On Stability of Scaling Vectors

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Abstract. The paper generalizes Lawton's criteria for scaling vectors by means of Kronecker products. Necessary and sufficient conditions for the stability (orthonormality) of scaling vectors are provided in terms of their two-scale symbols. The paper is based on the results of Shen [14].

§1. Introduction

Usually, the construction of multiwavelets is based on a multiresolution analysis (MRA) with higher multiplicity. In order to generate the MRA, a refinable function vector $\mathbf{\Phi} = (\phi_0, \ldots, \phi_{r-1})^T$ ($r \in \mathbb{N}$, $r \geq 1$) is needed, such that $\mathcal{B}(\mathbf{\Phi}) := \{\phi_\nu(\cdot - l) : l \in \mathbb{Z}, \nu = 0, \ldots, r-1\}$ forms an L^2-stable basis of its span. Moreover, the components of $\mathbf{\Phi}$ are often desired to be compactly supported, regular and symmetric (or antisymmetric) such that multiwavelets with similar properties can be derived.

The compactly supported scaling vector $\mathbf{\Phi}$ can be considered as a solution vector of a matrix refinement equation

$$\mathbf{\Phi}(x) = \sum_{l=0}^{N} \boldsymbol{P}_l \, \mathbf{\Phi}(2x - l), \tag{1}$$

where $\boldsymbol{P}_l$ are complex $(r \times r)$-coefficient matrices. Hence, the question occurs of how the L^2-stability of $\mathcal{B}(\mathbf{\Phi})$ for the solution vector $\mathbf{\Phi}$ of (1) can be ensured, just by appropriate choice of the (two–scale) symbol

$$\boldsymbol{P}(\omega) := \sum_{l=0}^{N} \boldsymbol{P}_l \, e^{-i\omega l}.$$

This problem has also been studied very recently in [3,4,9,14,15].
By Fourier transform of (1), we have

$$\hat{\mathbf{\Phi}}(\omega) = \boldsymbol{P}(\frac{\omega}{2}) \, \hat{\mathbf{\Phi}}(\frac{\omega}{2}), \tag{2}$$

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where $\hat{\mathbf{\Phi}}$ is taken componentwisely, *i.e.*, $\hat{\mathbf{\Phi}}(\omega) := (\hat{\phi}_0(\omega), \ldots, \hat{\phi}_{r-1}(\omega))^T$ with $\hat{\phi}_\nu(\omega) := \int_{-\infty}^{\infty} \phi_\nu(x)\, e^{-i\omega x}\, \mathrm{d}x$ $(\nu = 0, \ldots, r-1)$.

We say that a function vector $\mathbf{\Phi}$ is L^2*-stable* if there are constants $0 < A \le B < \infty$, such that

$$A \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^T \overline{\boldsymbol{c}}_l \le \| \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^T\, \mathbf{\Phi}(\cdot - l)\|_{L^2}^2 \le B \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^T \overline{\boldsymbol{c}}_l \tag{3}$$

for any vector sequence $\{\boldsymbol{c}_l\}_{l\in\mathbb{Z}} \in l_2^r$. Here l_2^r denotes the set of sequences of vectors $(\boldsymbol{c}_l)_{l\in\mathbb{Z}}$ $(\boldsymbol{c}_l \in \mathbb{C}^r)$ with $\sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^T \overline{\boldsymbol{c}_l} < \infty$. Introducing the autocorrelation symbol

$$\mathbf{\Omega}(\omega) := \sum_{l=-\infty}^{\infty} \hat{\mathbf{\Phi}}(\omega + 2\pi l)\, \hat{\mathbf{\Phi}}(\omega + 2\pi l)^\star \tag{4}$$

with $\hat{\mathbf{\Phi}}(\omega)^\star := \overline{\hat{\mathbf{\Phi}}(\omega)}^T$, (3) is equivalent with the following condition (see [6]):

$$0 < A \le \rho_{\min}(\mathbf{\Omega}) \le \rho_{\max}(\mathbf{\Omega}) \le B < \infty,$$

where

$$\rho_{\min}(\mathbf{\Omega}) := \min_{\omega\in[-\pi,\pi)} \{|\lambda| : \det\,(\mathbf{\Omega}(\omega) - \lambda \boldsymbol{I}) = 0\},$$

$$\rho_{\max}(\mathbf{\Omega}) := \max_{\omega\in[-\pi,\pi)} \{|\lambda| : \det\,(\mathbf{\Omega}(\omega) - \lambda \boldsymbol{I}) = 0\}.$$

