An Orthonormal Bivariate Algebraic Polynomial Basis for $C(I^2)$ of Low Degree

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Abstract

For any fixed $\varepsilon > 0$, we construct an orthonormal Schauder basis $\{P_\mu\}_{\mu=0}^{\infty}$ for $C(I^2)$ consisting of bivariate algebraic polynomials P_μ with $(\deg_q P_\mu)^2 \leq \mathrm{c}_q(1+\varepsilon)\mu$ as a generalization of the corresponding univariate basis constructed in [1]. The orthogonality is with respect to the bivariate Chebyshev weight.

1 Introduction

The question of polynomial Schauder bases for spaces of continuous functions with the uniform norm has a long history. For a more detailed discussion, we refer to the monograph [2, Chap. 5] and to [1, 3] where some final results are proved. Here we follow the approach described in [1]. The main idea to construct an univariate basis $\{p_\mu\}$ consists of an appropriate frequency splitting. I.e., writing p_μ in terms of Chebyshev polynomials, all p_μ with $n_1 < \mu < n_2$ are built from Chebyshev polynomials T_k with $n_1(1-\varepsilon) < k < n_2(1+\varepsilon)$ (cf. (2.2) and Definition 2.1). This decomposition of the natural numbers in overlapping dyadic intervals will now be generalized in the bivariate case to overlapping rectangular blocks. This means that the bivariate polynomials are chosen as appropriate tensor products of the univariate polynomials (see Algorithm 3.1). The important properties of these polynomials will be summarized in our main Theorem 3.4. In this context, the basic question is how to measure the degree of the bivariate polynomials. For this reason, we consider ℓ^q-(quasi) norms with $0 < q \leq \infty$ of the corresponding directional degrees (see (3.1)). Particular cases are $q = 1$, the (total) degree and $q = \infty$, the tensor product degree. Our construction could be easily applied also for $q = 0$. Namely, if one takes the limit $q \to 0$ in (3.1) one obtains $|(k,\ell)|_0 = \sqrt{k\ell}$, which yields polynomials corresponding to so-called hyperbolic crosses. However, this would not result in a basis for $C(I^2)$. In particular, the orthogonal projection operator is not bounded in this case.

2 The Univariate Basis

We recall the construction of an univariate polynomial Schauder basis for $C(I)$ with $I := [-1,1]$ where its elements are orthogonal with respect to the weighted inner product

$$\langle f,g\rangle_I \;=\; \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}\,.$$

Then, the basis polynomials were given in terms of Chebyshev polynomials $T_n(x) = \cos n \arccos x$ $(n \in \mathbb{N}_0)$ which fulfil the orthogonality relations

$$\langle T_n, T_m\rangle_I \;=\; \begin{cases} 2 & \text{for } n=m=0,\\ 1 & \text{for } n=m>0,\\ 0 & \text{otherwise.}\end{cases}$$

For arbitrary $\delta > 0$, a polynomial sequence $\{p_\mu\}_{\mu\in\mathbb{N}_0}$ was defined such that the degree $\deg p_\mu \le \mu(1+\delta)$. To this end, we introduce a natural number η satisfying

$$\begin{array}{cl} \eta = 3, & \text{for } \ \frac{1}{2}\le\delta,\\ \frac{3}{2^{\eta}-2}\le\delta<\frac{3}{2^{\eta-1}-2}, & \text{for } \ 0<\delta<\frac{1}{2}.\end{array} \tag{2.1}$$

For fixed δ, these conditions uniquely determine η, with $\eta \ge 3$. Then every index $\mu \in \mathbb{N}$, $\mu > 2^{\eta+1}$ determines uniquely the triplet of integers j, λ, s such that

$$\begin{array}{rl} \mu \;=\; & 2^j + (\lambda-1)2^{j-\eta+1} + s\,,\\ \text{with} \quad j\ge\eta+1\,, & 0\le\lambda\le 2^{\eta-1}-2\,,\quad 1\le s\le 2^{j-\eta+1}\,.\end{array} \tag{2.2}$$

