A PROPERTY OF COLORED COMPLEXES AND THEIR DUALS

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Abstract. A ranked poset P is a Macaulay poset if there is a linear order $\prec$ of the elements of P such that for any m, i the set $C(m, i)$ of the m (with respect to $\prec$) smallest elements of rank i has minimum–sized shadow among all m–element subsets of the i–th level, and the shadow of $C(m, i)$ consists of the smallest elements of the $(i-1)$–st level. P is called shadow–increasing if for all m, i the shadow of $C(m, i)$ is not smaller than the shadow of $C(m, i-1)$. We show that colored complexes and their duals, the star posets, are shadow–increasing.

1. Introduction

1.1. **Macaulay posets.** Let P be a ranked poset with the associated partial order $\leq$, and let $N_i(P)$ be its i-th level. (For ranked poset see [9].) For any $x \in N_i(P)$ the *shadow* of x is the set $\Delta\, x$ of all $y \in N_{i-1}(P)$ such that $y \leq x$, and for $X \subseteq N_i(P)$ the shadow $\Delta\, X$ is the union of all $\Delta\, x$ with $x \in X$.

P is said to be a *Macaulay poset* if there exists a linear order $\prec$ of its elements such that for all natural numbers i and $1 \leq m \leq |N_i(P)|$ the set $C(m, N_i(P))$ of the (with respect to $\prec$) smallest m elements of $N_i(P)$ satisfies the two below conditions:

(1) $|\Delta\, C(m, N_i(P))| \leq |\Delta\, X|$ holds for all $X \subseteq N_i(P)$ with $|X| = m$.

(2) $\Delta\, C(m, N_i(P))$ consists of the smallest $|\Delta\, C(m, N_i(P))|$ elements of $N_{i-1}(P)$.

In the following we always assume that P is a Macaulay poset. For $X \subseteq N_i(P)$ we abbreviate the set of the smallest $|X|$ elements of $N_i(P)$ by $C(X, N_i(P))$ and call it the *compression* of X. We say that X is *compressed* if $C(X, N_i(P)) = X$. If there is no danger of ambiguity we write N_i, $C(X)$, and $C(m, i)$ instead of $N_i(P)$, $C(X, N_i(P))$, and $C(m, N_i(P))$, respectively.

As an example we mention that, by a theorem of Clements and Lindström [7], cartesian products of chains are Macaulay posets. These

Date: June 1996.

posets are known to posess also several other properties which we will introduce now.

1.2. **Little–submodularity.** We say that P is *little–submodular* if for all i and $1 \le m \le m' \le |N_i|$ the following inequality is fulfilled:

$$|\Delta\,(m,i)| \,+\, |\Delta\,(m',i)| \\ \ge \begin{cases} |\Delta\, C(m+m',i)| & \text{if } m+m' \le |N_i| \\ |N_{i-1}| \,+\, |\Delta\, C(m+m'-|N_i|,i)| & \text{if } m+m' > |N_i|. \end{cases}$$

The little–submodularity of chain products was proven by Clements [3] (for a simplified proof see [9]).

1.3. **Additivity.** $X \subseteq N_i$ is said to be a *segment* if X consists of (w.r.t. $\prec$) consecutive elements of N_i. Particularly, $C(X)$ is called *initial segment* and the set $L(X)$ of the greatest $|X|$ elements of N_i *final segment.*

Let $X \subseteq N_i$ be a segment and Y the set of those elements of N_i preceding each element of X. Then the *new–shadow* of X is defined by

$$\Delta_{new}\, X := \; \Delta\,(Y \cup X) \,-\, \Delta\, Y.$$

P is *additive* if for every segment X (and every i) the following inequality holds:

$$|\Delta_{new}\, C(X)| \;\ge\; |\Delta_{new}\, X| \;\ge\; |\Delta_{new}\, L(X)|.$$

The interesting observation that P is additive if and only if P is little–submodular is due to Engel [9].

1.4. **Shadow–increasingness.** The Macaulay poset P is called *shadow–increasing* if the inequality

$$|\Delta\, C(m,i-1)| \;\le\; |\Delta\, C(m,i)|$$

is satisfied for all i and $1 \le m \le \min\{\,|N_{i-1}|, |N_i|\,\}$.

