Some Error Estimates for Periodic Interpolation on Full and Sparse Grids

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Abstract. We give a unified approach to error estimates for periodic interpolation on full and sparse grids in certain Sobolev spaces. We impose 'periodic' Strang–Fix conditions on the underlying functions in order to obtain error bounds with explicit constants.

§1. Introduction

The approximation and interpolation of bivariate periodic functions have been studied for some time. While periodic interpolation by translates on full grids is well investigated for several function spaces, especially Sobolev spaces $H_2^\alpha(\mathbb{T}^2)$ (see e.g. [2,3,8]), error estimates for periodic interpolation on sparse grids are studied for functions from Korobov spaces $E^\alpha(\mathbb{T}^2)$ (see e.g. [1,4]).

In this paper, we give a unified approach to error estimates for periodic interpolation on full and sparse grids for functions from the usual Sobolev spaces $H_2^\alpha(\mathbb{T}^2)$ and Sobolev spaces $S_{2,2}^{\alpha,\beta}H(\mathbb{T}^2)$ with dominating mixed smoothness. We consider tensor product interpolation and j-th order blending interpolation. Applying the concept of 'periodic' Strang–Fix conditions on the underlying univariate fundamental interpolant, we are able to give error estimates with explicit constants.

§2. Univariate Periodic Interpolation by Translates

Let $d \in \mathbb{N}$ be fixed. Put $d_j := 2^j d,\ j \in \mathbb{N}_0$. We consider continuous 2π-periodic functions Λ_j defined as fundamental interpolants on the equidistant interpolation grid $\mathcal{T}_j := \{t_{j,k} := 2\pi k/d_j \ :\ k = 0, \ldots, d_j - 1\}$, that means, $\Lambda_j(t_{j,k}) = \delta_{k,0},\ k = 0, 1, \ldots, d_j - 1$. The corresponding interpolation operator L_j is given as

$$L_j f := \sum_{k=0}^{d_j-1} f(t_{j,k})\, \Lambda_j(\cdot - t_{j,k}).$$

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Furthermore, we assume the functions Λ_j, $j \in \mathbb{N}_0$ satisfy $\operatorname{Im} L_j \subset \operatorname{Im} L_{j+1}$. Let the remainder of an arbitrary operator P be defined as $P^c := I - P$, where I denotes the identity operator in the function space under consideration. For instance, we investigate the error for functions from Sobolev spaces of order $\alpha \geq 0$ of periodic functions

$$H_2^\alpha(\mathbb{T}) := \{f \in L_2(\mathbb{T}) \ : \ \|f\|_{H_2^\alpha(\mathbb{T})} < \infty\}$$

with

$$\|f\|^2_{H_2^\alpha(\mathbb{T})} := \sum_{k \in \mathbb{Z}} (1+k^2)^\alpha \, |c_k(f)|^2,$$

where $c_k(f) := \langle f, e^{ik\cdot}\rangle_{L_2(\mathbb{T})}$ denote the Fourier coefficients. Let I_j for $j \in \mathbb{N}_0$ denote the index set $I_j := \{k \in \mathbb{Z} \ : \ -d_j/2 < k \leq d_j/2\}$.

To obtain error estimates in the case of multivariate periodic interpolation by translates, the 'periodic' Strang–Fix conditions were introduced in [2]. In our setting the univariate fundamental interpolants Λ_j $(j \in \mathbb{N}_0)$ satisfy the *'periodic' Strang–Fix conditions of order* $m > 0$, if for $k \in I_j$ and all $j \in \mathbb{N}_0$ the following inequalities hold:

$$\begin{aligned} |1 - d_j c_k(\Lambda_j)| &\leq a_0 \left|\frac{2\pi k}{d_j}\right|^m \\ |d_j c_{k+d_j\ell}(\Lambda_j)| &\leq a_\ell \left|\frac{2\pi k}{d_j}\right|^m, \qquad \ell \in \mathbb{Z}\backslash\{0\}, \end{aligned} \tag{1}$$

where $\{a_\ell\}_{\ell \in \mathbb{Z}}$ with $\gamma_1^2 := \sum_{\ell \in \mathbb{Z}} a_\ell^2 < \infty$ is a series of non-negative constants independent of j.