Furthermore, the two functions,

$$g_1(x) \;:=\; \begin{cases} \frac{2+x}{\sqrt{2+2(x+1)^2}} & \text{for } -2\le\ x\ <0,\\ \frac{2-x}{\sqrt{2+2(x-1)^2}} & \text{for } \ \ 0\le\ x\ \le 2,\end{cases}$$

and

$$g_0(x) \;:=\; \begin{cases} 0 & \text{for } -2\le\ x\ <-\frac{3}{2},\\ \frac{x+\frac{3}{2}}{\sqrt{\frac{1}{2}+2(x+1)^2}} & \text{for } -\frac{3}{2}\le\ x\ <-\frac{1}{2},\\ 1 & \text{for } -\frac{1}{2}\le\ x\ <0,\\ \frac{2-x}{\sqrt{2+2(x-1)^2}} & \text{for } \ \ 0\le\ x\ \le 2,\end{cases}$$

were needed, the sampling of which gives polynomial coefficients.

Definition 2.1. (cf. [1]) *Let $\delta > 0$. Then with η as in (2.1), the polynomials p_μ are given by*

$$\begin{aligned} p_0 &:= \frac{1}{\sqrt{2}}\,, \qquad p_k := T_k\,, \quad \text{for } k = 1,\ldots,2^{\eta+1}-2\,, \\ p_{2^{\eta+1}-1} &:= T_{2^{\eta+1}}\,, \qquad p_{2^{\eta+1}} := \frac{3}{\sqrt{10}} T_{2^{\eta+1}-1} + \frac{1}{\sqrt{10}} T_{2^{\eta+1}+1}\,, \end{aligned}$$

and for $\mu > 2^{\eta+1}$ with j, λ, s as in (2.2), by

$$p_\mu := 2^{(\eta-j)/2} \sum_{k=-3\cdot 2^{j-\eta}+1}^{2^{j-\eta}-1} g_1\left(1+\frac{k}{2^{j-\eta}}\right) \sin\left(\frac{k(2s-1)\pi}{2^{j-\eta+2}}\right) T_{2^j+\lambda 2^{j-\eta+1}+k}\,,$$

if λ is odd, and by

$$p_\mu := 2^{(\eta-j)/2} \sum_{k=-3\cdot 2^{j-\eta}+1}^{2^{j-\eta}-1} g_\lambda\left(1+\frac{k}{2^{j-\eta}}\right) \cos\left(\frac{k(2s-1)\pi}{2^{j-\eta+2}}\right) T_{2^j+\lambda 2^{j-\eta+1}+k}\,,$$

if λ is even, with $g_\lambda := g_1$ for all $\lambda > 0$.

Theorem 2.2. (cf. [1, 4]) *Let $\delta > 0$ be given. Then $\{p_\mu\}_{\mu\in\mathbb{N}_0}$ is an orthonormal polynomial Schauder basis of optimal degree for $C(I)$. I.e., we have for all $\mu, \nu \in \mathbb{N}_0$*

$$\begin{aligned} \deg\, p_\mu &\le \mu(1+\delta)\,, \\ \langle p_\mu\,,\, p_\nu\rangle_I &= \delta_{\mu,\nu}\,, \end{aligned}$$

and for all $f \in C(I)$

$$\left\| f - \sum_{s=0}^{\mu} \langle f\,,\, p_s\rangle_I\, p_s \right\|_{C(I)} \le \mathrm{c}\; E_{\lfloor \mu(1-\delta)\rfloor}(f, C(I))\,.$$