Clements [3] showed that chain products are shadow–increasing. Another, very simple proof is contained in [6] where the authors make use of an idea of Kleitman. They embed a chain product P into another chain product Q of higher dimension and apply the additivity of Q in a very elegant way (see also [9]). We shall argue in a similar manner in section 2.

1.5. **An application.** Which antichain of given cardinality generates an ideal of minimum weight?

This question was answered for chain products (and the usually considered weight functions) by Clements [5] generalizing earlier results of Daykin [8] and Kleitman [12]. His result was again generalized significantly by Engel [9] who was able to answer the above question for the class of all Macaulay posets P such that P and its dual P^* are graded, little–submodular, and shadow–increasing, where the weight functions considered are nonnegative–valued, constant on the levels, and increasing with the levels. Since the dual of a chain product is again a chain product his theorem can be applied to these Macaulay posets.

We should mention that the dual P^* of P is the poset on the same set of elements given by $x \leq^* y \iff x \geq y$. It is well–known that the dual of a Macaulay poset is again a Macaulay poset (see Bezrukov [2]), where one can take the reverse of $\prec$ as the corresponding linear order $\prec^*$.

In the next section we will prove shadow–increasingness for another pair (P, P^*) which is known to posess the other properties for the above application.

2. Results

2.1. **Colored complexes.** Let n and $1 \leq k_1 \leq \cdots \leq k_n$ be natural numbers. Furthermore, for $j = 1, \ldots, n$ let the *color class* A_j be the set of the smallest k_j positive natural numbers which are contained in the j–th coset modulo n. The *colored complex* $Col(k_1, \ldots, k_n)$ consists of all subsets of the *ground set* $A := \bigcup_{j=1}^n A_j$ which meet every color class at most once. The corresponding partial order is the usual set inclusion, and the i–th level consists of all i–sets. $Col(k_1, \ldots, k_n)$ can be represented as the following cartesian product.

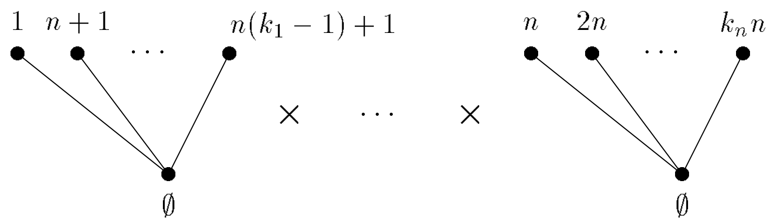

FIGURE 1. The colored complex $Col(k_1, \ldots, k_n)$.

Colored complexes were introduced by Frankl, Füredi, and Kalai [10], who showed for the special case $k_n - k_1 \leq 1$ that $Col(k_1, \ldots, k_n)$ is a Macaulay poset with the reverse lexicographic order as the associated linear order $\prec$. (The reverse lexicographic order may be defined by $B \prec C \iff \max[(B \cup C) - (B \cap C)] \in C$.) A simpler proof in the same case is due to London [17]. In the $k_1 = \ldots = k_n = 1$ case their result reduces to the famous Kruskal–Katona theorem [13, 11].

Engel [9] proved that their result remains valid without the restriction $k_n - k_1 \leq 1$ by observing that colored complexes are the duals of the star posets (to be defined in the next paragraph) for which the corresponding statement was known to be true. Using a notion from [10], we will refer to Engel's result as the *colored Kruskal–Katona theorem.*

The proof of the little–submodularity of $Col(k_1, \ldots, k_n)$ is also due to Engel [9].

