Interpolation and Wavelets on Sparse Gauß–Chebyshev Grids

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Abstract

Nested spaces of multivariate functions on the square forming a non-stationary multiresolution analysis are investigated. The scaling functions of these spaces are fundamental Lagrange interpolants on a sparse Gauß–Chebyshev grid. The approach based on Boolean sums leads to sample spaces of significantly lower dimension. The algorithms for complete decomposition and reconstruction are of simple structure and low complexity.

1 Introduction

We are interested here in wavelet methods on a bounded domain, in our case on the square $[-1,1]^2$, which are based on the construction of univariate wavelets on an interval. Therefore, we consider special bivariate sample and wavelet spaces of functions on the square. As our starting point we take univariate interpolatory scaling functions and wavelets similar to those introduced by Kilgore and Prestin [11], which were recently further developed and generalized by Plonka, Selig, and Tasche [12, 17]. This approach works for the weighted Hilbert space $L^2_w([-1,1])$ and with generalized shifts for functions on the interval instead of special scaling functions and wavelets at the boundary as in e.g. [4, 5, 13]. We want to introduce here new bivariate interpolatory scaling functions and wavelets by investigating Boolean sums of interpolation operators. In this way, we overcome the problem of a very fast growth of the dimensions that we have in the ordinary tensor product case. That means, we obtain sample spaces with dimension $\mathcal{O}(j\,2^j)$ while tensor products yield dimension $\mathcal{O}(2^{2j})$ for the level j. However, because of the simple structure of the spaces one can take use of fast algorithms provided in [12] for the univariate case. The concept of Boolean sums and sparse grids is often applied in interpolation and approximation (see e.g. [1, 6]) as well as in finite element methods (see e.g. [9, 10, 20]). It uses essentially the interpolatory properties of the underlying univariate scaling functions and wavelets. Similar approaches are the approximation of periodic functions by trigonometric polynomials with frequencies from the hyperbolic cross (see e.g. [8, 18, 19]) and the investigation of hyperbolic wavelets [7] in the cardinal case.

Applying the Boolean sum method, we are able to construct multivariate scaling functions and wavelets from univariate interpolatory scaling functions and wavelets. We obtain the bivariate sample spaces

$$V_j^B = \sum_{r=0}^{j} V_r \otimes V_{j-r},$$

and the bivariate wavelet spaces

$$W_{j+1}^B = \bigoplus_{r=0}^{j+1}{}^{\perp}\; W_r \otimes W_{j+1-r},$$

from the underlying univariate sample spaces V_r and wavelet spaces W_r, respectively. The univariate sample spaces V_r are images of interpolation operators L_r. Analogously, the bivariate sample spaces V_j^B are images of blending interpolation operators B_j defined by j-th order Boolean sums.

We can give the algorithms for complete decomposition and reconstruction which are based on the algorithms for the univariate case. They have simple structure and low complexity of $\mathcal{O}(j^2\, 2^j)$ essential operations for the level j.

2 Univariate Interpolation and Sample and Wavelet Spaces on the Interval

Let $d \in \mathbb{N}$ be a fixed parameter. Put $d_j := d2^j, j \in \mathbb{N}_0$. We denote our reference interval by $I := [-1, 1]$ and the Chebyshev weight by $w(x) := (1 - x^2)^{-1/2}\,(x \in (-1, 1))$. Let $L^2_w(I)$ be the weighted Hilbert space of all measurable functions $f : I \longrightarrow \mathbb{R}$ with

$$\int_I f(x)^2 w(x)\, dx < \infty.$$

For $f, g \in L^2_w(I)$, the corresponding inner product is given by

$$\langle f, g\rangle_I := \frac{2}{\pi} \int_I f(x)g(x)w(x)\, dx\,.$$

By Π_n we denote the set of all real valued polynomials of degree at most n restricted on I. Furthermore let $T_n \in \Pi_n$ be the Chebyshev polynomials $T_n(x) := \cos(n \arccos x)$ and $\mathcal{G}_j$ the grid of the Gauß–Chebyshev nodes

$$\mathcal{G}_j := \{g_{j,k} := \cos k\tfrac{\pi}{2^j}\,;\ k = 0, \ldots, d_j\}.$$

The Chebyshev shift (cf. [3]) $s_h f$ of a function f is defined as

$$(s_h f)(x) := \tfrac{1}{2}\, f(xh - \tfrac{1}{w(x)w(h)}) + \tfrac{1}{2}\, f(xh + \tfrac{1}{w(x)w(h)}), \qquad x \in I.$$