Here, $E_n(f, C(I))$ means the best approximation of f in the uniform norm on I by algebraic polynomials of degree n. In [4], the constant $\mathrm{c} = \mathrm{c}(\delta)$ was estimated as

$$\mathrm{c} \le \frac{4}{\pi^2}\, \log\left(\frac{1}{\delta}\right) + 2700\,.$$

3 The Construction of the Bivariate Basis

Now, we want to construct a bivariate orthonormal algebraic polynomial Schauder basis $\{P_\mu\}_{\mu\in\mathbb{N}_0}$ similar to the univariate basis in the section before, i.e., we want orthonormality with respect to the bivariate Chebyshev weight

$$\langle f,g\rangle_{I^2} := \frac{4}{\pi^2} \iint_{I^2} f(x,y)g(x,y)\,\frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}\,,$$

reproduction of certain polynomials and low increasing of q-degree $\deg_q P_\mu$ for a fixed $q>0$. For nonnegative k,ℓ, we define the ℓ^q-(quasi) norm as

$$|(k,\ell)|_q := \begin{cases} (k^q+\ell^q)^{1/q} & \text{if } 0<q<\infty\,,\\ \max\{k,\ell\} & \text{if } q=\infty\,. \end{cases} \tag{3.1}$$

Then, we introduce the q-degree of a bivariate polynomial P as

$$\deg_q P = \min\left\{\mu\in\mathbb{N}_0 \;:\; P(x,y) = \sum_{|(k,\ell)|_q\le\mu} c_{k,\ell}\,T_k(x)\,T_\ell(y)\right\}\,.$$

Given $\varepsilon>0$, we want our basis polynomials to satisfy

$$\left(\deg_q P_\mu\right)^2 \le (1+\varepsilon)\,\frac{\mu}{A(q)}\,,$$

where analogously to the univariate case the additional factor on the right-hand side is the reciprocal of the area

$$\begin{aligned} A(q) &:= \left|\left\{(x,y)\in\mathbb{R}^2_+ \;;\; |(x,y)|_q\le 1\right\}\right| \\ &= \begin{cases} \frac{\Gamma^2(\frac{1}{q})}{2q\,\Gamma(\frac{2}{q})} & \text{if } 0<q<\infty\,,\\ 1 & \text{if } q=\infty\,. \end{cases} \end{aligned}$$

The construction of the bivariate basis is based on tensor products of polynomials p_μ from the univariate polynomial basis given in the previous section. Therefore, we choose

$$\delta := \min\left\{\frac{\varepsilon}{7},\frac{1}{2}\right\} \tag{3.2}$$

and η and $\{p_\mu\}_{\mu\in\mathbb{N}_0}$ related to this δ by (2.1) and Definition 2.1. We want to give our basis in a constructive way. This can be described in two steps, the first is the initial step, the second is the general step. Because the curves $x^q + y^q = 1$ are convex for $0 < q < 1$ and concave for $q \geq 1$ we consider two different initialization steps. In the sequel, we set $1/q = 0$ for $q = \infty$.

Algorithm 3.1. (Defining algorithm for the sequence $\{P_\mu\}_{\mu \in \mathbb{N}_0}$)

1.a) Initialization step for $q \geq 1$

1.1. $P_0 := p_0 \otimes p_0$, $\mu := 1$, $K_0 := 2^{1/q}\, 2^{\eta+1}$
1.2. for $i := 1, \ldots, 2^{\eta+1}$ do
  for $m := 1, \ldots, i$ do
    for $n := 1, \ldots, i$ do
      if $i - 1 < |(m,n)|_q \leq i$ then $P_\mu := p_m \otimes p_n$, $\mu := \mu + 1$
1.3. for $i := 2^{\eta+1} + 1, \ldots, \lceil K_0 \rceil$ do
  for $m := 1, \ldots, 2^{\eta+1}$ do
    for $n := 1, \ldots, 2^{\eta+1}$ do
      if $i - 1 < |(m,n)|_q \leq i$ then $P_\mu := p_m \otimes p_n$, $\mu := \mu + 1$
1.4. $j_0 := \eta + 1$, $\lambda_0 := 0$

1.b) Initialization step for $0 < q < 1$

1.1. $P_0 := p_0 \otimes p_0$, $\mu := 1$
1.2. find j_0, λ_0 such that

$$2\,(2^{j_0} + \lambda_0 2^{j_0-\eta+1})^q + 2^{q(\eta+1)} \leq 2\,(2^{j_0} + (\lambda_0+1)2^{j_0-\eta+1})^q$$