Theorem 1. *Colored complexes are shadow–increasing.*

Proof. Let $P := Col(k_1, \ldots, k_n)$. Further let $2 \leq i \leq n$ and $1 \leq m \leq \min\{\,|N_{i-1}(P)|, |N_i(P)|\,\}$ be natural numbers. For brevity we define

$$\begin{aligned} \mathcal{F}_1 &:= \; C(m, N_{i-1}(P)), \\ \mathcal{F}_2 &:= \; C(m, N_i(P)). \end{aligned}$$

We have to show:

$$|\Delta\, \mathcal{F}_1| \;\leq\; |\Delta\, \mathcal{F}_2|.$$

Consider $Q := Col(k_1, \ldots, k_{n-1}, k_n, k_n)$. Let a be the largest element of its ground set and

$$\mathcal{F}_1' := \;\{F \cup \{a\} \mid F \in \mathcal{F}_1\}\,.$$

Obviously, $\mathcal{F}_2, \mathcal{F}_1' \subset N_i(Q)$. Moreover, $\mathcal{F}_1'$ is a segment which consists of the lexicographically smallest m elements of $N_i(Q)$ containing a. Hence the new–shadow of $\mathcal{F}_1'$ is given by

$$\Delta_{new}\, \mathcal{F}_1' \;=\; \{F \cup \{a\} \mid F \in \Delta\, \mathcal{F}_1\},$$

and by the additivity of Q and the colored Kruskal–Katona theorem we obtain

$$|\Delta\, \mathcal{F}_1| \;=\; |\Delta_{new}\, \mathcal{F}_1'| \;\leq\; |\Delta\, C(m, N_i(Q))| \;\leq\; |\Delta\, \mathcal{F}_2|. \qquad \square$$

2.2. **Star posets.** Let again n and $1 \leq k_1 \leq \cdots \leq k_n$ be natural numbers. The elements of the *star poset* $T(k_1, \ldots, k_n)$ are all vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfying $k_n - k_j \leq x_j \leq k_n$ for all j. With respect to the corresponding partial order $\boldsymbol{x} \leq \boldsymbol{y}$ holds, if and only if $\boldsymbol{x}$ can be obtained from $\boldsymbol{y}$ by replacing some entrees equal to k_n by other admissible entrees. The i-th level is formed by those elements having exactly i components equal to k_n. The star poset $T(k_1, \ldots, k_n)$ can be considered as the cartesian product given in figure 2.

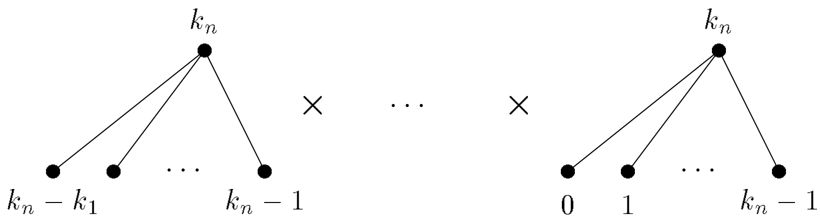

FIGURE 2. The star poset $T(k_1, \ldots, k_n)$.

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in T(k_1, \ldots, k_n)$ and $j \in \{1, \ldots, k_n\}$ let us now introduce the sets $\boldsymbol{x}(j) := \{\ell \mid x_\ell = j\}$. The linear order $\prec$ is defined by $\boldsymbol{x} \prec \boldsymbol{y}$ if with respect to the lexicographic order (see paragraph 2.1) $\boldsymbol{y}(j)$ is smaller than $\boldsymbol{x}(j)$, where j is the smallest number satisfying $\boldsymbol{x}(j) \neq \boldsymbol{y}(j)$.

$T(k_1, \ldots, k_n)$ is a Macaulay poset with the corresponding linear order $\prec$ (for a proof see [9] or [15]). Special cases of this statement have been proven earlier: In the $k_1 = k_n = 1$ case we again have the Kruskal–Katona theorem [13, 11], the case $k_1 = k_n = 2$ was esentially solved by Lindström [16], the $k_1 = k_n$ case has been settled by Leeb [14] and independently by Bezrukov [1].

Clements [4] showed that star posets are additive, his proof was simplified by Engel [9] who used the embedding idea of Kleitman.

