Strongly Fejer monotone Mappings
Part I : Relaxations

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Summary. *We consider the general class of strongly Fejer monotone mappings and some of their basic properties. These properties are useful for a convergence theory of corresponding iterative methods which are widely used to solve convex problems* (see e.g. [3], [6], [7], [10]). *In part I we study the relation between these mappings and their relaxations.*

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1 Strongly Fejer monotone mappings

Let H be a (real) Hilbert space. We consider a nonempty, convex and closed subset Q of H (set of feasible elements) and set-valued mappings (multioperators) $g : Q \to \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ contains all nonempty subsets of Q. For g we introduce sets of weak and strong fixed points, namely

$$F_-(g) := \{x \in Q : x \in g(x)\} \quad , \quad F_+(g) := \{x \in Q : \{x\} = g(x)\} ,$$

where $F_+(g) \subseteq F_-(g)$. As usual mappings (operators) $g : Q \to Q$ are integrated as imbeddings. Here both kinds of fixed point sets coincide with $F(g) := \{x \in Q : x = g(x)\}$. We exclude the uninteresting special case that g is the identity I ($g \neq I$). Now we turn to the basic concepts.

Definition 1.1. For a mapping g and a (real) number λ we call the modified mapping

$$g_\lambda := (1-\lambda)\, I + \lambda\, g\ , \tag{1}$$

relaxation of g with the parameter λ.

Although the above concept is usually used only for values of λ in $(0,\infty)$ or even in $(0,2)$ we omit this restriction here.

Definition 1.2. Let M be a nonempty subset of Q and α a positive number. The mapping g is said to be *α-strongly M-Fejer monotone* ($g \in \mathbb{F}^\alpha(M)$) iff

$$\|y-x\|^2 - \|z-x\|^2 \geq \alpha\,\|y-z\|^2 \quad \forall x \in M\ ,\ \forall y \in Q\ ,\ \forall z \in g(y) \tag{2}$$

and

$$y \notin g(y) \quad \forall y \in Q \backslash M \tag{3}$$

hold. Besides, g is called *α-strongly Fejer monotone* ($g \in \mathbb{F}^\alpha$) iff g is α-strongly M-Fejer monotone for any M. Finally, g is called *strongly (M-)Fejer monotone* ($g \in \mathbb{F}_s$, $g \in \mathbb{F}_s(M)$) iff g is α-strongly (M-)Fejer monotone for any $\alpha > 0$.

Remark 1.3. General *M-Fejer monotone* mappings $g \in \mathbb{F}(M)$ satisfy (2) with the limit value $\alpha = 0$. Here we have the relation $M \subseteq F_+(g)$. If also (3) is added, then g is said to be *regularly M-Fejer monotone* ($g \in \mathbb{F}_r(M)$). By the way, $\mathbb{F}_r(M)$ can be regarded as the limit class $\mathbb{F}^0(M)$ of $\mathbb{F}^\alpha(M)$. In principle also negative α can be allowed. In the regular case M is uniquely determined by g. Namely, we get the convex and closed set

$$M = F_+(g) = F_-(g)$$

(see [7], [8]). All the more this holds for strongly Fejer monotone mappings. In Definition 1.2 the condition (2) is fulfilled automatically for $y \in M$ if $M = F_+(g)$. Besides, (3) can be omitted for $M = F_-(g)$.

Remark 1.4. Obviously the sets of strongly Fejer monotone mappings satisfy

$$\mathbb{F}^{\beta}(M) \subseteq \mathbb{F}^{\alpha}(M) \subseteq \mathbb{F}_s(M) \subseteq \mathbb{F}_r(M) \quad \text{for} \quad \beta \geq \alpha > 0 \ .$$

So they form a descending chain. Moreover the relations

$$\mathbb{F}^{\alpha} = \bigcup_{M \neq \emptyset} \mathbb{F}^{\alpha}(M) \quad , \quad \mathbb{F}_s(M) = \bigcup_{\alpha > 0} \mathbb{F}^{\alpha}(M)$$

are fulfilled. A sharper result will follow in section 5 (see Theorem 5.7).

Lemma 1.5. Let A be an index set of numbers $\alpha > 0$ with $\alpha^* = \sup A$. Then

$$g \in \mathbb{F}^{\alpha} \quad \forall \alpha \in A \quad \Longleftrightarrow \quad g \in \mathbb{F}^{\alpha^*} \ .$$

Proof. Both parts of the equivalence imply $M = F_-(g) \neq \emptyset$, that is (3). Choose fixed elements x, y and z. If (2) holds for each α in A, then also for α^*. So the right-hand implication follows. The left-hand implication is a direct consequence of Remark 1.4. ■

Definition 1.6. Let be $g \in \mathbb{F}_s$. Then the number

$$\alpha^* = \alpha^*_F(g) := \sup\{\alpha : g \in \mathbb{F}^{\alpha}\}$$

is said to be the *F-index* of g. (Mappings $g \in \mathbb{F}_r \setminus \mathbb{F}_s$ obtain the F-index 0.)

Remark 1.7. The F-index $\alpha^*_F(g)$ of $g \in \mathbb{F}_s$ is determined by

$$\alpha^*_F(g) = \inf \left\{ \frac{\|y-x\|^2 - \|z-x\|^2}{\|y-z\|^2} \ : \ x \in M \, , \, y \in Q \setminus M \, , \, z \in g(y) \right\} ,$$

where $M = F_-(g)$. The restriction for y ensures that the quotients are always defined. Because of Lemma 1.5 we obtain

$$\alpha^*_F(g) = \max\{\alpha : g \in \mathbb{F}^{\alpha}\} \ .$$

Moreover, the set

$$\mathbb{F}_*^\alpha := \mathbb{F}^\alpha \setminus \bigcup_{\beta>\alpha} \mathbb{F}^\beta$$

contains all mappings g with index $\alpha_F^*(g) = \alpha$. In section 5 we will show $\mathbb{F}_*^\alpha \neq \emptyset$ (see proof of Theorem 5.7).

The geometrical background of the introduced concepts is given in [9]. The following convergence result illustrates the importance of strongly Fejer monotone mappings g since they fulfil at least the first two conditions (see [9]).

Theorem 1.8. Under the assumptions

a) $g \in \mathbb{F}_r(M)$,

b) g asymptotically regular ,

c) $g' = I - g$ demiclosed (I identity)

the iterative method (x_k) defined by $x_{k+1} \in g(x_k)$ converges weakly to an element x^* in M.

2 Classes of strongly Fejer monotone mappings

We start with mappings which are slight generalizations of contractive operators (operators with Lipschitz norm $q \in [0,1)$).

Definition 2.1. The mapping g is said to be *Fejer q-contractive* for a number $q \in [0,1)$ ($g \in \mathbb{F}_c^q$) iff g has a fixed point x ($x \in F_-(g)$) such that

$$\|z - x\| \leq q \, \|y - x\| \quad \forall y \in Q \ , \ \forall z \in g(y) \tag{4}$$

holds.