For every level j, we need the shifts $\sigma_{j,k} := s_{g_{j,k}}$, $k = 0, \ldots, d_j$. We choose scaling functions ϕ_j as modified Langrange interpolants on the Gauß–Chebyshev grid, i.e.,

$$(\sigma_{j,k}\phi_j)(g_{j,\ell}) = \frac{1}{\varepsilon_{j,k}}\, \delta_{k,l}, \tag{2.1}$$

where here and in the following

$$\varepsilon_{j,k} := \begin{cases} \frac{1}{2} & \text{if} \quad k = 0 \text{ or } k = d_j, \\ 1 & \text{if} \quad k = 1, 2, \ldots, d_j - 1. \end{cases}$$

We define the sample spaces V_j by

$$V_j := \operatorname{span}\{\phi_{j,k} := \sigma_{j,k}\phi_j\,;\ k = 0, \ldots, d_j\} \tag{2.2}$$

with the dimension $\dim V_j = d_j + 1$. We assume the spaces V_j to form a non–stationary multiresolution of the weighted Hilbert space $L_2(I)$, i.e.,

$$V_j \subset V_{j+1},\ j \in \mathbb{N}_0 \quad \text{and} \quad \operatorname{clos}_{L^2_w(I)} \bigcup_{j\in\mathbb{N}_0} V_j = L^2_w(I).$$

The interpolation operator L_j projecting onto $\operatorname{Im} L_j = V_j$ with the interpolation nodes $\mathcal{G}_j$ is given as

$$L_j f := \sum_{k=0}^{d_j} \varepsilon_{j,k}\, f(g_{j,k})\, \phi_{j,k}$$

Now let us consider the univariate wavelet spaces W_{j+1} defined by

$$V_{j+1} = V_j \oplus^{\perp} W_{j+1},$$

where $\oplus^{\perp}$ denotes the orthogonal sum with respect to the inner product in $L^2_w(I)$. Assume the wavelet spaces W_{j+1} to be generated by the shifts of a single function ψ_{j+1}

$$W_{j+1} := \operatorname{span}\{\psi_{j+1,2k+1} := \sigma_{j+1,2k+1}\psi_{j+1}\,;\ k = 0, 1, \ldots, d_j - 1\} \tag{2.3}$$

with wavelet functions $\psi_{j+1,2k+1}$ satisfying also interpolatory conditions

$$\psi_{j+1,2m+1}\left(g_{j+1,2\ell+1}\right)=\delta_{\ell,m}, \qquad \ell=0,1,\ldots,d_j-1. \tag{2.4}$$

Then, we want to establish reconstruction and decomposition relation in matrix form. Therefore, we define the function vectors $\boldsymbol{\phi}_{j-1}:=(\phi_{j-1,k})_{k=0}^{d_{j-1}}$, $\boldsymbol{\psi}_j:=(\psi_{j,2k+1})_{k=0}^{d_{j-1}-1}$, and $\boldsymbol{P}_j\,\boldsymbol{\phi}_j:=(\phi_{j,0},\phi_{j,2},\ldots,\phi_{j,d_j},\phi_{j,1},\phi_{j,3},\ldots,\phi_{j,d_j-1})^T$, the following knot evaluation and permutation matrices

$$\begin{aligned}
\boldsymbol{K}_{j-1} &:= \left(\phi_{j-1,k}(g_{j,2\ell+1})\right)_{k=0,\,\ell=0}^{d_{j-1},\,d_{j-1}-1}, \quad \boldsymbol{P}_j=\left(\delta_{k,2\ell}+\delta_{k+d_{j-1}+1,2\ell+1}\right)_{k,\ell=0}^{d_{j-1}},\\
\boldsymbol{L}_j &:= \left(\psi_{j,2k+1}(g_{j,2\ell})\right)_{k=0,\,\ell=0}^{d_{j-1}-1,\,d_{j-1}}, \quad \tilde{\boldsymbol{P}}_{j-1}=\left(\delta_{k,d_{j-1}-\ell}\right)_{k,\ell=0}^{d_{j-1}-1},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{C}_j^e &:= \left(\cos\tfrac{k\ell\pi}{d_j}\right)_{k,\ell=0}^{d_j}, \qquad \boldsymbol{\mathcal{E}}_j:=\operatorname{diag}\left(\varepsilon_{j,k}\right)_{k=0}^{d_j},\\
\mathbf{C}_j^o &:= \left(\cos\tfrac{(2k+1)\ell\pi}{d_j}\right)_{k,\ell=0}^{d_j-1}, \qquad \tilde{\boldsymbol{\mathcal{E}}}_j:=\operatorname{diag}\left(\varepsilon_{j,k}\right)_{k=0}^{d_j-1}.
\end{aligned}$$