Applying the estimates for multivariate periodic interpolation by translates proved in [2,8], we obtain in our special case the following theorem.

Theorem 1. *Let $f \in H_2^\alpha(\mathbb{T})$ with $\alpha > 1/2$ be given. Further, let the fundamental interpolants Λ_j $(j \in \mathbb{N}_0)$ with $\{c_k(\Lambda_j)\}_{k \in \mathbb{Z}} \in \ell_1(\mathbb{Z})$ satisfy the 'periodic' Strang–Fix conditions* (1) *of order m. Let $\varrho := \min\{m, \alpha\}$. Then the following estimate holds*

$$\|f - L_j f\|_{L_2(\mathbb{T})} \leq \mathrm{c}_\varrho \, d_j^{-\varrho} \, \|f\|_{H_2^\alpha(\mathbb{T})},$$

where

$$\mathrm{c}_\varrho := (2\pi)^\varrho \begin{cases} \gamma_1 + \pi^{-\alpha} + \gamma_2\gamma_3 & \text{for } \alpha \geq m, \\ (\sqrt{2}\,\pi)^{m-\alpha}\gamma_1 + \pi^{-\alpha} + \gamma_2\,\gamma_3 & \text{for } \alpha < m. \end{cases}$$

The constant γ_1 is given by (1) *and γ_2, γ_3 are defined by*

$$\gamma_2^2 := \sup_{j \in \mathbb{N}_0} d_j^2 \max_{k \in I_j} \sum_{\ell \in \mathbb{Z}} |c_{k+d_j\ell}(\Lambda_j)|^2, \quad \gamma_3^2 := \max_{k \in I_j} \sum_{\ell \in \mathbb{Z}\backslash\{0\}} |2\pi(k/d_j + \ell)|^{-2\alpha}.$$

All constants are independent of j.

Remark: The constant γ_3 is independent of d_j ($j \in \mathbb{N}_0$), because for $\ell \in \mathbb{Z}\setminus\{0\}$ we obtain $\max_{k\in I_j} |2\pi(k/d_j+\ell)|^{-2\alpha} = (\pi(2|\ell|-1))^{-2\alpha}$. In [8] the estimate $\gamma_3^2 \le (\pi^{2\alpha})^{-1}(4\alpha-1)(2\alpha-1)^{-1}$ was given for $\alpha > 1/2$. If $\alpha \in \mathbb{N}$, then we get the sharper estimate $\gamma_3^2 \le (2^{2\alpha}-1)((2\alpha)!)^{-1}B_\alpha$, where B_α denotes the Bernoulli number.

Corollary 2. *Assume the fundamental interpolants Λ_j with $\{c_k(\Lambda_j)\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ ($j \in \mathbb{N}_0$) satisfy the 'periodic' Strang–Fix conditions* (1) *of order m. Let $\alpha > 1/2$, $\varrho := \min\{m,\alpha\}$. Then*

$$\|L_j^c\|_{H_2^\alpha(\mathbb{T})\to L_2(\mathbb{T})} \le \mathrm{c}_\varrho\, d_j^{-\varrho}$$

with c_ϱ defined in Theorem 1.

§3. Examples

Lagrange functions constructed from B–Splines or trigonometric polynomials are often used for bivariate interpolation on full and sparse grids which we consider in the next section. Therefore, we want to investigate the 'periodic' Strang–Fix conditions (1) for B–Splines and special de la Vallée Poussin means. The constant γ_1 we obtain here occurs in the error estimates of Theorem 1 for univariate interpolation as well as those of Theorem 5 and 6 for the bivariate case.