Theorem 2. *Star posets are shadow–increasing.*

PROOF. Let $P = T(k_1, \ldots, k_n)$. The numbers i, m are like in the the proof of theorem 1, and the sets F_1, F_2 are defined by

$$F_1 := \; C(m, N_{i-1}(P)),$$
$$F_2 := \; C(m, N_i(P)).$$

Consider $Q = T(k_1, \ldots, k_{n-1}, k_n, k_n)$. For $\boldsymbol{x} = (x_1, \ldots, x_n) \in P$ and $j \in \{0, 1, \ldots, k_n\}$ denote the vector $(x_1, \ldots, x_n, j)$ by $(\boldsymbol{x}, j)$. Further we define

$$F_1' := \; \{(\boldsymbol{x}, k_n) \mid \boldsymbol{x} \in F_1\}.$$

Let $\boldsymbol{e}$ be the greatest element of F_1' with respect to $\preceq$, and for $j \in \{0, 1, \ldots, k_n\}$ we introduce the notations

$$\begin{aligned} G'_j &:= \{(\boldsymbol{x}, j) \in N_i(Q) \mid (\boldsymbol{x}, j) \preceq \boldsymbol{e}\}, \\ G_j &:= \{\boldsymbol{x} \mid (\boldsymbol{x}, j) \in G'_j\}, \\ G &:= \textstyle\bigcup_{j=1}^{k_n} G'_j. \end{aligned}$$

As immediate consequences of these definitions we obtain that G is a segment in $N_i(Q)$, where $G \cap G'_0 = \emptyset$, and $G \cup G'_0$ is compressed. Therefore the new–shadow of G consists of all elements of ΔG whose last components are different from 0. Since $(\boldsymbol{x}, j) \in \Delta G'_j$ for all $\boldsymbol{x} \in \Delta G_j$, we may infer

$$|\Delta_{new} G| \;\geq\; \textstyle\sum_{j=1}^{k_n} |\Delta G_j|. \tag{1}$$

By the definition of $\prec$, it is not difficult to observe that $G_1, \ldots, G_{k_n-1}$ are compressed subsets of $N_i(P)$. Furthermore, $G_{k_n} = F_1$ holds. Introducing $g_j := |G_j|$ for $j = 1, \ldots, k_n - 1$, it follows that

$$\textstyle\sum_{j=1}^{k_n} |\Delta G_j| \;=\; \sum_{j=1}^{k_n-1} |\Delta C(g_j, N_i(P))| + |\Delta F_1|. \tag{2}$$

$C(G, N_i(Q))$ is the initial segment of $N_i(Q)$ of size $\sum_{j=1}^{k_n-1} g_j + m$, and the little–submodularity of Q implies

$$|\Delta C(G, N_i(Q))| \leq \textstyle\sum_{j=1}^{k_n-1} |\Delta C(g_j, N_i(Q))| + |\Delta C(m, N_i(Q))|. \tag{3}$$

On the other hand, the additivity of Q yields

$$|\Delta C(G, N_i(Q))| \;\geq\; |\Delta_{new} G|. \tag{4}$$

Putting together (1), (2), (3), and (4) gives

$$\begin{aligned} &\textstyle\sum_{j=1}^{k_n-1} |\Delta C(g_j, N_i(P))| + |\Delta F_1| \\ &\qquad \leq \textstyle\sum_{j=1}^{k_n-1} |\Delta C(g_j, N_i(Q))| + |\Delta C(m, N_i(Q))|. \end{aligned} \tag{5}$$

By the definition of $\prec$, for $j \in \{1, \ldots, k_n - 1\}$ the set $C(g_j, N_i(Q))$ consists of all $(\boldsymbol{x}, 0)$ with $\boldsymbol{x} \in C(g_j, N_i(P))$, and $C(m, N_i(Q))$ consists of all $(\boldsymbol{x}, 0)$ with $\boldsymbol{x} \in F_2$. This implies

$$|\Delta C(g_j, N_i(P))| \;=\; |\Delta C(g_j, N_i(Q))| \qquad \forall j \in \{1, \ldots, k_n - 1\} \tag{6}$$

and

$$|\Delta C(m, N_i(Q))| \;=\; |\Delta F_2|. \tag{7}$$

Now the assertion follows by (5), (6), and (7). $\square$

Acknowledgement

The author wishes to thank Professor Konrad Engel for pointing his attention to the subject and for several useful discussions.

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