Theorem 2.2. A Fejer q-contractive mapping g has exactly one fixed point x which is even strong ($F_-(g) = F_+(g) = \{x\}$). Besides, g is α-strongly M-Fejer monotone with $M = \{x\}$ and

$$\alpha = \alpha(q) := \frac{1-q}{1+q} \,.$$

Proof. Let x be a fixed point of $g \in \mathbb{F}_c^q$. Choosing $y = x$ in (4) yields $z = x$ such that x is a strong fixed point. Let x' be a fixed point of g, too. Choosing $y = x'$ in (4) supplies $\|x' - x\| \leq q\,\|x' - x\|$ which is only possible for $x' = x$, since $q \in [0,1)$. Hence $M = \{x\}$ contains all fixed points of g. Moreover, in view of (4) and the triangle inequality we get the estimates

$$\|y - z\| \leq \|y - x\| + \|z - x\| \leq (1+q)\,\|y - x\|$$

and

$$\begin{aligned} \|y-x\|^2 - \|z-x\|^2 &\geq (1-q^2)\,\|y-x\|^2 \\ &\geq \frac{1-q^2}{(1+q)^2}\,\|y-z\|^2 = \frac{1-q}{1+q}\,\|y-z\|^2 \,. \end{aligned}$$

Consequently g is α-strongly M-Fejer monotone with the given α and M. ∎

Remark 2.3. The function $\alpha(q)$ is strictly monotone decreasing with $\alpha(0) = 1$ and the left limit $\alpha(1-0) = 0$. So $\alpha(q)$ produces only the range $(0,1]$ for α. But observe the relation $\alpha_F^*(g) \geq \alpha(q)$ for $g \in \mathbb{F}_c^q$. We assume that the infimum characterizing the F-index $\alpha_F^*(g)$ is attained (see Remark 1.7). An analysis of the above proof shows that the equality in the estimates holds, that is $\alpha_F^*(g) = \alpha(q)$, iff there exist elements $y \in Q$ and $z \in g(y)$ different from x with the property

$$\|z - x\| = q\,\|y - x\| \quad , \quad \|y - z\| = \|y - x\| + \|z - x\| \,.$$

Since H is strictly normed the triangle equality is equivalent to

$$\exists \gamma > 0 \;:\; z - x = \gamma\,(x - y)$$

(see e.g. [5, p. 14]). So the first equality supplies $\gamma = q$. This situation occurs iff x is a convex combination of y and z :

$$x = (1-\kappa)\,y + \kappa\, z \quad , \quad \kappa = \frac{1}{1+q} \in (0.5, 1) \, .$$

But there are examples for mappings $g \in \mathbb{F}_c^q$ with $\alpha_F^*(g) > 1 \geq \alpha(q)$. By the way, the F-index can become arbitrarily great (see [9]).
The reversion of Theorem 2.2 is not true. A mapping g which is α-strongly M-Fejer monotone with $\alpha = \alpha(q)$ and $M = \{x\}$ need not to be Fejer q-contractive (see [9]).

Now we turn to special classes of nonexpansive operators.

Definition 2.4. The operator g is said to be *α-strongly nonexpansive* for a positive number α ($g \in \mathbb{L}^\alpha$) iff $F(g) \neq \emptyset$ and

$$\|y-x\|^2 - \|g(y)-g(x)\|^2 \geq \alpha\, \|g'(y)-g'(x)\|^2 \quad \forall x\,,\, y \in Q\,, \tag{5}$$

where g' denotes the complement $I - g$ of g. Besides, g is called *strongly nonexpansive* ($g \in \mathbb{L}_s$) iff g is α-strongly nonexpansive for any $\alpha > 0$.

Remark 2.5. Sometimes it is useful to specify the fixed point set $F(g)$ of the operator $g \in \mathbb{L}^\alpha$. Then we write $g \in \mathbb{L}^\alpha(M)$, where $M = F(g)$. This appears in accordance with the notation $\mathbb{F}^\alpha(M)$ in Definition 1.2. At all, it will turn out that the aspects we are interested in contain a great analogy between both classes of mappings. Therefore results for the second case are often omitted or only outlined in the following.
Observe that operators $g \in \mathbb{L}_s$ can not possess different elements x and y with the property $g(y) - g(x) = x - y$ as (5) shows. The limit case $\alpha = 0$ in (5) characterizes *nonexpansive* operators g (operators with Lipschitz norm less or equal to 1). If the fixed point property is added ($F(g) \neq \emptyset$), then we speak of *regularly nonexpansive* operators g ($g \in \mathbb{L}_r =: \mathbb{L}^0$) which are also regularly M-Fejer monotone with $M = F(g)$ (see [8]). Fejer q-contractive mappings need not to be α-strongly nonexpansive and vice versa (see [9]).

Remark 2.6. Evidently the sets of strongly nonexpansive operators fulfil the descending chain relations

$$\mathbb{L}^{\beta}(M) \subseteq \mathbb{L}^{\alpha}(M) \subseteq \mathbb{L}_s(M) \subseteq \mathbb{L}_r(M) \quad \text{for} \quad \beta \geq \alpha > 0 \ .$$

Besides, it is

$$\mathbb{L}^{\alpha} = \bigcup_{M \neq \emptyset} \mathbb{L}^{\alpha}(M) \quad , \quad \mathbb{L}_s(M) = \bigcup_{\alpha > 0} \mathbb{L}^{\alpha}(M) \ .$$

Lemma 2.7. Let A be an index set of numbers $\alpha > 0$ with $\alpha^* = \sup A$. Then

$$g \in \mathbb{L}^{\alpha} \quad \forall \alpha \in A \quad \Longleftrightarrow \quad g \in \mathbb{L}^{\alpha^*} \ .$$

Definition 2.8. Let be $g \in \mathbb{L}_s$. Then the number

$$\alpha^* = \alpha_L^*(g) := \sup\{\alpha : g \in \mathbb{L}^{\alpha}\}$$

is called *L-index* of g. (Operators $g \in \mathbb{L}_r \setminus \mathbb{L}_s$ obtain the L-index 0.)

Remark 2.9. The L-index $\alpha_L^*(g)$ of $g \in \mathbb{L}_s$ is given by

$$\alpha_L^*(g) = \inf \left\{ \frac{\|y - x\|^2 - \|g(y) - g(x)\|^2}{\|g'(y) - g'(x)\|^2} \ : \ y \, , \, x \in Q \, , \, g'(y) \neq g'(x) \right\} \ .$$

By Lemma 2.7 we get

$$\alpha_L^*(g) = \max\{\alpha : g^{\alpha} \in \mathbb{L}^{\alpha}\} \ .$$

Moreover, the set

$$\mathbb{L}_*^{\alpha} := \mathbb{L}^{\alpha} \setminus \bigcup_{\beta > \alpha} \mathbb{L}^{\beta}$$

contains all operators g with L-index $\alpha_L^*(g) = \alpha$.

Theorem 2.10. For arbitrary $\alpha > 0$ it holds

$$\mathbb{L}^\alpha(M) \subseteq \mathbb{F}^\alpha(M) ,$$

that is, α-strongly nonexpansive operators are α-strongly Fejer monotone.