Let $M_{m,j}$ ($j \in \mathbb{N}_0$) denote the 2π–periodic centred B–Spline of order $m \in \mathbb{N}$ given by its Fourier coefficients as $c_k(M_{m,j}) = d_j^{-1}(\mathrm{sinc}(\pi k/d_j))^m$ ($k \in \mathbb{Z}$, $j \in \mathbb{N}_0$) with $\mathrm{sinc}\, t := \sin t/t$. Then the corresponding fundamental interpolants Λ_j are known [7] to be

$$c_k(\Lambda_j) = \frac{c_k(M_{m,j})}{\sum_{\ell\in I_j} M_{m,j}(2\pi\ell/d_j)e^{-2\pi i\, k\ell/d_j}}, \qquad k \in \mathbb{Z},\ j \in \mathbb{N}_0. \tag{2}$$

The following 'periodic' Strang–Fix conditions for B–Splines were shown in [8]. The order of the Strang–Fix conditions coincides with the known approximation order of B–Splines. Theorem 3 also provides explicit constants in the error estimates.

Theorem 3. *Let Λ_j ($j \in \mathbb{N}_0$) be the fundamental interpolant of the 2π–periodic B–Spline of order $m \in \mathbb{N}$ given by* (2)*. Then Λ_j satisfies the 'periodic' Strang–Fix conditions* (1) *of order m with the constants*

$$a_0 = \begin{cases} 2^{-(m+1)} & \text{for } m \text{ odd,} \\ (2(2^m-1))^{-1} & \text{for } m \text{ even,} \end{cases}$$

$$a_\ell = \frac{1}{\pi^m(2|\ell|-1)^m} \times \begin{cases} 1 & \text{for } m = 1, \\ \dfrac{(m-1)!}{E_{(m-1)/2}} & \text{for } m > 1 \text{ odd,} \\ \dfrac{m!}{2^m(2^m-1)B_{m/2}} & \text{for } m \text{ even} \end{cases}$$

and

$$\gamma_1^2 = \begin{cases} \dfrac{5}{16} & \text{for } m = 1, \\ \dfrac{1}{2^{2(m+1)}} + \left(\dfrac{(m-1)!}{E_{(m-1)/2}}\right)^2 \dfrac{2^{2m}-1}{(2m)!} B_m & \text{for } m > 1 \text{ odd}, \\ \dfrac{1}{4(2^m-1)^2} + \left(\dfrac{m!}{2^m(2^m-1)B_{m/2}}\right)^2 \dfrac{2^{2m}-1}{(2m)!} B_m & \text{for } m \text{ even}. \end{cases}$$

Here, B_n and E_n ($n \in \mathbb{N}$) denote the corresponding Bernoulli and Euler number, respectively.

The interpolation by trigonometric polynomials may serve as another example. For $N, M \in \mathbb{N}, N > M$, the de la Vallée Poussin means φ_N^M of the Dirichlet kernel are given as

$$\varphi_N^M(x) := \frac{1}{4MN} \sum_{\ell=N-M}^{N+M-1} \sum_{k=-\ell}^{\ell} e^{ikx}\,.$$

For $j \in \mathbb{N}_0$, we define the fundamental polynomial Λ_j as a special de la Vallée Poussin mean

$$\Lambda_j := \begin{cases} 1 & \text{for } j = 0,\ d = 1, \\ \varphi_{N_j}^{M_j} & \text{otherwise}, \end{cases} \tag{3}$$

where

$$N_j := d\,2^{j-1} \quad \text{and} \quad M_j := \begin{cases} 2^{j-1-\lambda} & \text{for } j > \lambda, \\ 1 & \text{for } j \le \lambda, \end{cases} \qquad \lambda \in \mathbb{N}, \lambda \ge 2.$$

Theorem 4. *Let Λ_j ($j \in \mathbb{N}_0$) be the special de la Vallée Poussin mean given by* (3). *Then Λ_j satisfies the 'periodic' Strang–Fix conditions* (1) *of order m for arbitrary m with the constants*

$$a_\ell = \begin{cases} (4/3\pi)^m/2 & \text{for } \ell = -1, 0, 1, \\ 0 & \text{otherwise} \end{cases}$$

and $\gamma_1^2 = 3\,(4/3\pi)^{2m}/4$.