  $K_0 := 2^{1/q}\,(2^{j_0} + \lambda_0 2^{j_0-\eta+1})$
  $j_1 := j_0$, $\lambda_1 := \lambda_0 + 1$, $K_1 := 2^{1/q}\,(2^{j_1} + \lambda_1 2^{j_0-\eta+1})$
1.3. for $i := 1, \ldots, \lceil K_0 \rceil$ do
  for $m := 1, \ldots, i$ do
    for $n := 1, \ldots, i$ do
      if $i - 1 < |(m,n)|_q \leq i$ then $P_\mu := p_m \otimes p_n$, $\mu := \mu + 1$
1.4. for $j := 0, \ldots, j_1 + \lceil 1/q \rceil$ do
  for $k := 0, \ldots, j_1 + \lceil 1/q \rceil$ do
    for $\lambda := 0, \ldots, 2^{\eta-2} - 2$ do
      for $\nu := 0, \ldots, 2^{\eta-2} - 2$ do
        if $|(2^j + \lambda 2^{j-\eta+1}, 2^k + \nu 2^{k-\eta+1})|_q \leq K_1$
          then for $r := 1, \ldots, 2^{j-\eta+1}$ do
            $m := 2^j + (\lambda - 1)2^{j-\eta+1} + r$
            for $s := 1, \ldots, 2^{k-\eta+1}$ do
              $n := 2^k + (\nu - 1)2^{k-\eta+1} + s$
              if $|(m,n)|_q > K_0$ then $P_\mu := p_m \otimes p_n$, $\mu := \mu + 1$
1.5. $j_0 := j_1$, $\lambda_0 := \lambda_1$, $K_0 := K_1$

2. General Step

2.1. j_0, λ_0, K_0, μ from the previous step

2.2. if $\lambda_0 < 2^{\eta-1}$

then $\lambda_1 = \lambda_0 + 1$, $j_1 := j_0$ else $\lambda_1 = 0$, $j_1 := j_0 + 1$

$K_1 := 2^{1/q}\,(2^{j_1} + \lambda_1 2^{j_1-\eta+1})$

2.3 for $j := 0, \ldots, j_1 + \lceil 1/q \rceil$ do

for $k := 0, \ldots, j_1 + \lceil 1/q \rceil$ do

for $\lambda := 0, \ldots, 2^{\eta-2} - 2$ do

for $\nu := 0, \ldots, 2^{\eta-2} - 2$ do

if $K_0 < |(2^j + \lambda 2^{j-\eta+1}, 2^k + \nu 2^{k-\eta+1})|_q \le K_1$

then for $r := 1, \ldots, 2^{j-\eta+1}$ do

$m := 2^j + (\lambda - 1)2^{j-\eta+1} + r$

for $s := 1, \ldots, 2^{k-\eta+1}$ do

$n := 2^k + (\nu - 1)2^{k-\eta+1} + s$

$P_\mu := p_m \otimes p_n$, $\mu := \mu + 1$

2.4. $j_0 := j_1$, $\lambda_0 := \lambda_1$, $K_0 := K_1$

2.5. goto 2.1.

Let us summarize properties of the P_μ defined in this way. By construction, we obtain orthonormality

$$\begin{aligned}\langle P_\mu, P_\nu \rangle_{I^2} &= \langle p_{m_1} \otimes p_{n_1}, p_{m_2} \otimes p_{n_2} \rangle_{I^2} \\ &= \langle p_{m_1}, p_{m_2} \rangle_I \, \langle p_{n_1}, p_{n_2} \rangle_I = \delta_{m_1,m_2} \delta_{n_1,n_2} = \delta_{\mu,\nu} \,. \end{aligned} \tag{3.3}$$

Now, we fix a number μ and the values K_0, K_1 of the step (1b.4. or 2.3) of Algorithm 3.1 in which we counted μ. (The result (3.6) holds trivially for the μ counted earlier in steps 1a.2, 1a.3 or 1b.3.) By construction (see step 2.2 or 1b.2), we can estimate

$$K_0 \;\le\; K_1 \;\le\; K_0\,(1 + 2^{-\eta}) \;\le\; K_0 \left(1 + \frac{\delta}{3}\right) . \tag{3.4}$$