The matrices for the discrete cosine transforms are given by

$$\begin{aligned}
\mathbf{C}_j^{\mathrm{I}} &:= \mathbf{C}_j^e\,\boldsymbol{\mathcal{E}}_j && : \text{ type-I-DCT of length } d_j+1 \text{ (DCT-I } (d_j+1)),\\
\mathbf{C}_j^{\mathrm{II}} &:= \mathbf{C}_j^o && : \text{ type-II-DCT of length } d_j \text{ (DCT-II } (d_j)),\\
\mathbf{C}_j^{\mathrm{III}} &:= (\mathbf{C}_j^e)^T\,\tilde{\boldsymbol{\mathcal{E}}}_j && : \text{ type-III-DCT of length } d_j \text{ (DCT-III } (d_j)).
\end{aligned}$$

They satisfy the relations

$$\mathbf{C}_j^{\mathrm{I}}\,\mathbf{C}_j^{\mathrm{I}}=\tfrac{d_j}{2}\,\boldsymbol{I}_{d_j+1} \qquad \text{and} \qquad \mathbf{C}_j^{\mathrm{II}}\,\mathbf{C}_j^{\mathrm{III}}=\mathbf{C}_j^{\mathrm{III}}\,\mathbf{C}_j^{\mathrm{II}}=\tfrac{d_j}{2}\,\boldsymbol{I}_{d_j}.$$

Fast and numerically stable algorithms for the DCT-I (d_j+1), DCT-II (d_j) and DCT-III (d_j) working in real arithmetic are described in [2, 16].

We use the notation $\boldsymbol{A}\boxplus\boldsymbol{B}:=\operatorname{diag}(\boldsymbol{A},\boldsymbol{B})$ for the direct sum of matrices.

Theorem 2.1. (i) *The two-scale or reconstruction relation is given by*

$$\begin{pmatrix}\boldsymbol{\phi}_{j-1}\\ \boldsymbol{\psi}_j\end{pmatrix}=\boldsymbol{C}_j\,\boldsymbol{\phi}_j,$$

with

$$\begin{aligned}
\boldsymbol{C}_j &= \begin{pmatrix} \boldsymbol{I}_{d_{j-1}+1} & \boldsymbol{K}_{j-1} \\ \tilde{\boldsymbol{\mathcal{E}}}_{j-1}\boldsymbol{L}_j & \boldsymbol{I}_{d_{j-1}} \end{pmatrix} \boldsymbol{P}_j \\
&= \tfrac{1}{d_{j-1}} \left((\mathbf{C}^{\mathrm{I}}_{j-1})^T \boxplus (\mathbf{C}^{\mathrm{II}}_{j-1})^T \right) \, \boldsymbol{C}_{j,\lambda} \, (\mathbf{C}^{\mathrm{I}}_j)^T .
\end{aligned}$$

Here $\boldsymbol{C}_{j,\lambda}$ *is the following sparse matrix*

$$\boldsymbol{C}_{j,\lambda} = \begin{pmatrix} \operatorname{diag}(\alpha_{j-1,k})_{k=0}^{d_{j-1}-1} & \mathbf{0} & \operatorname{diag}(\alpha_{j-1,k})_{k=d_{j-1}+1}^{d_j} \tilde{\boldsymbol{P}}_{j-1} \\ \mathbf{0} & \alpha_{j-1,d_{j-1}} & \mathbf{0} \\ \operatorname{diag}(\beta_{j,k})_{k=0}^{d_{j-1}-1} & \mathbf{0} & -\operatorname{diag}(\beta_{j,k})_{k=d_{j-1}+1}^{d_j} \tilde{\boldsymbol{P}}_{j-1} \end{pmatrix},$$

where

$$\begin{aligned}
(\alpha_{j-1,k})_{k=0}^{d_j} &:= \mathbf{C}^{\mathrm{I}}_j \left(\phi_{j-1}(g_{j,k}) \right)_{k=0}^{d_j}, \\
(\beta_{j,k})_{k=0}^{d_j} &:= \mathbf{C}^{\mathrm{I}}_j \left(\psi_j(g_{j,k}) \right)_{k=0}^{d_j}.
\end{aligned}$$