§4. Bivariate Periodic Interpolation

Starting from univariate interpolation, we want to investigate bivariate interpolation on full and sparse grids. Therefore, we define the corresponding interpolation projectors. The tensor product operator $L_j \otimes L_k$ realizes the interpolation on a full grid $\mathcal{T}_j \times \mathcal{T}_k$ with $d^2 2^{j+k}$ interpolation points. The j-th order Boolean sum operator is defined by

$$B_j := \bigoplus_{r=0}^{j} L_r \otimes L_{j-r},$$

where $A \oplus B := A + B - AB$. The j-th order Boolean sum B_j is an interpolation projector on the sparse grid $\bigcup_{r=0}^{j} \mathcal{T}_r \times \mathcal{T}_{j-r}$ which contains $d^2(j2^{j-1} + 2^{j+1})$ interpolation points.

We want to measure the error of interpolation for functions from certain Sobolev spaces. With the Fourier coefficients $c_{k,\ell}(f) := \langle f, e^{ik\cdot} \otimes e^{i\ell\cdot} \rangle_{L_2(\mathbb{T}^2)}$, the usual Sobolev space of order $\alpha \geq 0$ of bivariate periodic functions is defined by

$$H_2^\alpha(\mathbb{T}^2) := \{f \in L_2(\mathbb{T}^2) \; : \; \|f\|_{H_2^\alpha(\mathbb{T}^2)} < \infty\}$$

with

$$\|f\|^2_{H_2^\alpha(\mathbb{T}^2)} := \sum_{k,\ell \in \mathbb{Z}} (1 + k^2 + \ell^2)^\alpha \, |c_{k,\ell}(f)|^2.$$

Another important function space is the Sobolev space of order (α, β), $\alpha, \beta \geq 0$ of bivariate periodic functions with mixed smoothness. It is defined (see [9]) as

$$S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2) := \{f \in L_2(\mathbb{T}^2) \; : \; \|f\|_{S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2)} < \infty\}$$

with

$$\|f\|^2_{S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2)} := \sum_{k,\ell \in \mathbb{Z}} (1 + k^2)^\alpha \, (1 + \ell^2)^\beta \, |c_{k,\ell}(f)|^2.$$

The spaces $S_{2,2}^{\alpha,\alpha} H(\mathbb{T}^2)$ are strongly related to the Korobov spaces [6]

$$E^\alpha(\mathbb{T}^2) := \Big\{ f \in L_2(\mathbb{T}^2) \; : \; |c_{k,\ell}(f)| = \mathcal{O}\left(((1 + |k|)(1 + |\ell|))^{-\alpha} \right), \\ \text{for } |k|, |\ell| \longrightarrow \infty \Big\},$$

which are often used in error estimates for j-th order blending interpolation (e.g. [1,4]) and numerical integration based on number theoretical methods (e.g. [6]). It holds that

$$S_{2,2}^{\alpha,\alpha} H(\mathbb{T}^2) \subset E^\alpha(\mathbb{T}^2) \subset S_{2,2}^{\alpha-1/2-\varepsilon, \alpha-1/2-\varepsilon} H(\mathbb{T}^2)$$

for arbitrary $\varepsilon > 0$. With the help of these definitions, we obtain immediately $L_2(\mathbb{T}^2) = H_2^0(\mathbb{T}^2) = S_{2,2}^{0,0} H(\mathbb{T}^2)$ and $S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2) = H_2^\alpha(\mathbb{T}) \hat{\otimes} H_2^\beta(\mathbb{T}) \subset H_2^{\min\{\alpha,\beta\}}(\mathbb{T}^2) \subset L_2(\mathbb{T}^2)$.