Because of Definition 2.1 of the underlying univariate polynomial basis we obtain for the q-degree

$$\deg_q P_\mu \;\le\; K_1\;(1 + \delta)\,.$$

Furthermore, we use the geometry of the problem for estimating from below the number of polynomials already counted

$$\mu \;\ge\; A(q)\; K_0^2\; (1 - 2^{-\eta})^2 \;\ge\; A(q)\; K_0^2 \left(1 - \frac{\delta}{3}\right)^2 . \tag{3.5}$$

Hence,

$$\frac{\left(\deg_q P_\mu\right)^2}{\mu} \leq \frac{1}{A(q)} \frac{K_1^2\,(1+\delta)^2}{K_0^2\,\left(1-\frac{\delta}{3}\right)^2} \leq \frac{1}{A(q)} \frac{K_0^2\,(1+\delta)^2\,\left(1+\frac{\delta}{3}\right)^2}{K_0^2\,\left(1-\frac{\delta}{3}\right)^2}.$$

Exploiting additionally the fact $\delta \leq 1/2$, we obtain after some simple calculations

$$\frac{(1+\delta)^2\,\left(1+\frac{\delta}{3}\right)^2}{\left(1-\frac{\delta}{3}\right)^2} \leq (1+7\,\delta)\ .$$

With the definition (3.2) of δ, we obtain the desired result

$$\left(\deg_q P_\mu\right)^2 \leq (1+\varepsilon)\,\frac{\mu}{A(q)} \tag{3.6}$$

for the degrees of the polynomials in the sequence $\{P_\mu\}_{\mu\in\mathbb{N}_0}$. The way of ordering the polynomials will turn out to be an essential ingredient in proving the main result. As in [1], we collect the univariate polynomials in sample spaces

$$V = V_{2^j+(\lambda-1)2^{j-\eta+1}}^{2^{j-\eta}} = \operatorname{span}\{p_\mu\ ;\ \mu = 0,\ldots,2^j+(\lambda-1)2^{j-\eta+1}\}$$

and wavelet packet spaces

$$W = W_{2^j,\lambda}^{2^{j-\eta}} = \operatorname{span}\{p_{2^j+(\lambda-1)2^{j-\eta+1}+s}\ ;\ s = 1,\ldots,2^{j-\eta+1}\}\,.$$

Observe that all polynomials in a wavelet packet space $W_{2^j,\lambda}^{2^{j-\eta}}$ are of the same degree. The ordering uses this structure. That means, the algorithm in the general step first collects all polynomials in one bivariate wavelet packet space $W_{2^j,\lambda}^{2^{j-\eta}} \otimes W_{2^k,\nu}^{2^{k-\eta}}$ before it proceeds to the next packet space.

Now we can prove some basic properties of the polynomial sequence $\{P_\mu\}_{\mu\in\mathbb{N}_0}$.

Lemma 3.2. *There exists an index μ_0, such that for all $\mu \geq \mu_0$ and every polynomial Q with*

$$\deg_\infty Q \leq 2^{-1/q}\sqrt{\frac{\mu}{A(q)}}\,(1-\varepsilon)\,,$$

it holds that

$$Q \in \mathit{span}\{P_k\ ;\ k = 0,\ldots,\mu\}\,.$$

Proof. The sequence $\{P_\mu\}_{\mu\in\mathbb{N}_0}$ was defined by Algorithm 3.1. Assume, we had already run Step 1 and once Step 2. We take the last counted parameter μ and denote it by μ_0. Now, we fix $\mu \geq \mu_0$. That μ was counted in a certain Step 2 with the parameters j_0, λ_0, K_0, and K_1. That means, with $N := 2^{j_0} + (\lambda_0 - 1)2^{j_0-\eta+1}$ and $M := 2^{j_0-\eta}$, we have

$$V_N^M \otimes V_N^M \;\subset\; \mathrm{span}\,\{P_k\;;\; k = 0,\ldots,\mu\}\,.$$