Proof. If g is α-strongly nonexpansive, then $M = F(g) \neq \emptyset$. Further we obtain for $x \in M$ and $z = g(y)$

$$\begin{aligned} \|y-x\|^2 &- \|g(y)-g(x)\|^2 - \alpha\|g'(y)-g'(x)\|^2 \\ &= \|y-x\|^2 - \|z-x\|^2 - \alpha\|z-y\|^2 . \end{aligned}$$

This shows the assertion if the Definitions 1.2 and 2.4 concerning $g \in \mathbb{F}^\alpha$ and $g \in \mathbb{L}^\alpha$ are taken into account. ∎

Remark 2.11. For $g \in \mathbb{L}_s$ we have the index relation

$$\alpha_F^*(g) \geq \alpha_L^*(g) .$$

Namely, by Theorem 2.10 it is $g \in \mathbb{F}^\alpha$ with $\alpha = \alpha_L^*(g)$. By definition of the F-index the assertion follows. There are examples with $\alpha_F^*(g) > \alpha_L^*(g)$ (see [9]).

3 Auxiliary results

First we list some auxiliary statements which are useful to show the results of the next two sections. An important part will play the functional $d^\alpha : H \times H \to \mathbb{R}$ defined for $\alpha \in \mathbb{R}$ by

$$d^\alpha(u,v) := \|u\|^2 - \|v\|^2 - \alpha\,\|u-v\|^2 \quad , \quad u \in H \ , \ v \in H \tag{6}$$

which is related to (2) in Definition 1.2 and to (5) in Definition 2.4.

Lemma 3.1. The functional d^α in (6) has the equivalent form

$$d^\alpha(u,v) = (1-\alpha)\,\|u\|^2 - (1+\alpha)\,\|v\|^2 + 2\,\alpha\,(u,v) .$$

Proof. The assertion is an immediate consequence of

$$\|u-v\|^2 = (u-v, u-v) = \|u\|^2 - 2\,(u,v) + \|v\|^2 \,. \ \blacksquare$$

Next we define for two elements u , v in a linear space V and a number λ the linear combination (straight line through u and v)

$$w_\lambda = w_\lambda(u,v) := (1-\lambda)\,u + \lambda\,v \tag{7}$$

which is related to the relaxation (1) in Definition 1.1, namely it is $g_\lambda = w_\lambda(I,g)$. In the special case $V = \mathbb{R}$ we use the notation w^*_λ . The operation w_λ has simple, but interesting properties listed in the next lemma.

Lemma 3.2. The following relations hold for elements u, v, u', v' and numbers λ, μ, ν, a, b :

a) $w_0(u,v) = u \quad , \quad w_1(u,v) = v \quad , \quad w_\lambda(u,v) = w_{1-\lambda}(v,u)$,

b) $w_\lambda(u,u) = u \quad , \quad w_\lambda(\nu\,u, \nu\,v) = \nu\,w_\lambda(u,v)$,

c) $w_\lambda(u+u', v+v') = w_\lambda(u,v) + w_\lambda(u',v')$,

d) $w_\lambda(u,v) - w_\mu(u,v) = (\lambda-\mu)\,(v-u)$,

e) $w_{a\,\lambda + b\,\mu}(u,v) = a\,w_\lambda(u,v) + b\,w_\mu(u,v) \quad \text{for} \quad a+b=1$,

f) $w_{\lambda\mu}(u,v) = w_\mu(u, w_\lambda(u,v))$,

g) $\|w_\lambda(u,v)\|^2 = w^*_\lambda(\|u\|^2, \|v\|^2) - \lambda\,(1-\lambda)\,\|u-v\|^2$.

Proof. It is easy to check the above identities by use of (7). $\blacksquare$

Now we study for fixed but arbitrary elements u and v the nonnegative (straight line distance) function

$$r(\lambda) = r(\lambda; u, v) := \|w_\lambda(u,v)\| \,. \tag{8}$$

Because of Lemma 3.2 e) and the norm properties $r(\lambda)$ is convex. Observing that H is strictly normed, $r(\lambda)$ is even strictly convex iff u and v are linearly independent. In this case there exists a unique positive minimum at

$$\lambda^* = \lambda^*(u,v) := \frac{(u, u-v)}{\|u-v\|^2} \ . \tag{9}$$

This can easily be checked if the quadratic function

$$s(\lambda) = s(\lambda; u, v) := r^2(\lambda; u, v)$$

instead of $r(\lambda)$ is considered which has the same minimizing argument. So $\lambda^*(u,v)$ is the zero of the derivative

$$s'(\lambda) = (w_\lambda, v-u) = (u, v-u) + \lambda \, \|u-v\|^2 = 0 \ .$$

If $v = cu$ is satisfied for a number c, then

$$r(\lambda) = |1 + (c-1)\,\lambda| \, \|u\| \ .$$

For $u \neq v$ and $c \neq 1$, respectively, $r(\lambda)$ has a unique minimum, too, and it is attained at the zero

$$\lambda^*(u,v) := \frac{1}{1-c} \ .$$

A check shows that this value is the specification of (9). Finally, $r(\lambda) = \|u\|$ arizes for $u = v$. The properties of $r(\lambda)$ can also be revealed, if Lemma 3.2 g) is applied and quadratic completion is joined. By use of (9) we get

$$\begin{aligned} s(\lambda) &= (1-\lambda)\|u\|^2 + \lambda\|v\|^2 - \lambda(1-\lambda)\|u-v\|^2 \\ &= \|u-v\|^2 \, \lambda^2 - 2(u, u-v) \, \lambda + \|u\|^2 \\ &= \|u-v\|^2((\lambda-\lambda^*)^2 - (\lambda^*)^2) + \|u\|^2 \end{aligned}$$

and

$$s(\lambda^*) = \|u\|^2 - (\lambda^*)^2\|u-v\|^2 = \|u\|^2 - \frac{(u, u-v)^2}{\|u-v\|^2} \ .$$

(For $u = v$ the minimum λ^* is not unique, but arbitrary.) So we recognize the symmetry properties

$$s(\lambda^* - \lambda) = s(\lambda^* + \lambda) \quad , \quad r(\lambda^* - \lambda) = r(\lambda^* + \lambda)$$

and the positivity of $s(\lambda^*)$ and $r(\lambda^*)$ if u and v are linearly independent. Namely, in this case the Schwarz inequality for scalar products holds strictly. For $u \neq v$ the minimizing argument λ^* separates the left strictly monotone decreasing part of $r(\lambda)$ from the right strictly monotone increasing part. If also the symmetry of $r(\lambda)$ with respect to λ^* is observed, then we get with the definition

$$\mu^* = \mu^*(u, v) := 2\lambda^*(u, v) - \mu \tag{10}$$

the (extended) monotony relations

$$r(\lambda) < r(\mu) \quad \text{for} \quad \mu < \min(\lambda^*, \lambda) < \mu^* \quad , \quad r(\mu^*) = r(\mu)\ . \tag{11}$$

Further, we can show important relations combining the functions $r(\lambda)$ in (8) and d^α in (6).