Theorem 5. *Let $f \in S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2)$ with $\alpha, \beta > 1/2$ be given. Further, let the fundamental interpolants Λ_j ($j \in \mathbb{N}_0$) with $\{c_k(\Lambda_j)\}_{k \in \mathbb{Z}} \in \ell_1(\mathbb{Z})$ satisfy the 'periodic' Strang–Fix conditions* (1) *of order m. Let $\varrho := \min\{m, \alpha\}$, $\sigma := \min\{m, \beta\}$, $\tau := \min\{m, \alpha, \beta\}$, $\ell := \min\{\varrho j, \sigma k\}$. Then the following estimates hold:*

$$\|f - (L_j \otimes L_k) f\|_{L_2(\mathbb{T}^2)} \leq (\mathrm{c}_\varrho \, d_j^{-\varrho} + \mathrm{c}_\sigma \, d_k^{-\sigma} + \mathrm{c}_\varrho \, \mathrm{c}_\sigma \, d_j^{-\varrho} \, d_k^{-\sigma}) \, \|f\|_{S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2)},$$

$$\|f - B_j f\|_{L_2(\mathbb{T}^2)} \leq \big(\mathrm{c}_\varrho \, d_j^{-\varrho} + \mathrm{c}_\sigma \, d_j^{-\sigma} + \mathrm{c}_\varrho \, \mathrm{c}_\sigma \sum_{r=0}^{j} d_r^{-\varrho} \, d_{j-r}^{-\sigma} \\ + \mathrm{c}_\varrho \, \mathrm{c}_\sigma \sum_{r=0}^{j-1} d_r^{-\varrho} \, d_{j-r-1}^{-\sigma} \big) \, \|f\|_{S_{2,2}^{\alpha,\beta} H(\mathbb{T}^2)},$$

with $\mathrm{c}_\varrho, \mathrm{c}_\sigma$ *as defined in Theorem 1. Summarizing, we obtain*

$$\begin{aligned} \|f-(L_j\otimes L_k)f\|_{L_2(\mathbb{T}^2)} &\le \mathrm{c}_{\tau,T}\, d^{-\tau}\, 2^{-\ell}\, \|f\|_{S^{\alpha,\beta}_{2,2}H(\mathbb{T}^2)}\,, \\ \|f-B_jf\|_{L_2(\mathbb{T}^2)} &\le \mathrm{c}_{\tau,B}\, d_j^{-\tau}\, \|f\|_{S^{\alpha,\beta}_{2,2}H(\mathbb{T}^2)}\,, \end{aligned}$$

where

$$\mathrm{c}_{\tau,T} := \mathrm{c}_\varrho\, d^{\tau-\varrho}\, 2^{\ell-\varrho j} + \mathrm{c}_\sigma\, d^{\tau-\sigma}\, 2^{\ell-\sigma k} + \mathrm{c}_\varrho\, \mathrm{c}_\sigma\, d^{\tau-\varrho-\sigma}\, 2^{\ell-\varrho j-\sigma k}$$

$$\begin{aligned} \mathrm{c}_{\tau,B} = \mathrm{c}_{\tau,B}(j) := \mathrm{c}_\varrho\, d_j^{\tau-\varrho} + \mathrm{c}_\sigma\, d_j^{\tau-\sigma} + \mathrm{c}_\varrho\, \mathrm{c}_\sigma \sum_{r=0}^{j} d_j^{\tau}\, d_r^{-\varrho}\, d_{j-r}^{-\sigma} \\ + \mathrm{c}_\varrho\, \mathrm{c}_\sigma \sum_{r=0}^{j-1} d_j^{\tau}\, d_r^{-\varrho}\, d_{j-r-1}^{-\sigma}. \end{aligned}$$

We note that $\mathrm{c}_{\tau,B}(j)$ depends on j. For $\varrho = \sigma$, we obtain $\mathrm{c}_{\tau,B}(j) = \mathrm{c}_\varrho^2\, d^{-\varrho}(1+2^\varrho)j + 2\mathrm{c}_\varrho - \mathrm{c}_\varrho^2\, d^{-\varrho}\, 2^\varrho = \mathcal{O}(j)$. That means that in this case, we lose only a logarithmic term in approximation order if we compare the interpolation by B_j on a sparse grid with $\mathcal{O}(j2^j)$ points and the interpolation by $L_j \otimes L_j$ on a full grid with $\mathcal{O}(2^{2j})$ points.