From the univariate construction, we can use the fact that

$$\Pi_{\lfloor N(1-\delta)\rfloor} \;\subset\; V_N^M$$

which gives

$$\Pi_{\lfloor N(1-\delta)\rfloor} \otimes \Pi_{\lfloor N(1-\delta)\rfloor} \;\subset\; \mathrm{span}\,\{P_k\;;\; k = 0,\ldots,\mu\}\,. \tag{3.7}$$

With (3.4) and (3.5)

$$A(q)\,K_0^2\left(1-\frac{\delta}{3}\right)^2 \;\leq\; \mu \;\leq\; A(q)\,K_1^2 \;\leq\; A(q)\,K_0^2\left(1+\frac{\delta}{3}\right)^2,$$

we estimate

$$\begin{aligned}
\sqrt{\mu} &\leq \sqrt{A(q)}\;K_0\left(1+\frac{\delta}{3}\right)\\
\sqrt{\frac{\mu}{A(q)}}\;\frac{1}{1+\frac{\delta}{3}} &\leq K_0\\
\sqrt{\frac{\mu}{A(q)}}\left(1-\frac{\delta}{3}\right) &\leq K_0 \;=\; 2^{1/q}\,N\\
2^{-1/q}\sqrt{\frac{\mu}{A(q)}}\left(1-\frac{\delta}{3}\right)(1-\delta) &\leq N\,(1-\delta)\\
2^{-1/q}\sqrt{\frac{\mu}{A(q)}}\,(1-\varepsilon) &\leq N\,(1-\delta)\,.
\end{aligned}$$

By the assumption on Q, we conclude

$$Q \;\in\; \Pi_{\lfloor N(1-\delta)\rfloor} \otimes \Pi_{\lfloor N(1-\delta)\rfloor}$$

which together with (3.7) proves the lemma.

Let the partial sum operator for $f \in C(I^2)$ be defined by

$$S_\mu f := \sum_{k=0}^{\mu} \langle f, P_k \rangle_{I^2} \, P_k \, .$$

Lemma 3.3. *The norm of the partial sum operator S_μ is bounded*

$$\|S_\mu\|_{C(I^2)\to C(I^2)} \;\le\; \mathrm{C}(q,\varepsilon)\,, \tag{3.8}$$

independently of μ.

Proof. Choose μ_0 as in the proof of Lemma 3.2. For $\mu < \mu_0$ it holds that

$$\|S_\mu\|_{C(I^2)\to C(I^2)} \;\le\; \|S_{\mu_0}\|_{C(I^2)\to C(I^2)} \, .$$

Now, let $\mu \ge \mu_0$ where μ was counted in a certain Step 2 with j_0, λ_0, K_0 and K_1. Denote $f_s := \langle f, P_s\rangle_{I^2}$. Because Algorithm 3.1 first collects all polynomials in one bivariate wavelet packet space $W^{2^{j-\eta}}_{2^j,\lambda} \otimes W^{2^{k-\eta}}_{2^k,\nu}$ before it defines polynomials P_μ from the next packet space we can split the partial sum in the following way

$$\begin{aligned}\left\|\sum_{s=0}^{\mu} f_s\,P_s\right\|_{C(I^2)} \;&\le\; \left\|\sum_{P_s\in VV} f_s\,P_s\right\|_{C(I^2)} + \sum_{VW\subset\mathcal{VW}} \left\|\sum_{P_s\in VW} f_s\,P_s\right\|_{C(I^2)} \\ &\quad + \sum_{WV\subset\mathcal{WV}} \left\|\sum_{P_s\in WV} f_s\,P_s\right\|_{C(I^2)} + \left\|\sum_{\substack{P_s\in WW,\\ s\le\mu}} f_s\,P_s\right\|_{C(I^2)} .\end{aligned}$$