Here $\boldsymbol{I}$ denotes the unit matrix of size r. In particular, for compactly supported $\mathbf{\Phi}$, supp $\mathbf{\Phi} \subseteq [0, N]$, $\mathbf{\Omega}(\omega)$ is a matrix of trigonometric polynomials of degree (at most) $N-1$, and (3) is already satisfied if det $\mathbf{\Omega}(\omega) \ne 0$ for all $\omega \in [-\pi, \pi]$. The function vector $\mathbf{\Phi}$ is called *orthonormal* if (3) is satisfied with constants $A = B = 1$, or equivalently, if the autocorrelation symbol is the unit matrix, i.e., $\mathbf{\Omega}(\omega) = \boldsymbol{I}$. Applying the refinement equation (2) in (4), we find

$$\mathbf{\Omega}(2\omega) = \boldsymbol{P}(\omega)\, \mathbf{\Omega}(\omega)\, \boldsymbol{P}(\omega)^\star + \boldsymbol{P}(\omega+\pi)\, \mathbf{\Omega}(\omega+\pi)\, \boldsymbol{P}(\omega+\pi)^\star. \tag{5}$$

Hence, the orthonormality of $\mathbf{\Phi}$ implies that, for all $\omega \in [-\pi, \pi]$,

$$\boldsymbol{P}(\omega)\, \boldsymbol{P}(\omega)^\star + \boldsymbol{P}(\omega+\pi)\, \boldsymbol{P}(\omega+\pi)^\star = \boldsymbol{I}. \tag{6}$$

Analogously, a necessary condition on $\boldsymbol{P}(\omega)$ for L^2-stability of $\mathbf{\Phi}$ is that $\boldsymbol{P}(\omega)\, \boldsymbol{P}(\omega)^\star + \boldsymbol{P}(\omega+\pi)\, \boldsymbol{P}(\omega+\pi)^\star$ is positive definite for all $\omega \in [-\pi, \pi]$, i.e.,

$$\boldsymbol{y}^T\, (\boldsymbol{P}(\omega)\, \boldsymbol{P}(\omega)^\star + \boldsymbol{P}(\omega+\pi)\, \boldsymbol{P}(\omega+\pi)^\star)\, \overline{\boldsymbol{y}} > 0 \quad (\boldsymbol{y} \in \mathbb{R}^r,\, \boldsymbol{y} \ne 0) \tag{6'}$$

(see e.g. [6,8,9]). The condition (6) (or (6)') is known to be not sufficient to ensure that $\mathbf{\Phi}$ is orthonormal (L^2-stable). In the case $r = 1$, conditions could be given, being necessary and sufficient for orthonomality (or L^2-stability) of $\mathbf{\Phi}$ (see e.g. [1,5,11,13]).

The purpose of this paper is to present a generalization of Lawton's condition (see [5,13]) for the matrix coefficients of the symbol $\boldsymbol{P}(\omega)$, such that the corresponding solution vector $\mathbf{\Phi}$ of (1) is orthonormal and L^2-stable, respectively. We fundamentally use the results in [14] on the transfer operator corresponding to $\boldsymbol{P}(\omega)$.