Proof: The remainder of the tensor product interpolation operator can be written as

$$(L_j\otimes L_k)^c = L_j^c\otimes I + I\otimes L_k^c - L_j^c\otimes L_k^c.$$

Since $S^{\alpha,\beta}_{2,2}H(\mathbb{T}^2) = H_2^\alpha(\mathbb{T})\hat{\otimes}H_2^\beta(\mathbb{T})$ we can estimate the operator norm (compare [10, Chap. 8]) of the remainder

$$\begin{aligned} \|(L_j\otimes L_k)^c\|_{S^{\alpha,\beta}_{2,2}H(\mathbb{T}^2)\to L_2(\mathbb{T}^2)} \le \|L_j^c\|_{H_2^\alpha(\mathbb{T})\to L_2(\mathbb{T})} + \|L_k^c\|_{H_2^\beta(\mathbb{T})\to L_2(\mathbb{T})} \\ + \|L_j^c\|_{H_2^\alpha(\mathbb{T})\to L_2(\mathbb{T})}\, \|L_k^c\|_{H_2^\beta(\mathbb{T})\to L_2(\mathbb{T})}. \end{aligned}$$

Applying Corollary 2, we obtain the assertion for tensor product interpolation. The assertion for j-th order blending interpolation follows from the representation of the remainder of the j-th order Boolean sum (see [4, Chap. 1])

$$B_j^c = L_j^c\otimes I + I\otimes L_j^c - \sum_{r=0}^{j} L_r^c\otimes L_{j-r}^c + \sum_{r=0}^{j-1} L_r^c\otimes L_{j-r-1}^c$$

and the same arguments as before. ■

With the next theorem, we give error estimates for functions from Sobolev spaces $H_2^\alpha(\mathbb{T}^2)$. The estimate for tensor product interpolation can be derived from a result on multivariate interpolation by translates [2,8]. Using the tensor product structure of the involved operators and function spaces, we can prove it here in a shorter way and obtain a similar result for the j-th order blending interpolation.

Theorem 6. *Let $f \in H_2^\alpha(\mathbb{T}^2)$ with $\alpha > 1$ be given. Further, let the fundamental interpolants Λ_j $(j \in \mathbb{N}_0)$ with $\{c_k(\Lambda_j)\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ satisfy the 'periodic' Strang–Fix conditions* (1) *of order m. Let $\varrho := \min\{m, \alpha\}$, $\sigma := \min\{m, \alpha/2\}$, $\ell := \min\{j, k\}$. Then the following estimates hold:*

$$\|f - (L_j \otimes L_k)f\|_{L_2(\mathbb{T}^2)} \le \tilde{c}_{\varrho,T}\, d_\ell^{-\varrho}\, \|f\|_{H_2^\alpha(\mathbb{T}^2)},$$
$$\|f - B_j f\|_{L_2(\mathbb{T}^2)} \le \tilde{c}_{\varrho,B}\, d_j^{-\sigma}\, \|f\|_{H_2^\alpha(\mathbb{T}^2)},$$

where

$$\tilde{c}_{\varrho,T} := c_\varrho\, 2^{\varrho(\ell-j)} + c_\varrho\, 2^{\varrho(\ell-k)} + c_\sigma^2\, d^{\varrho-2\sigma}\, 2^{\varrho\ell-\sigma k-\sigma\ell},$$
$$\tilde{c}_{\varrho,B} = \tilde{c}_{\varrho,B}(j) := 2\, c_\varrho\, d_j^{\sigma-\varrho} + 2\, c_\sigma^2\, d^{-\sigma}(j + (j-1)\, 2^\sigma)$$

and c_ϱ is as in Theorem 1.