With the notations $VV, \mathcal{VW}, \mathcal{WV}, WW$, we abbreviate the following spaces

$$\begin{aligned} VV \;&:=\; V^{2^{j_0-\eta}}_{2^{j_0}+(\lambda_0-1)2^{j_0-\eta+1}} \otimes V^{2^{j_0-\eta}}_{2^{j_0}+(\lambda_0-1)2^{j_0-\eta+1}}\,, \\ \mathcal{VW} \;&:=\; \bigcup V^{2^{j-\eta}}_{2^j+(\lambda-1)2^{j-\eta+1}} \otimes W^{2^{k-\eta}}_{2^k,\nu}\,, \\ &\qquad \text{where } 2^{j_0} + (\lambda_0-1)2^{j_0-\eta+1} \le 2^k + (\nu-1)2^{k-\eta+1}\,, \\ &\qquad \text{all polynomials in } VW := V^{2^{j-\eta}}_{2^j+(\lambda-1)2^{j-\eta+1}} \otimes W^{2^{k-\eta}}_{2^k,\nu} \\ &\qquad \text{were already counted before } \mu\,, \\ &\qquad j,\lambda \text{ are chosen as big as possible}\,, \\ \mathcal{WV} \;&:=\; \bigcup W^{2^{j-\eta}}_{2^j,\lambda} \otimes V^{2^{k-\eta}}_{2^k+(\nu-1)2^{k-\eta+1}}\,, \\ &\qquad \text{where } 2^{j_0} + (\lambda_0-1)2^{j_0-\eta+1} \le 2^j + (\nu-1)2^{j-\eta+1}\,, \end{aligned}$$

$$\begin{array}{rcl} & & \text{all polynomials in } WV := W^{2^{j-\eta}}_{2^j,\lambda} \otimes V^{2^{k-\eta}}_{2^k+(\nu-1)2^{k-\eta+1}} \\ & & \text{were already counted before } \mu\,, \\ & & k,\nu \text{ are chosen as big as possible}\,, \\ WW & := & W^{2^{j-\eta}}_{2^j,\nu} \otimes W^{2^{k-\eta}}_{2^k,\nu}\,, \\ & & \text{where } P_\mu \in WW\,. \end{array}$$

Using the bounds for the univariate projection norms (cf. [4])

$$\begin{aligned} \left\| \sum_{p_s\in V} \langle 1, |p_s|\rangle_I \; p_s \right\|_{C(I)} &=: \; \mathrm{C}_V \le \frac{4}{\pi^2}\, \log\left(\frac{1}{\delta}\right) + 35\,, \\ \left\| \sum_{p_s\in W} \langle 1, |p_s|\rangle_I \; p_s \right\|_{C(I)} &=: \; \mathrm{C}_W \le 2625\,, \end{aligned}$$

we estimate

$$\begin{aligned} \left\| \sum_{P_s\in VV} f_s P_s \right\|_{C(I^2)} &\le \; \|f\|_{C(I^2)}\; \mathrm{C}_V^2\,, \\ \left\| \sum_{P_s\in VW(WV)} f_s P_s \right\|_{C(I^2)} &\le \; \|f\|_{C(I^2)}\; \mathrm{C}_V\,\mathrm{C}_W\,, \\ \left\| \sum_{P_s\in WW} f_s P_s \right\|_{C(I^2)} &\le \; \|f\|_{C(I^2)}\; \mathrm{C}_W^2\,. \end{aligned}$$

Hence, we obtain

$$\|S_\mu f\|_{C(I^2)} \le \|f_s\|_{C(I^2)} \left(\mathrm{C}_V^2 + \sum_{VW\subset\mathcal{VW}} \mathrm{C}_V\,\mathrm{C}_W + \sum_{WV\subset\mathcal{WV}} \mathrm{C}_W\,\mathrm{C}_V + \mathrm{C}_W^2 \right)\,.$$