Lemma 3.3. The following equation is fulfilled for arbitrary λ and μ :

$$s(\lambda; u, v) - s(\mu; u, v) = (\mu - \lambda)\, d^{\lambda+\mu-1}\,(u, v)\ .$$

Besides we have for $\lambda > \mu$ the equivalence

$$r(\lambda; u, v) \leq r(\mu; u, v) \quad \Longleftrightarrow \quad d^{\lambda+\mu-1}\,(u, v)\ \geq 0\ ,$$

where the equality in one of the relations implies also the equality in the other.

Proof. Using the properties g) and d) in Lemma 3.2 for w_λ and w^*_λ, respectively, we obtain for $s(\lambda) = s(\lambda; u, v) = r^2(\lambda)$

$$\begin{aligned} s(\lambda) - s(\mu) &= w^*_\lambda(\|u\|^2, \|v\|^2) - w^*_\mu(\|u\|^2, \|v\|^2) \\ &\quad + (\mu\,(1-\mu) - \lambda\,(1-\lambda))\|u - v\|^2 \\ &= (\lambda - \mu)\,(\|v\|^2 - \|u\|^2) + (\mu - \lambda - \mu^2 + \lambda^2)\|u - v\|^2 \\ &= (\lambda - \mu)\,(\|v\|^2 - \|u\|^2 + (\lambda + \mu - 1)\,\|u - v\|^2) \\ &= (\mu - \lambda)\, d^{\lambda+\mu-1}\,(u, v)\ . \end{aligned}$$

This is the first assertion. The second assertion is now a simple consequence. ∎

Remark 3.4. Obviously $d^\alpha(u,v)$ is for fixed, different elements u, v a linear function of the superscript α with the zero

$$\alpha^* = \alpha^*(u,v) := \frac{\|u\|^2 - \|v\|^2}{\|u-v\|^2} = \frac{(u+v,u-v)}{\|u-v\|^2} \; .$$

Hence Lemma 3.3 supplies

$$r(\lambda) = r(\mu) \quad , \quad \lambda \neq \mu \quad \Longleftrightarrow \quad \lambda + \mu = \alpha^* + 1 \; .$$

But, using (9) and considering

$$\alpha^* + 1 = \frac{(u+v,u-v) + (u-v,u-v)}{\|u-v\|^2} = 2\,\frac{(u,u-v)}{\|u-v\|^2} = 2\,\lambda^* \; , \qquad (12)$$

the symmetry of $r(\lambda)$ relative to λ^* is again verified. Besides, Lemma 3.3 allows to compute $s'(\lambda)$. A limit transition shows

$$s'(\lambda;u,v) = -d^{2\lambda-1}(u,v) \; .$$

This confirms again the above relation between α^* and λ^*.

The next result shows how the superscript of d is influenced by changing the second argument from v to w_λ. Besides, δ will play the part of a perturbation.

Lemma 3.5. Let be $\lambda \neq 0$ and δ arbitrary. Then the equation

$$d^{\beta+\delta}(u,w_\lambda) = \lambda\, d^{\alpha+\lambda\delta}(u,v)$$

holds with $w_\lambda = w_\lambda(u,v)$ if the parameters α, β and λ fulfil the condition

$$(1+\beta)\,\lambda = 1 + \alpha \; .$$

Moreover, we get under this condition for $\lambda > 0$

$$d^{\alpha+\lambda\delta}(u,v) \geq 0 \quad \Longleftrightarrow \quad d^{\beta+\delta}(u,w_\lambda) \geq 0 \; .$$

Proof. At first let be $\delta = 0$. Then we start with the right-hand side of the equation and substitute

$$v = \frac{\lambda - 1}{\lambda}\, u + \frac{1}{\lambda}\, w_\lambda$$

from (7). Using Lemma 3.1 this leads to

$$
\begin{aligned}
\lambda\, d^{\alpha}(u,v) &= \lambda\,(1-\alpha)\,\|u\|^2 - \lambda\,(1+\alpha)\,\|\frac{\lambda-1}{\lambda}\,u + \frac{1}{\lambda}\,w_\lambda\|^2 \\
&\quad +2\,\lambda\,\alpha(u, \frac{\lambda-1}{\lambda}\,u + \frac{1}{\lambda}\,w_\lambda) \\
&= \left((1-\alpha)\,\lambda - (1+\alpha)\frac{(\lambda-1)^2}{\lambda} + 2\alpha\,(\lambda-1)\right)\|u\|^2 \\
&\quad -(1+\alpha)\frac{1}{\lambda}\,\|w_\lambda\|^2 + 2\left(\alpha - (1+\alpha)\,\frac{\lambda-1}{\lambda}\right)(u,w_\lambda) \\
&= \frac{2\lambda-1-\alpha}{\lambda}\,\|u\|^2 - \frac{1+\alpha}{\lambda}\,\|w_\lambda\|^2 + 2\,\frac{1+\alpha-\lambda}{\lambda}\,(u,w_\lambda) \\
&= (1-\beta)\,\|u\|^2 - (1+\beta)\,\|w_\lambda\|^2 + 2\,\beta\,(u,w_\lambda) \\
&= d^{\beta}(u,w_\lambda)\,.
\end{aligned}
$$

This is the first assertion for $\delta = 0$. Now let δ be arbitrary. Then the general equation follows immediately if

$$(1+\beta+\delta)\,\lambda = (1+\beta)\,\lambda + \delta\,\lambda = 1+\alpha+\lambda\,\delta$$

is considered. The second assertion is a simple consequence. ■

Remark 3.6. For $\alpha > 0$ and $\beta > 0$ the relation between the parameters show that λ can vary in the open interval $(0, 1+\alpha)$. If the limit case $\beta = 0$ is integrated, then $\lambda = 1+\alpha$ is possible, too.

4 Norm relations for relaxations

We want to characterize the considered mappings g by certain norm relations of corresponding relaxations. Therefore we choose a mapping $g \in \mathbb{F}_r(M)$ and study for arbitrary elements of

$$J_g = \{(x,y,z) : \; x \in M \;,\; y \in Q \setminus M \;,\; z \in g(y)\,\}$$

the properties of the specialized (straight line distance) functions

$$r(\lambda) = r(\lambda; y-x, z-x) = \|w_\lambda(y-x, z-x)\| = \|z_\lambda - x\| \tag{13}$$

from (8) with $u = y - x$ and $v = z - x$, where $z_\lambda = w_\lambda(y,z) \in g_\lambda(y)$. Namely, by Lemma 3.2 b) , c) we win

$$w_\lambda(y-x, z-x) = w_\lambda(y,z) - w_\lambda(x,x) = z_\lambda - x \; .$$

Now observe

$$r(0) = \|y-x\| \quad , \quad r(1) = \|z-x\| \; ,$$

such that

$$r(1) \leq r(0) \quad \forall (x,y,z) \in J_g \; .$$

The excluded case $y \in M$ corresponds to $z = y$ (compare Remark 1.3) and produces only the uninteresting constant function $r(\lambda) = \|y - x\|$. For $z \neq y$ or in other words for $(x,y,z) \in J_g$ the unique minimum is attained at

$$\lambda^* = \lambda^*(y-x, z-x) = \frac{(y-x, y-z)}{\|y-z\|^2}$$

(see (9) and the following remarks). If we define for g the characterizing number

