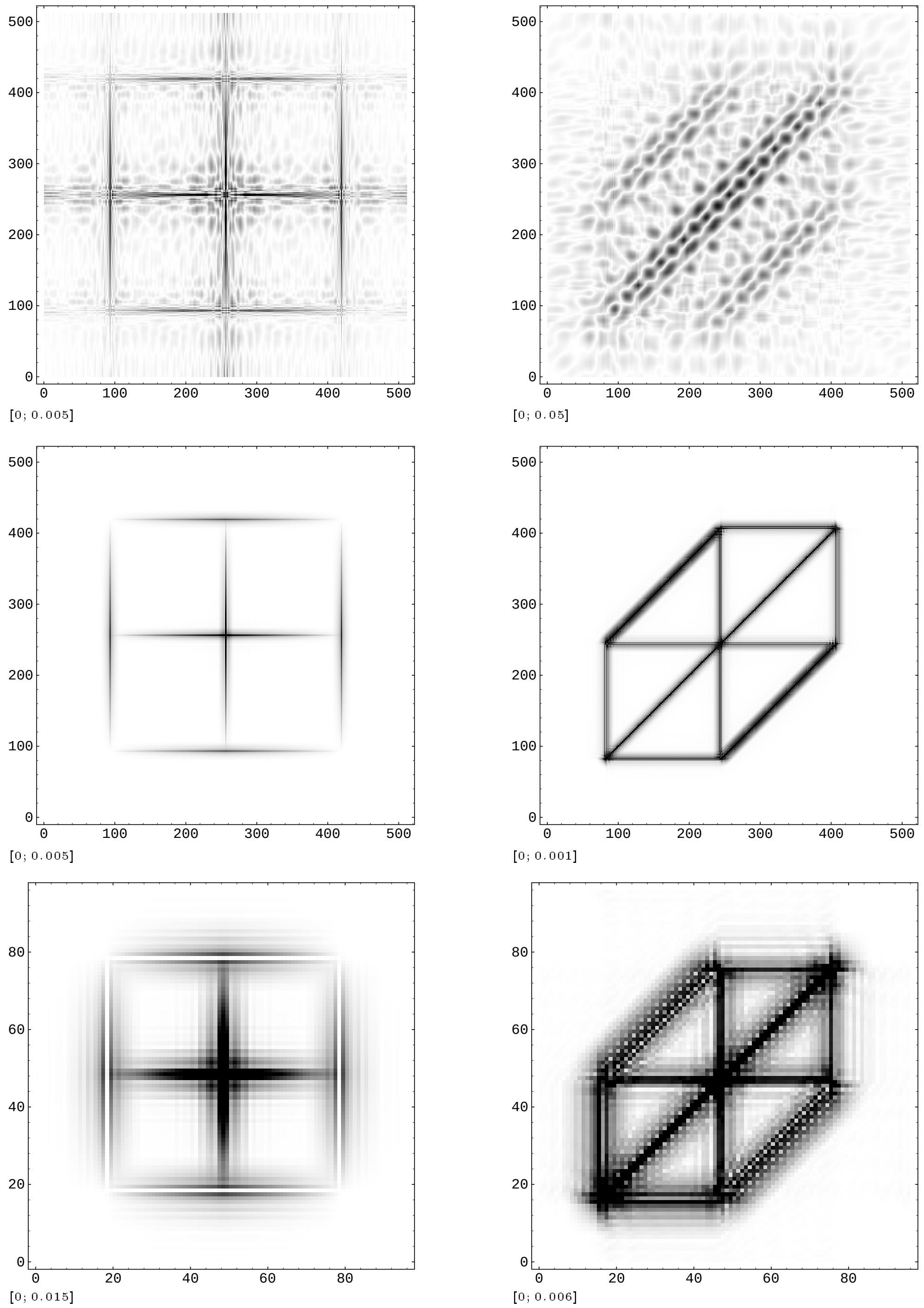


Figure 6. Some wavelet decompositions.

which is the orthogonal difference of the sample space of level $j$ (corresponding to $\mathcal{T}_j \times \mathcal{T}_j$) and the sample space of level $j-1$ (corresponding to $\mathcal{T}_{j-1} \times \mathcal{T}_{j-1}$).

Now we proceed to the first row. The range of the wavelet part of the second test function is ten times as large as the wavelet part of the first test function. The interpolated function are not in the same Sobolev spaces with dominated mixed smoothness. In this sense, the box spline is only "half as smooth" as the tensor product B–Spline. So the interpolation error for $f_2$ has half the order of the interpolation error for $f_1$. This again effects the wavelet part shown here. Furthermore, the regions of discontinuities seem to be better localized for $f_1$. Already in the usual tensor product case discontinuities parallel to the axes are better localized. This effect seems to be strengthed for the sparse grid wavelets. There the wavelet functions are tensor products of univariate interpolatory wavelets where the step sizes in $x$- or $y$-direction, respectively, can be very different from each other. If one compares the first and the third row the sparse grid wavelets seem to be a useful tool for analyzing bivariate functions (signals, pictures) at least in the case of mixed smoothness (only edges, discontinuities parallel to the axes).

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