§2. Generalization of Lawton's condition

For a square matrix $\boldsymbol{M}$ (or a linear operator) let us introduce the following

Condition E. *The spectral radius of $\boldsymbol{M}$ is less than or equal to 1, i.e. $\rho(\boldsymbol{M}) \leq 1$, and 1 is the only eigenvalue of $\boldsymbol{M}$ on the unit circle. Moreover, 1 is a simple eigenvalue.*

As shown in [4,10], we have:

Proposition 1. *Let $\boldsymbol{\Phi}$ be a stable L_1-solution vector of (1). Then for the corresponding symbol $\boldsymbol{P}(\omega)$ we have:*

a) *$\boldsymbol{P}(0)$ satisfies Condition E.*

b) *The solution vector $\boldsymbol{\Phi}$ provides approximation order 1, i.e., we have*

$$\boldsymbol{y}^T \sum_{l=-\infty}^{\infty} \boldsymbol{\Phi}(\cdot - l) = c,$$

where $\boldsymbol{y}$ is a left eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1, and c is a nonvanishing constant. Equivalently, we have $\boldsymbol{y}^T \boldsymbol{P}(0) = \boldsymbol{y}^T$ and $\boldsymbol{y}^T \boldsymbol{P}(\pi) = \boldsymbol{0}^T$.

Observe that the necessary conditions for $\boldsymbol{P}(\omega)$ in Proposition 1 are also assumed in [14]; there they are called *basic conditions.*

Let $\boldsymbol{P}(\omega)$ satisfy the basic conditions, and let $\boldsymbol{a}$ be a right eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1. Then the infinite product

$$\hat{\boldsymbol{\Phi}}(\omega) := \lim_{L\to\infty} \prod_{j=1}^{L} \boldsymbol{P}\left(\frac{\omega}{2^j}\right) \boldsymbol{a}$$

converges uniformly on compact sets, and $\hat{\boldsymbol{\Phi}}$ is a solution of (2) (see [2,7,8]). Hence, if we speak about a solution vector of (1) or (2) corresponding to a symbol $\boldsymbol{P}(\omega)$, we mean the vector determined by this infinite product. Further, if $\boldsymbol{\Phi}$ is supposed to be compactly supported, then $\operatorname{supp} \boldsymbol{\Phi} \subseteq [0, N]$.

We want to generalize the following result (see [5,13]):

Lawton's condition. *Let $r = 1$, and assume that $\boldsymbol{P}(\omega) = \frac{1}{2}\sum_{n=0}^{N} p_n e^{-i\omega n}$ is a trigonometric polynomial satisfying the condition (6) and $\boldsymbol{P}(0) = 1$. Then we have: The solution of (1) corresponding to $\boldsymbol{P}(\omega)$ is orthonormal if and only if the $(2N-1)\times(2N-1)$ matrix*

$$\boldsymbol{M} := \frac{1}{2}\left(\sum_{n=0}^{N} p_n \, \overline{p_{k-2l+n}}\right)_{k,l=-N+1}^{N-1}$$

possesses a simple eigenvalue 1.

Let $\mathbb{H} = \mathbb{H}_{N-1}$ be the space of trigonometric polynomials of degree at most $N-1$, i.e., the elements of $\mathbb{H}$ are of the form $h(\omega) = \sum_{-N+1}^{N-1} h_n \, e^{-i\omega n}$ $(h_n \in \mathbb{C})$. We introduce the following transfer operator $T : \mathbb{H}^{r\times r} \to \mathbb{H}^{r\times r}$,

$$T\boldsymbol{H}(\omega) := \boldsymbol{P}(\frac{\omega}{2})\,\boldsymbol{H}(\frac{\omega}{2})\,\boldsymbol{P}(\frac{\omega}{2})^\star + \boldsymbol{P}(\frac{\omega}{2}+\pi)\,\boldsymbol{H}(\frac{\omega}{2}+\pi)\boldsymbol{P}(\frac{\omega}{2}+\pi)^\star,$$

acting on $(r \times r)$-matrices $\boldsymbol{H}(\omega)$ with elements of $\mathbb{H}$ as entries. For $r = 1$, the following equivalence is known: Lawton's condition is satisfied if and only if the only trigonometric polynomials invariant under T are constants, or equivalently, if and only if T possesses a simple eigenvalue 1 (see [5], p. 189–190). Recently, Shen [14] presented the following generalization for scaling vectors:

Theorem 2 ([14]). *Let $\boldsymbol{P}(\omega)$ be an $(r \times r)$-matrix of trigonometric polynomials, $\boldsymbol{P}(\omega) = \frac{1}{2}\sum_{n=0}^{N} \boldsymbol{P}_n \, e^{-i\omega n}$ $(\boldsymbol{P}_n \in C^{r\times r})$, satisfying the basic conditions of Proposition 1, and let $\boldsymbol{\Phi}$ be a corresponding compactly supported solution vector of (1).*

(a) *If $\boldsymbol{P}$ satisfies (6) then $\boldsymbol{\Phi}$ is orthonormal if and only if the transfer operator T satisfies Condition E.*

(b) *The vector $\boldsymbol{\Phi}$ is L^2-stable if and only if the transfer operator T satisfies Condition E and the eigenmatrix corresponding to the eigenvalue 1 is nonsingular on $[-\pi, \pi]$.*

Observe that, by (5), the autocorrelation symbol $\boldsymbol{\Omega}(\omega) \in \mathbb{H}^{r\times r}$ is an eigenmatrix of the transfer operator T corresponding to the eigenvalue 1, and it is uniquely determined if the eigenvalue 1 of T is simple.

For applications, it seems to be also convenient to have a direct generalization of Lawton's criteria. To this end, let us introduce the Kronecker product of matrices $\boldsymbol{A} = (a_{ij})_{i,j=1}^{n} \in \mathbb{C}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{C}^{n\times n}$,

$$\boldsymbol{A} \otimes \boldsymbol{B} := \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{B} & \dots & a_{nn}\boldsymbol{B} \end{pmatrix}.$$

Further, for a matrix $\boldsymbol{A} = (\boldsymbol{A}_1, \dots, \boldsymbol{A}_n)$ with columns $\boldsymbol{A}_j \in \mathbb{C}^n$, let

$$\operatorname{vec} \boldsymbol{A} := \begin{pmatrix} \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_n \end{pmatrix} \in \mathbb{C}^{n^2}.$$

Then, it is well–known that, for $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X} \in \mathbb{C}^{n\times n}$,

$$\operatorname{vec}\,(\boldsymbol{A}\,\boldsymbol{X}\,\boldsymbol{B}) = (\boldsymbol{B}^T \otimes \boldsymbol{A}) \operatorname{vec} \boldsymbol{X}$$

(see e.g. [12], p. 410, Proposition 4). We find

Theorem 3. *Let $\boldsymbol{P}(\omega)$ be an $(r \times r)$-matrix of trigonometric polynomials of the form $\boldsymbol{P}(\omega) = \frac{1}{2}\sum_{n=0}^{N} \boldsymbol{P}_n e^{-i\omega n}$, and let $\boldsymbol{\Phi}$ be a corresponding compactly supported solution vector of (1). Assume that $\boldsymbol{P}(\omega)$ satisfies the basic conditions of Proposition 1.*

a) *Let $\boldsymbol{P}(\omega)$ satisfy (6). Then the scaling vector $\boldsymbol{\Phi}$ is orthonormal if and only if the matrix $\boldsymbol{M}$,*

$$\boldsymbol{M} := \left(\frac{1}{2} \sum_{n=0}^{N} \overline{\boldsymbol{P}_{n-2\mu+l}} \otimes \boldsymbol{P}_n \right)_{\mu,l=-N+1}^{N-1} \in C^{r^2(2N-1)}, \qquad (7)$$

has a simple eigenvalue 1.

b) *The scaling vector $\mathbf{\Phi}$ is L^2-stable if and only if we have: The matrix $\boldsymbol{M}$ in (7) satisfies the Condition E with a right eigenvector $\boldsymbol{w} = (\mathrm{vec}\ \boldsymbol{A}_l)_{l=-N+1}^{N-1}$ ($\boldsymbol{A}_l \in \boldsymbol{C}^{r\times r}$) corresponding to the eigenvalue 1, and the matrix polynomial $\sum_{l=-N+1}^{N-1} \boldsymbol{A}_l\, e^{-i\omega l}$ is nonsingular for all $\omega \in [-\pi, \pi]$.*

Proof: 1. We show that the condition E for the matrix $\boldsymbol{M}$ in (7) is equivalent with the condition E for the transfer operator T, i.e., that, for an appropriate basis of the finite dimensional space $\mathbb{H}^{r\times r}$, $\boldsymbol{M}$ is the representing matrix of T. Then the assertion follows from Theorem 2.