$$\lambda_F^*(g) := \inf\{\lambda^*(y-x, z-x) \; : \; (x,y,z) \in J_g\} \; , \tag{14}$$

then the (extended) monotony property (11) of $r(\lambda)$ in section 3 supplies

$$r(\lambda) < r(\mu) \quad \forall (\lambda, \mu) : \mu < \lambda \leq \mu_F^*(g) \quad , \quad \forall (x,y,z) \in J_g \; ,$$

where $\mu < \lambda_F^*(g)$ and

$$\mu_F^*(g) := \inf\{\mu^*(y-x, z-x) \; : \; (x,y,z) \in J_g\} \; .$$

But this property can also be used to integrate the mapping g into a certain class $\mathbb{F}^\alpha$. At first we get for $\alpha > 0$ the characterization

$$g \in \mathbb{F}^\alpha \iff d^\alpha(y-x, z-x) \geq 0 \quad \forall (x,y,z) \in J_g \tag{15}$$

by the functional d^α in (6) if $M = F_-(g) \neq \emptyset$ is supposed (see Definition 1.2 and Remark 1.3). Observe that $y \in M$ can also be admitted. Moreover, the statements $g \in \mathbb{F}^\alpha$ and $g \in \mathbb{F}^\alpha(M)$ are equivalent. Now the central statement of this section follows which can be exploited in various ways.

Theorem 4.1. Let be $\lambda - \mu > 0$ and $\lambda + \mu > 1$. If $g \in \mathbb{F}_r$, then

$$g \in \mathbb{F}^{\lambda+\mu-1} \quad \Longleftrightarrow \quad r(\lambda) \le r(\mu) \quad \forall (x,y,z) \in J_g \; .$$

Proof. For $g \in \mathbb{F}_r(M)$ we have $M = F_-(g) \neq \emptyset$. With the above mentioned choices $u = y - x$, $v = z - x$, where $(x,y,z) \in J_g$, Lemma 3.3 reads for $\lambda > \mu$

$$r(\lambda) \le r(\mu) \quad \Longleftrightarrow \quad d^{\lambda+\mu-1}(y-x, z-x) \ge 0 \; .$$

But, if the left part holds for all $(x,y,z) \in J_g$, then also the second. By the characterization (15) the latter corresponds to $g \in \mathbb{F}^{\lambda+\mu-1}$. ■

Remark 4.2.

a) The conditions for λ and μ in Theorem 4.1 can also be expressed by $\lambda > \max\{\mu, 1-\mu\}$.

b) Putting $\lambda = 1$ in Theorem 4.1 we obtain for $\mu \in (0,1)$

$$g \in \mathbb{F}^{\mu} \quad \Longleftrightarrow \quad r(1) \le r(\mu) \quad \forall (x,y,z) \in J_g \; .$$

Putting $\mu = 0$ we arrive for $\lambda > 1$ at

$$g \in \mathbb{F}^{\lambda-1} \quad \Longleftrightarrow \quad r(\lambda) \le r(0) \quad \forall (x,y,z) \in J_g \; .$$

Finally, choosing $\lambda + \mu = 2$, we win for any $\mu \in (0,1)$

$$g \in \mathbb{F}^{1} \quad \Longleftrightarrow \quad r(2-\mu) \le r(\mu) \quad \forall (x,y,z) \in J_g \; .$$

c) In Theorem 4.1 also the limit case $\lambda + \mu = 1$ is admissible. So the choice $\mu = 0$ and $\lambda = 1$ is possible in the first two relations of b).

d) The equality $r(\lambda) = r(\mu)$ in Theorem 4.1 holds only in special cases (see Remark 3.4). But observe that the index set J_g can also be replaced by the extension

$$J_g^0 = \{(x,y,z) \;:\; x \in M \;,\; y \in Q \;,\; z \in g(y)\}$$

since for $(x,y,z) \in J_g^0 \setminus J_g$ and $z = y$, respectively, the equation $r(\lambda) = r(\mu)$ is fulfilled automatically.

Corollary 4.3. If we introduce the sets

$$\begin{aligned} \Lambda_\alpha^* &:= \{(\lambda,\mu) : \lambda-\mu>0\ ,\ \lambda+\mu=\alpha+1\} \\ \Lambda_\alpha &:= \{(\lambda,\mu) : \lambda-\mu>0\ ,\ \lambda+\mu\le\alpha+1\}\ , \end{aligned}$$

then we get for any $\alpha > 0$ and any subset Λ_α^- of Λ_α with the property $\sup\{\lambda+\mu\ :\ (\lambda,\mu)\in\Lambda_\alpha^-\}=\alpha+1$:

$$\begin{aligned} g\in\mathbb{F}^\alpha \quad &\Longleftrightarrow \quad \exists(\lambda,\mu)\in\Lambda_\alpha^*\quad \forall(x,y,z)\in J_g\ :\ r(\lambda)\le r(\mu) \\ &\Longleftrightarrow \quad \forall(\lambda,\mu)\in\Lambda_\alpha^-\quad \forall(x,y,z)\in J_g^0\ :\ r(\lambda)\le r(\mu)\ . \end{aligned}$$

Proof. The assertions follow from Theorem 4.1 if the substitution $\alpha = \lambda+\mu-1$ is used. For the second equivalence we need additionally Lemma 1.5 and Remark 4.2 d). ■

Corollary 4.4. The following conditions are equivalent to $g\in\mathbb{F}^\alpha$ for any $\alpha>0$:

a) $r(\lambda)\le r(\mu)\quad \forall\lambda\,,\,\mu\,:\mu<\lambda<\frac{\alpha+1}{2}\quad,\quad \forall(x,y,z)\in J_g^0\ ,$

b) $r(\frac{\alpha+1}{2})\le r(\mu)\quad \forall\mu<\frac{\alpha+1}{2}\quad,\quad \forall(x,y,z)\in J_g^0\ ,$

c) $r(\lambda)\le r(\frac{\alpha-1}{2})\quad \forall\lambda\in(\frac{\alpha-1}{2},\frac{\alpha+3}{2})\quad,\quad \forall(x,y,z)\in J_g^0\ .$

Proof. In a) the pairs (λ,μ) form a set fulfilling the properties of Λ_α^- in Corollary 4.3. The same is true for the pairs $(\frac{\alpha+1}{2},\mu)$ in b) and $(\lambda,\frac{\alpha-1}{2})$ in c). So Corollary 4.4 is a direct consequence of Corollary 4.3. ■

Remark 4.5. The mentioned properties of $r(\lambda)$ show that the inequalities for r in Corollary 4.4 hold even strictly if the triples (x,y,z) vary in J_g instead of J_g^0. On the other hand, if the equality for r is admitted, then by Corollary 4.3 the endpoints of the open intervals regarding the arguments λ and μ can be included. Besides, the range can be modified as long as the pairs (λ,μ) form a set of the kind Λ_α^-.
Corollary 4.4 means that for $g\in\mathbb{F}^\alpha$ and $(x,y,z)\in J_g$ the function $r(\lambda)$ is

monotone decreasing up to $\frac{\alpha+1}{2}$ and attains in $[\frac{\alpha-1}{2}, \frac{\alpha+3}{2}]$ its maximum at the left endpoint. So the relation $\frac{\alpha+1}{2} \leq \lambda_F^*(g)$ is true (see (14)). This suggests that perhaps for the index $\alpha = \alpha_F^*(g)$ the equality is satisfied. Indeed, the next theorem will show this.