Proof: We will use that $H^\alpha(\mathbb{T}^2) = S_{2,2}^{\alpha,0}H(\mathbb{T}^2) \cap S_{2,2}^{0,\alpha}H(\mathbb{T}^2)$ (compare [5]) with the following norm equivalence

$$2^{1-3\alpha}\, \|\cdot\|_{H^\alpha(\mathbb{T}^2)}^2 \le \|\cdot\|_{S_{2,2}^{\alpha,0}H(\mathbb{T}^2)}^2 + \|\cdot\|_{S_{2,2}^{0,\alpha}H(\mathbb{T}^2)}^2 \le 2\, \|\cdot\|_{H^\alpha(\mathbb{T}^2)}^2$$

and the inequalities

$$\|\cdot\|_{S_{2,2}^{\alpha,0}H(\mathbb{T}^2)} \le \|\cdot\|_{H^\alpha(\mathbb{T}^2)} \qquad \text{and} \qquad \|\cdot\|_{S_{2,2}^{0,\alpha}H(\mathbb{T}^2)} \le \|\cdot\|_{H^\alpha(\mathbb{T}^2)}\,.$$

Furthermore, $H^\alpha(\mathbb{T}^2) \subset S_{2,2}^{\alpha/2,\alpha/2}H(\mathbb{T}^2)$ and

$$\|\cdot\|_{S_{2,2}^{\alpha/2,\alpha/2}H(\mathbb{T}^2)} \le \|\cdot\|_{H^\alpha(\mathbb{T}^2)}\,.$$

This leads to the following estimates

$$\begin{aligned}
&\|f - (L_j \otimes L_k)f\|_{L_2(\mathbb{T}^2)} \\
&\qquad = \|(L_j^c \otimes I + I \otimes L_k^c - (L_j^c \otimes I)(I \otimes L_k^c))\, f\|_{L_2(\mathbb{T}^2)} \\
&\qquad \le (\|L_j^c\|_{H_2^\alpha(\mathbb{T})\to L_2(\mathbb{T})} + \|L_k^c\|_{H_2^\alpha(\mathbb{T})\to L_2(\mathbb{T})} \\
&\qquad\quad + \|L_j^c\|_{H_2^{\alpha/2}(\mathbb{T})\to L_2(\mathbb{T})}\, \|L_k^c\|_{H_2^{\alpha/2}(\mathbb{T})\to L_2(\mathbb{T})})\, \|f\|_{H_2^\alpha(\mathbb{T}^2)}.
\end{aligned}$$

Applying Corollary 2, we obtain the assertion in the case of tensor products. The Boolean sum case can be handled analogously. ■

While it might be quite useful in tensor product interpolation to use different grids $\mathcal{T}_j$ and $\mathcal{T}_k$ in x- and y-direction, respectively, for $f \in S_{2,2}^{\alpha,\beta}H(\mathbb{T}^2)$, the previous result shows that it doesn't make any sense for $f \in H_2^\alpha(\mathbb{T}^2)$. If we interpolate on the grid $\mathcal{T}_j \times \mathcal{T}_j$ we achieve the same approximation order already for functions in $H_2^\alpha(\mathbb{T}^2)$ as we have for the smoother functions in $S_{2,2}^{\alpha,\alpha}H(\mathbb{T}^2)$.

Furthermore, we can use j-th order blending interpolation with essentially less interpolation knots than for tensor product interpolation. We obtain reasonable approximation order only for functions which are smooth enough, but for the classes $S_{2,2}^{\alpha,\alpha}H(\mathbb{T}^2)$, the approximation order for interpolation on sparse and full grid only differ in a logarithmic term.

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