Using the properties of Kronecker product, it follows from the definition of T that

$$\begin{aligned}
&(\mathrm{vec}\ T)(\mathrm{vec}\ \boldsymbol{H})(w) = \\
&\qquad = \mathrm{vec}\ \left(\boldsymbol{P}(\frac{\omega}{2})\, \boldsymbol{H}(\frac{\omega}{2})\, \boldsymbol{P}(\frac{\omega}{2})^\star + \boldsymbol{P}(\frac{\omega}{2}+\pi)\, \boldsymbol{H}(\frac{\omega}{2}+\pi)\, \boldsymbol{P}(\frac{\omega}{2}+\pi)^\star\right) \\
&\qquad = \left(\overline{\boldsymbol{P}(\frac{\omega}{2})} \otimes \boldsymbol{P}(\frac{\omega}{2})\right) \mathrm{vec}\ \boldsymbol{H}(\frac{\omega}{2}) \\
&\qquad + \left(\overline{\boldsymbol{P}(\frac{\omega}{2}+\pi)} \otimes \boldsymbol{P}(\frac{\omega}{2}+\pi)\right) \mathrm{vec}\ \boldsymbol{H}(\frac{\omega}{2}+\pi),
\end{aligned}$$

i.e., vec T maps $\mathbb{H}^{r^2}$ into $\mathbb{H}^{r^2}$. For simplicity, let us assume that $\boldsymbol{P}_l = \boldsymbol{0}$ if $l < 0$ and $l > N$. Then we observe that

$$\begin{aligned}
&\overline{\boldsymbol{P}(\omega)} \otimes \boldsymbol{P}(\omega) + \overline{\boldsymbol{P}(\omega+\pi)} \otimes \boldsymbol{P}(\omega+\pi) \\
&= \frac{1}{4} \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \left(\overline{\boldsymbol{P}_n} \otimes \boldsymbol{P}_m + (-1)^{m-n} \left(\overline{\boldsymbol{P}_n} \otimes \boldsymbol{P}_m\right)\right) e^{-i\omega(m-n)} \\
&= \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\overline{\boldsymbol{P}_{m-2k}} \otimes \boldsymbol{P}_m\right)\, e^{-2i\omega k},
\end{aligned}$$

where we have used the substitution $n := m - 2k$. Analogously,

$$\overline{\boldsymbol{P}(\omega)} \otimes \boldsymbol{P}(\omega) - \overline{\boldsymbol{P}(\omega+\pi)} \otimes \boldsymbol{P}(\omega+\pi) = \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\overline{\boldsymbol{P}_{m-2k-1}} \otimes \boldsymbol{P}_m\right)\, e^{-i\omega(2k+1)}.$$

Let $\boldsymbol{e}_\nu := (\underbrace{0,\ldots,0}_{\nu-1}, 1, 0, \ldots, 0)^T \in \mathbb{R}^{r^2}$ ($\nu = 1, \ldots, r^2$) and consider the basis $\boldsymbol{u}_{\nu,l} := \boldsymbol{e}_\nu\, e^{-i\omega l}$ ($\nu = 1, \ldots, r^2$, $l = -N+1, \ldots, N-1$) of $\mathbb{H}^{r^2}$. Then, we find for $\boldsymbol{u}_{\nu,2l}$ ($\nu = 1, \ldots, r^2$, $2l \in \{-N+1, \ldots, N-1\}$) with $\mu := k + l$:

$$\begin{aligned}
(\mathrm{vec}\ T)\boldsymbol{u}_{\nu,2l}(\omega) &= \left(\overline{\boldsymbol{P}(\frac{\omega}{2})} \otimes \boldsymbol{P}(\frac{\omega}{2}) + \overline{\boldsymbol{P}(\frac{\omega}{2}+\pi)} \otimes \boldsymbol{P}(\frac{\omega}{2}+\pi)\right) \boldsymbol{e}_\nu\, e^{-i\omega 2l/2} \\
&= \frac{1}{2} \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(\overline{\boldsymbol{P}_{m-2k}} \otimes \boldsymbol{P}_m\right)\, e^{-i\omega k}\, \boldsymbol{e}_\nu\, e^{-i\omega l} \\
&= \sum_{\mu\in\mathbb{Z}} \left(\frac{1}{2} \sum_{m\in\mathbb{Z}} \overline{\boldsymbol{P}_{m-2\mu+2l}} \otimes \boldsymbol{P}_m\right) \boldsymbol{u}_{\nu,\mu},
\end{aligned}$$

and for $\boldsymbol{u}_{\nu,2l-1}$ ($\nu = 1,\ldots,r^2$, $2l-1 \in \{-N+1,\ldots,N-1\}$), in the same manner,

$$\begin{aligned}
(\text{vec } T)\boldsymbol{u}_{\nu,2l-1}(\omega) &= \left(\overline{\boldsymbol{P}(\frac{\omega}{2})} \otimes \boldsymbol{P}(\frac{\omega}{2}) - \overline{\boldsymbol{P}(\frac{\omega}{2}+\pi)} \otimes \boldsymbol{P}(\frac{\omega}{2}+\pi)\right) \boldsymbol{e}_\nu e^{-i\omega(2l-1)/2} \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left(\overline{\boldsymbol{P}_{m-2k-1}} \otimes \boldsymbol{P}_m\right) \boldsymbol{e}_\nu \, e^{-i\omega(k+l)} \\
&= \sum_{\mu\in\mathbb{Z}} \left(\frac{1}{2}\sum_{m\in\mathbb{Z}} \overline{\boldsymbol{P}_{m-2\mu+2l-1}} \otimes \boldsymbol{P}_m\right) \boldsymbol{u}_{\nu,\mu}.
\end{aligned}$$

Thus, generally it follows that

$$(\text{vec } T)\boldsymbol{u}_{\nu,l}(\omega) = \sum_{\mu\in\mathbb{Z}} \left(\frac{1}{2}\sum_{m\in\mathbb{Z}} \overline{\boldsymbol{P}_{m-2\mu+l}} \otimes \boldsymbol{P}_m\right) \boldsymbol{u}_{\nu,\mu}, \tag{8}$$

and hence, $\boldsymbol{M} = (\boldsymbol{M}_{\mu,l})_{\mu,l=-N+1}^{N-1}$ with $\boldsymbol{M}_{\mu,l} := \frac{1}{2}\sum_{m\in\mathbb{Z}} \overline{\boldsymbol{P}_{m-2\mu+l}} \otimes \boldsymbol{P}_m$ is a representing matrix of T. If $\boldsymbol{P}(\omega)$ satisfies condition (6), then it follows that $\rho(\boldsymbol{M}) = 1$. Then Condition E simplifies to the assertion that the eigenvalue 1 of $\boldsymbol{M}$ is simple.