Remark 4.6. The mappings $g \in \mathbb{F}^1$ can be characterized by the following properties, where the definition (13) of r is used:

a) $\|z_\lambda - x\| \leq \|z_\mu - x\| \quad \forall \lambda\,,\, \mu \,:\, 0 \leq \mu \leq \lambda \leq 1\;,\; \forall (x,y,z) \in J_g^0\;,$
b) $\|z_\lambda - x\| \leq \|z_{2-\lambda} - x\| \leq \|y - x\| \quad \forall \lambda \in [1,2]\;,\; \forall (x,y,z) \in J_g^0\;,$
c) $\|z_{2-\lambda} - x\| \leq \|z_\lambda - x\| \quad \forall \lambda \in [0,1]\;,\; \forall (x,y,z) \in J_g^0\;.$

The first two properties arize if Corollary 4.3 and Corollary 4.4 a), respectively, are used with $\alpha = 1$. The third property is only a modification of Remark 4.2 b). These properties are described in [1, p. 85-86] for special relaxations, namely for so-called transfer operators g_λ of simultaneous projectors g which belong to $\mathbb{F}^1$ (see [9] and Example 6.4).

Theorem 4.7. For $g \in \mathbb{F}_r$ the equation

$$\alpha_F^*(g) + 1 = 2\,\lambda_F^*(g)$$

holds.

Proof. By (12) we obtain

$$\alpha^*(y-x, z-x) + 1 = 2\,\lambda^*(y-x, z-x)$$

for all $(x,y,z) \in J_g$. Now the assertion follows if we take the infimum over these elements. Namely, consider

$$\alpha_F^*(g) = \inf\{\alpha^*(y-x, z-x) \;:\; (x,y,z) \in J_g\}$$

(Remarks 1.7 and 3.4) and the definition of $\lambda_F^*(g)$.

Analogous results are obtained for the classes of nonexpansivity. We start with an operator $g \in \mathbb{L}_r$ and introduce the set

$$J_g' = \{(x,y) : x \in Q\;,\; y \in Q\;,\; g'(x) \neq g'(y)\}$$

where $g' = I - g$ again denotes the complement of g. Now we consider for arbitrary elements of J_g' the specialized functions

$$\begin{aligned} r(\lambda) &= r(\lambda; y-x, g(y)-g(x)) = \|w_\lambda(y-x, g(y)-g(x))\| \\ &= \|g_\lambda(y) - g_\lambda(x)\| \end{aligned} \tag{16}$$

from (8) with $u = y - x$ and $v = g(y) - g(x)$. The latter representation of r is verified by Lemma 3.2 b) , c) which supplies

$$w_\lambda(y-x, g(y)-g(x)) = w_\lambda(y, g(y)) - w_\lambda(x, g(x)) = g_\lambda(y) - g_\lambda(x) .$$

The excluded case $g'(y) = g'(x)$ supplies again the uninteresting constant function $r(\lambda) = \|y - x\|$. Observe that for $x \in F(g)$ we arrive at

$$r(\lambda) = \|z_\lambda - x\| \quad , \quad z_\lambda = g_\lambda(y)$$

such that for $g \in \mathbb{L}_r$ the family of functions r in (16) is an extension of the previously studied family of functions r in (13). Taking the special values

$$r(0) = \|y - x\| \quad , \quad r(1) = \|g(y) - g(x)\| .$$

the inequality $r(1) \leq r(0)$ holds for all $(x, y) \in J_g'$. Further, we can define

$$\begin{aligned} \lambda_L^*(g) &:= \inf\{\lambda^*(y-x, g(y)-g(x)) \ : \ (x,y) \in J_g'\} , \\ \mu_L^*(g) &:= \inf\{\mu^*(y-x, g(y)-g(x)) \ : \ (x,y) \in J_g'\} . \end{aligned}$$

Now we could list analogue results replacing formally $\mathbb{F}_r$ by $\mathbb{L}_r$, $\mathbb{F}^\alpha$ by $\mathbb{L}^\alpha$, J_g by J_g', $\alpha_F^*(g)$ by $\alpha_L^*(g)$, $\lambda_F^*(g)$ by $\lambda_L^*(g)$, $\mu_F^*(g)$ by $\mu_L^*(g)$ and so on. Here we restrict us to the characterization of $\mathbb{F}^\alpha$ by the functional d^α in (6) and to Theorem 4.1 which can be reformulated as follows:

The functional characterization of $\mathbb{L}^\alpha$ is given by

$$g \in \mathbb{L}^\alpha \quad \Longleftrightarrow \quad d^\alpha(y-x, g(y)-g(x)) \geq 0 \quad \forall (x,y) \in J'(g) \tag{17}$$

if $F(g) \neq \emptyset$ is supposed.

Theorem 4.1'. Let be $\lambda - \mu > 0$ and $\lambda + \mu > 1$. If $g \in \mathbb{L}_r$, then

$$g \in \mathbb{L}^{\lambda+\mu-1} \quad \Longleftrightarrow \quad r(\lambda) \leq r(\mu) \quad \forall (x,y) \in J_g' .$$

The analogue of Corollary 4.4 a) reads for $\alpha = 1$

$$g \in \mathbb{L}^1 \iff \begin{array}{l} r(\lambda) \le r(\mu) \quad \forall \lambda\,,\, \mu \ :\ 0 \le \mu \le \lambda \le 1 \\ \forall (x,y) \in Q \times Q \end{array}$$

if the endpoints of the λ-interval are included and μ is restricted to the nonnegative domain (see Remark 4.5). The definition (16) of r supplies with the specification $\lambda = 1$

$$g \in \mathbb{L}^1 \iff \begin{array}{l} \|g(y) - g(x)\| \le \|g_\mu(y) - g_\mu(x)\| \\ \forall \mu \in [0,1]\ ,\ \forall (x,y) \in Q \times Q\ . \end{array}$$

Operators with this property play an important part in the fixed point theory and are called there *firmly nonexpansive* (see [4, p. 41-44]). So these operators turn out to be in our context nothing else than strongly nonexpansive (with L-indices at least 1). Finally, the equation

$$\alpha_L^*(g) + 1 = 2\,\lambda_L^*(g)$$

holds for $g \in \mathbb{L}_r$. If the analogue of Theorem 4.7 and the index relation $\alpha_L^*(g) \le \alpha_F^*(g)$ of Remark 2.11 is observed, we get as a byproduct

$$\lambda_L^*(g) \le \lambda_F^*(g)\ .$$

5 Scales of relaxations

As seen above, relaxations supply characterizations of mapping classes $\mathbb{F}^\alpha$ and $\mathbb{L}^\alpha$, respectively. But they also open the possibility to change between these classes. This will be investigated below. Again we formulate the results for $\mathbb{F}^\alpha$. The transformation to $\mathbb{L}^\alpha$ can be realized without difficulties.