It remains to estimate the number n_{VW} of the spaces VW in $\mathcal{VW}$ and WV in $\mathcal{WV}$. Assume $\lambda_0 < 2^{\eta-1}-2$ (the estimation for $\lambda_0 = 2^{\eta-1}-2$ can be done in a similar way). Then,

$$\begin{aligned} n_{VW} &\le \; 2^{-j_0+\eta} \left(2^{1/q}\,(2^{j_0} + (\lambda_0+1)\,2^{j_0-\eta+1}) - (2^{j_0} + \lambda_0\,2^{j_0-\eta+1}) \right) \\ &= \; 2\, \left((2^{1/q}-1)\,(2^{\eta-1}+\lambda_0) + 2^{1/q} \right) \\ &\le \; 2\, \left((2^{1/q}-1)\,(2^{\eta-1}+2^{\eta-1}-2) + 2^{1/q} \right) \end{aligned}$$

$$\begin{aligned} n_{VW} &\leq 2 \left((2^{1/q} - 1) \, (2^{\eta} - 2) + 2^{1/q} \right) \\ &\leq 2 \left(2 \, (2^{1/q} - 1) \, \left(\frac{3}{\delta} + 1 \right) + 2^{1/q} \right) \end{aligned}$$

leads to

$$\begin{aligned} \|S_\mu\|_{C(I^2) \to C(I^2)} &\leq \mathrm{C}_V^2 + \mathrm{C}_W^2 + 2 \left(2(2^{1/q} - 1) \left(\frac{3}{\delta} + 1 \right) + 2^{1/q} \right) \mathrm{C}_V \, \mathrm{C}_W \\ &=: \mathrm{C}(q, \varepsilon) \, , \end{aligned}$$

which proves the lemma.

Now we are able to state the main result of this paper.

Theorem 3.4. *Let $\varepsilon > 0$ be given. Then $\{P_\mu\}_{\mu \in \mathbb{N}_0}$ is an orthornormal polynomial Schauder basis of low degree for $C(I^2)$. I.e., the polynomial sequence $\{P_\mu\}_{\mu \in \mathbb{N}_0}$ satisfies for all $\mu, \nu \in \mathbb{N}_0$ the following properties*

$$\left(\deg_q P_\mu \right)^2 \leq \frac{\mu}{A(q)} \, (1 + \varepsilon) \, , \tag{3.9}$$

$$\langle P_\mu, P_\nu \rangle_{I^2} = \delta_{\mu,\nu} \tag{3.10}$$

and furthermore there exists $\mu_0 \in \mathbb{N}$ such that for all $\mu \geq \mu_0$ and all $f \in C(I^2)$

$$\|f - S_\mu f\|_{C(I^2)} \leq (1 + \mathrm{C}(q, \varepsilon)) \, E_{\mathcal{M}}(f, C(I^2)) \, , \tag{3.11}$$

where

$$\mathcal{M} := \left\lfloor 2^{-1/q} \sqrt{\frac{\mu}{A(q)}} \, (1 - \varepsilon) \right\rfloor .$$

Here, $E_n(f, C(I^2))$ denotes the best approximation of f in the uniform norm on I^2 by algebraic polynomials from $\Pi_n \otimes \Pi_n$.

From the construction, we have for the constants

$$\mathrm{C}(q, \varepsilon) \longrightarrow \infty \qquad \text{for } q \longrightarrow 0 \quad \text{or } \varepsilon \longrightarrow 0 \, .$$

Let us mention here that an estimate of the form (3.11) is sufficient to state the Schauder basis property. However, replacing the inequality in Lemma 3.2 by a statement for $\deg_r Q$ would yield more general error estimates than (3.11) with best approximation from different kinds of polynomial spaces.

Proof. The properties (3.9), (3.10) follow directly from the defining Algorithm 3.1 (see Equations (3.6), (3.3)). The approximation property (3.11) can be proved easily using the previous lemmata. With the same μ_0 as in the proof of Lemma 3.2, we estimate for arbitrary $\mu \geq \mu_0$ and $Q \in \Pi_{\mathcal{M}} \otimes \Pi_{\mathcal{M}}$

$$\begin{aligned} \|f - S_\mu f\|_{C(I^2)} &= \|f - Q + S_\mu(Q - f)\|_{C(I^2)} \\ &\leq (1 + \mathrm{C}(q, \varepsilon)) \ \|f - Q\|_{C(I^2)} \, . \end{aligned}$$

Choosing Q as the polynomial of best approximation proves the theorem.

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