2. The vector $\boldsymbol{\Phi}$ is L^2-stable if and only if the transfer operator T satisfies condition E, and the eigenmatrix corresponding to the eigenvalue 1 is nonsingular for $\omega \in [-\pi, \pi]$. As shown in the first part of the proof, T satisfies Condition E if and only if $\boldsymbol{M}$ satisfies Condition E. Let $\boldsymbol{w} \in \mathbb{C}^{r^2(2N-1)}$ be a right eigenvector of $\boldsymbol{M}$ corresponding to the eigenvalue 1, and let $\boldsymbol{A}_l$ ($l = -N+1,\ldots,N-1$) be constant $(r\times r)$-matrices formed by $\boldsymbol{w}$ such that $\boldsymbol{w} = (\text{vec } \boldsymbol{A}_l)_{l=-N+1}^{N-1}$, i.e., $\boldsymbol{A}_{-N+1}$ is formed by the first r^2 entries of $\boldsymbol{w}$, $\boldsymbol{A}_{-N+2}$ is formed by the second r^2 entries of $\boldsymbol{w}$ and so on. We show that

$$\boldsymbol{\Psi}(\omega) := \sum_{l=-N+1}^{N-1} \boldsymbol{A}_l \, e^{-i\omega l}$$

is an eigenmatrix of T corresponding to the eigenvalue 1. This follows by (8) from

$$\begin{aligned}
\text{vec } (T\boldsymbol{\Psi})(\omega) &= \sum_{l=-N+1}^{N-1} \text{vec } (T\boldsymbol{A}_l e^{-i\cdot l})(\omega) \\
&= \sum_{l=-N+1}^{N-1} \left(\sum_{\mu\in\mathbb{Z}} \boldsymbol{M}_{\mu,l} \, (\text{vec } \boldsymbol{A}_l) \, e^{-i\omega\mu}\right) \\
&= \sum_{\mu\in\mathbb{Z}} \left(\sum_{l=-N+1}^{N-1} \boldsymbol{M}_{\mu,l} \, \text{vec } (\boldsymbol{A}_l)\right) e^{-i\omega\mu} \\
&= \sum_{\mu\in\mathbb{Z}} \text{vec } \boldsymbol{A}_\mu \, e^{-i\omega\mu} = \text{vec } \boldsymbol{\Psi}(\omega). \quad \blacksquare
\end{aligned}$$

Remarks: 1. Observe, that $\Psi(\omega)$ is (up to normalization) the autocorrelation symbol of the solution vector $\boldsymbol{\Phi}$ of (1).
2. Let the matrix $\boldsymbol{U}$ be defined by $\boldsymbol{U} := (\delta_{l,-m})_{(-N+1)r^2 \le l,m \le (N-1)r^2}$. Then $\boldsymbol{M}$ in (7) satisfies $\boldsymbol{M}\,\boldsymbol{U} = \boldsymbol{U}\,\overline{\boldsymbol{M}}$. This fact can be used to simplify the computation of $\boldsymbol{M}$. In particular, if $\boldsymbol{w}$ is an eigenvector to the simple eigenvalue 1 of $\boldsymbol{M}$, then we have $\boldsymbol{w} = \boldsymbol{U}\overline{\boldsymbol{w}}$.

§3. An Example

Let us apply the result of Theorem 3 for checking the stability property for a solution vector $\boldsymbol{\Phi}$ with the symbol

$$\boldsymbol{P}(\omega) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\omega} \\ e^{-i\omega} & e^{-2i\omega} \end{pmatrix}.$$

Note that $\boldsymbol{P}(0)$ possesses the eigenvalues 1 and 0, and we find $(1,\,1)\,\boldsymbol{P}(0) = (1,\,1)$ and $(1,\,1)\boldsymbol{P}(\pi) = (0,\,0)$. Hence, $\boldsymbol{P}(\omega)$ satisfies the basic conditions of Proposition 1. However, with

$$\boldsymbol{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{P}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain

$$\boldsymbol{M} = \frac{1}{2}\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.$$

But $\boldsymbol{M}$ possesses the eigenvalue 1 with multiplicity 2, hence Condition E is not satisfied and the corresponding solution vector is not stable. One obtains the solution vector

$$\boldsymbol{\Phi} = \begin{pmatrix} \chi_{[0,3/2)} \\ \chi_{[1/2,2)} \end{pmatrix},$$

where $\chi_{[a,b)}$ denotes the characteristic function of the interval $[a,b)$. Indeed, $\boldsymbol{\Phi}$ is not L^2-stable (see also [9]).

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