Theorem 5.1. For a mapping g and parameters $\alpha \ge 0$, $\beta \ge 0$ and $\lambda > 0$ which are connected by the equation

$$(1 + \beta)\,\lambda = 1 + \alpha$$

the following statement holds:

$$g \in \mathbb{F}^{\alpha}(M) \quad \Longleftrightarrow \quad g_\lambda \in \mathbb{F}^{\beta}(M) \ .$$

Moreover, it is fulfilled:

$$\alpha = \alpha_F^*(g) \quad \Longleftrightarrow \quad \beta = \alpha_F^*(g_\lambda) \ .$$

Proof. We consider a mapping $g : Q \to \mathbb{P}(Q)$. In view of (7) , (1) and Lemma 3.2 b) , c) we have for $x \in M$, $y \in Q$ and $z \in g(y)$ the equations

$$w_\lambda(y-x, z-x) = z_\lambda - x \ ,$$

where $z_\lambda = w_\lambda(y,x) \in g_\lambda(y)$ (see also (13) and the passage after it). Putting $\delta = 0$ in Lemma 3.5 the equivalence

$$d^{\alpha}(y-x, z-x) \geq 0 \quad \Longleftrightarrow \quad d^{\beta}(y-x, z_\lambda - x) \geq 0$$

follows if the parameters satisfy the conditions given in this theorem. Now, iff $(x,y,z) \in J_g$ is related to $(x,y,z_\lambda) \in J_{g_\lambda}$, then a bijective mapping between these two triple sets is established. Considering the characterization (15) of $g \in \mathbb{F}^{\alpha}$ and $F_-(g) = F_-(g_\lambda)$ this corresponds to the first assertion. Choosing $\delta > 0$ in Lemma 3.5 the statement

$$d^{\beta+\delta}(y-x, z_\lambda - x) \geq 0 \quad \Longleftrightarrow \quad d^{\alpha+\delta\lambda}(y-x, z-x) \geq 0$$

for the above listed elements shows also the index result. ■

Corollary 5.2. The following relations are fulfilled:

$$0 < \lambda < 1 : \ g \in \mathbb{F}_r(M) \ \Longleftrightarrow \ g_\lambda \in \mathbb{F}^{\beta}(M) \quad \text{for} \quad \beta = \frac{1-\lambda}{\lambda} \ ,$$
$$0 < \lambda < 2 : \ g \in \mathbb{F}^{1}(M) \ \Longleftrightarrow \ g_\lambda \in \mathbb{F}^{\beta}(M) \quad \text{for} \quad \beta = \frac{2-\lambda}{\lambda} \ .$$

Proof. A rearrangement of the relation $(1+\beta)\,\lambda = 1+\alpha$ in Theorem 5.1 supplies $\beta = \frac{1}{\lambda}(1+\alpha-\lambda)$. Further, observe $\mathbb{F}_r(M) = \mathbb{F}^0(M)$ (see Remark 1.3). Now the assertions follow immediately by putting $\alpha = 0$ and $\alpha = 1$ in Theorem 5.1, respectively. ■

Corollary 5.3. Let be $g \in \mathbb{F}_r$. For $\lambda^* = \lambda^*_F(g)$ we get $g_{\lambda^*} \in \mathbb{F}^1_*$, that is

$$\alpha^*_F(g_{\lambda^*}) = 1 \ .$$

Proof. There is an index $\alpha = \alpha^* = \alpha^*_F(g)$ such that $g \in \mathbb{F}^\alpha_*$. Then by Theorem 5.1 and Theorem 4.7 the relation $g_{\lambda^*} \in \mathbb{F}^\beta$ holds with

$$\alpha^*_F(g_{\lambda^*}) = \beta = \frac{1}{\lambda^*}\,(1 + \alpha^* - \lambda^*) = \frac{1+\alpha^*}{\lambda^*} - 1 = 1 \ .$$

But this is the assertion. ■

Corollary 5.4. For a mapping g and parameters $\alpha \geq 0$, $\beta \geq 0$, $\lambda > 0$ and $\mu > 0$ which are connected by the equation

$$(1+\alpha)\,\lambda = (1+\beta)\,\mu$$

the following statements hold:

$$g_\lambda \in \mathbb{F}^\alpha(M) \quad \Longleftrightarrow \quad g_\mu \in \mathbb{F}^\beta(M) \ .$$

Proof. By Lemma 3.2 f) and $g_\lambda = w_\lambda(I, g)$ we have

$$g_\mu = (g_\lambda)_{\frac{\mu}{\lambda}} \ .$$

If we use Theorem 5.1 with g_λ instead of g and with $\mu' = \frac{\mu}{\lambda}$ instead of λ, then the asserted equivalences are fulfilled for the parameter relation $\mu'\,(1+\beta) = 1 + \alpha$. But this is the assertion. ■

We want to show now that appropriate relaxations g_λ of g create a complete system of representatives of the family $\mathbb{F}^\alpha$. We start with the following concept.

Definition 5.5. Let $\{N^\alpha : \alpha > 0\}$ be a family of subsets of a certain basic set N with the property

$$N^\beta \subset N^\alpha \subset N \quad \text{for} \quad \beta > \alpha > 0\ .$$

Besides, consider the sets

$$N_*^\alpha := N^\alpha \setminus \bigcup_{\beta > \alpha} N^\beta\ .$$

The family $\{p_\lambda : \lambda \in J \subseteq \mathbb{R}\}$ of elements $p_\lambda \in N$ is said to be a *scale* (a *strong scale*) of $\{N^\alpha\}$ iff there exists a bijective mapping $c : J \to (0, \infty)$ between the index sets such that $p_\lambda \in N^\alpha$ ($p_\lambda \in N_*^\alpha$) for $\alpha = c(\lambda)$.

Remark 5.6. Obviously the family $\{N^\alpha\}$ generates pairwise disjoint and nonempty sets N_*^α. If $\{p_\lambda\}$ is a strong scale of $\{N^\alpha\}$, then the elements p_λ are pairwise different. Namely, choose $p_\lambda \in N_*^\alpha$ and $p_\mu \in N_*^\beta$, where $\alpha = c(\lambda)$ and $\beta = c(\mu)$. Then $\alpha \neq \beta$ means $N_*^\alpha \neq N_*^\beta$ and therefore $p_\lambda \neq p_\mu$.

Theorem 5.7. Let be $g \in \mathbb{F}^\gamma(M)$ with $\gamma \geq 0$. Then $\{g_\lambda : \lambda \in (0, 1+\gamma)\}$ is a scale, for $\gamma = \alpha_F^*(g)$ even a strong scale of $\{\mathbb{F}^\alpha(M) \ : \ \alpha > 0\}$.

Proof. We consider the family of subsets $N^\alpha := \mathbb{F}^\alpha(M)$ of the basic set $N := \mathbb{F}_s(M)$. By Remark 1.4 the relation

$$\mathbb{F}^\beta(M) \subseteq \mathbb{F}^\alpha(M) \subseteq \mathbb{F}_s(M)$$

is fulfilled for $\beta > \alpha > 0$. Now we turn to the sets $N_*^\alpha = \mathbb{F}_*^\alpha(M)$ introduced in Definition 5.5 which contain mappings $g \in \mathbb{F}_s(M)$ with F-index α (see Remark 1.7). At first, the set $\mathbb{F}_*^1(M)$ is nonempty by Example 6.2. If we choose $g \in \mathbb{F}_*^1(M)$ then Theorem 5.1 implies $g_\lambda \in \mathbb{F}_*^\alpha(M)$ for arbitrary $\alpha > 0$ and $\lambda = \frac{2}{1+\alpha}$. So the sets $\mathbb{F}_*^\alpha(M)$ are all nonempty. This means that even the proper subset relation of Definition 5.5 is satisfied for the sets $\mathbb{F}^\alpha(M)$. Again by Theorem 5.1 the relation $g_\lambda \in \mathbb{F}^\alpha(M)$ holds for the bijective mapping

$$\alpha = c(\lambda) := \frac{1 + \gamma - \lambda}{\lambda}$$

if $g \in \mathbb{F}^{\gamma}(M)$. So $\{g_\lambda\}$ is a scale of $\{\mathbb{F}^{\alpha}(M)\}$. The second part of Theorem 5.1 shows that $\alpha^*_F(g_\lambda) = \alpha$ for $\alpha^*_F(g) = \gamma$. This means $g_\lambda \in \mathbb{F}^{\alpha}_*(M)$. Hence the scale is strong in this case. ■

Remark 5.8. An analogue of Theorem 5.7 holds if the fixed point set M of the operators in the statements is omitted. Namely, the relations $g_\lambda \in \mathbb{F}^{\alpha}$ and $g_\lambda \in \mathbb{F}^{\alpha}(M)$ are equivalent.

Corollary 5.9. Let be $g \in \mathbb{F}^{\gamma}_*$. Then the relaxations g_λ ($\lambda \in (0, 1+\gamma)$) of g are pairwise different.

Proof. By Theorem 5.7 $\{g_\lambda \, : \, \lambda \in (0, 1+\gamma)\}$ is a strong scale of $\{\mathbb{F}^{\alpha}\}$. So by Remark 5.6 the assertion follows. ■

As mentioned at the beginning of this section, analogous results can be formulated for the classes $\mathbb{L}^{\alpha}$ of strongly nonexpansive operators.

6 Applications

Example 6.1. Let $b \, : \, Q \to \mathbb{R}$ be convex and continuous. Then the set $N(b) = \{x \in Q \, : \, b(x) \leq 0\}$ is convex and closed. We assume $N(b)$ to be nonempty. Further, the *subgradient* ∂b is defined on Q. If b^+ denotes the positive part of b, we define for elements $y \in Q$ and $v \in H$

$$\begin{aligned} \mu(b, y, v) &:= \begin{cases} b^+(y)v/\|v\|^2 & \text{if } v \neq 0 \quad , \\ 0 & \text{if } v = 0 \quad , \end{cases} \\ t_b(y) &:= \{\, \mu(b, y, v) \, : \, v \in \partial b(y)\} \quad . \end{aligned} \tag{18}$$

Then the mapping g_b given by $g_b(y) := y - t_b(y)$ is 1-strongly $N(b)$-Fejer monotone, i.e. $\alpha^*_F(g_b) \geq 1$ (see [9]). So results of sections 4 and 5 hold for $g := g_b \in \mathbb{F}^1(N(b))$.

Example 6.2. For a convex and closed set $M \subset Q \subseteq H$ the metric projector $P_M : Q \to Q$ onto M is well-defined. Moreover, P_M is 1-strongly nonexpansive. More precisely, it is even $\alpha_L^*(P_M) = \alpha_F^*(P_M) = 1$ (see [9]). Further, results of sections 4 and 5 can be applied to $g := P_M \in \mathbb{L}_*^1(M)$.

Remark 6.3. We consider the mapping $g(y) = g_b(y) = y - t_b(y)$ with t_b given in (18). If we study the relaxed form (1)

$$g_\lambda(y) = (1-\lambda)\, y + \lambda\, g(y) = y - \lambda\, t_b(y) \ , \quad \lambda \in (0,2) \ ,$$

then the functions

$$r(\lambda) = \|z_\lambda - x\| \quad , \quad z_\lambda \in g_\lambda(y) \quad , \quad x \in N(b)$$

from (13) fulfil the properties of Corollaries 4.3 and 4.4 with $\alpha = 1$. Theorem 4.7 says that $\lambda_F^*(g_b) \geq 1$ holds. Moreover, Corollary 5.2 shows that g_λ is α-strongly Fejer monotone with $\alpha = (2-\lambda)/\lambda$. This is a generalization of a result in [2, p. 308], where only the strict Fejer monotony is proven. Similarly the relaxed projector

$$P_\lambda(y) := (1-\lambda)\, y + \lambda\, P_M(y) = y - \lambda\,(y - P_M(y)) \quad , \quad \lambda \in (0,2)$$

generates the functions

$$r(\lambda) = \|P_\lambda(y) - P_\lambda(x)\|$$

from (16) which fulfil the corresponding properties outlined for strongly nonexpansive operators g. Besides, P_λ is α-strongly nonexpansive with $\alpha = (2-\lambda)/\lambda = \alpha_F^*(P_\lambda)$ by the analogue of Corollary 5.2. This again generalizes results in [2, p. 307] and in [10, p. 47] , where P_λ is proven to be strictly Fejer monotone and nonexpansive in this case, respectively.

Example 6.4. Let M_i $(i = 1, \cdots, m)$ be convex and closed sets with the nonempty intersection $M := \cap_{i=1}^m M_i$. If $g_i \in \mathbb{F}^{\alpha_i}(M_i)$, then

$$g := g_m g_{m-1} \cdots g_1 \in \mathbb{F}^\alpha(M)$$

with

$$\alpha := \frac{1}{2^{m-1}} \min\{\alpha_i \ : \ i = 1, \cdots, m\}$$

and

$$\begin{aligned} g &:= \gamma_1 g_1 + \gamma_2 g_2 + \cdots + \gamma_m g_m \in \mathbb{F}^\alpha(M) \\ \gamma_i &\geq 0 \quad (i = 1, \cdots, m) \quad , \quad \gamma_1 + \gamma_2 + \cdots + \gamma_m = 1 \end{aligned}$$

with

$$\alpha = \min\{\alpha_i \ : \ i = 1, \cdots, m\} \quad \text{(see [9])} \ .$$

If P_i are the projectors onto M_i and

$$g_i = (1 - \lambda_i) I + \lambda_i P_i \quad , \quad 0 < \lambda_i < 2$$

are corresponding relaxations, then we have the relation $g \in \mathbb{F}^\alpha(M)$ with the above α and $\alpha_i = (2 - \lambda_i)/\lambda_i$ by Remark 6.3. Finally, a further relaxation of g leads to $g_\lambda \in \mathbb{F}^\beta(M)$ with β according to Theorem 5.1.

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