(ii) *The decomposition relation can be written as*

$$\boldsymbol{\phi}_j = \boldsymbol{D}_j \begin{pmatrix} \boldsymbol{\phi}_{j-1} \\ \boldsymbol{\psi}_j \end{pmatrix},$$

with

$$\boldsymbol{D}_j = \boldsymbol{C}_j^{-1} = \tfrac{2}{d_{j-1}} (\mathbf{C}^{\mathrm{I}}_j)^T \, \boldsymbol{D}_{j,\lambda} \left((\mathbf{C}^{\mathrm{I}}_{j-1})^T \boxplus (\mathbf{C}^{\mathrm{III}}_{j-1})^T \right) .$$

Here $\boldsymbol{D}_{j,\lambda}$ *denotes*

$$\boldsymbol{D}_{j,\lambda} = \begin{pmatrix} -\operatorname{diag}(\tilde{\alpha}_{j-1,k})_{k=0}^{d_{j-1}-1} & \mathbf{0} & \operatorname{diag}(\tilde{\beta}_{j,k})_{k=0}^{d_{j-1}-1} \tilde{\boldsymbol{P}}_{j-1} \\ \mathbf{0} & \tilde{\alpha}_{j-1,d_{j-1}} & \mathbf{0} \\ \operatorname{diag}(\tilde{\alpha}_{j-1,k})_{k=d_{j-1}+1}^{d_j} & \mathbf{0} & \operatorname{diag}(\tilde{\beta}_{j-1,k})_{k=d_{j-1}+1}^{d_j} \tilde{\boldsymbol{P}}_{j-1} \end{pmatrix},$$

where

$$\begin{aligned}
\tilde{\alpha}_{j-1,k} &:= \begin{cases} \frac{\alpha_{j-1,k}}{\alpha_{j-1,k}\beta_{j,d_j-k} + \alpha_{j-1,d_j-k}\beta_{j,k}}, & \textit{for} \quad k = 0, \ldots, d_{j-1}-1, \\ & \qquad\quad d_{j-1}+1, \ldots, d_j, \\ \frac{1}{\alpha_{j-1,d_{j-1}}}, & \textit{for } k = d_{j-1}, \end{cases} \\
\tilde{\beta}_{j,k} &:= \frac{\beta_{j,k}}{\alpha_{j-1,k}\beta_{j,d_j-k} + \alpha_{j-1,d_j-k}\beta_{j,k}}, \qquad \textit{for } k = 0, \ldots, d_j.
\end{aligned}$$

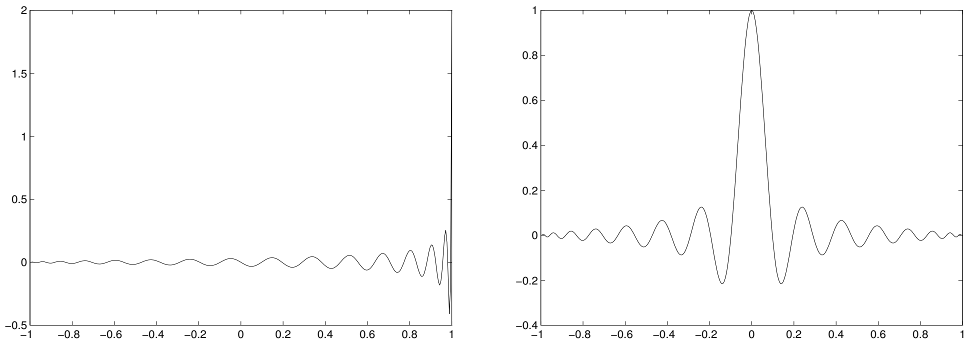

Figure 1: The polynomial scaling function ϕ_5^p and the Chebyshev shift $\phi_{5,16}^p$.

Proof. We derive the form of the reconstruction matrix $\boldsymbol{C}_j$ immediately from the interpolatory properties (2.1) and (2.4) of the scaling functions and the wavelets. The factorized forms of the matrices $\boldsymbol{C}_j$ and $\boldsymbol{D}_j$ are obtained by using the results in [12].

The polynomial scaling functions and wavelets introduced in [11] and generalized in [12, 17] may serve as an example here. The scaling function is given by

$$\phi_j^p := \frac{2}{d_j} \sum_{r=0}^{d_j} \varepsilon_{j,r}\, T_r,$$

with $d = 1$. Then $V_j = \Pi_{d_j}$. The corresponding wavelet function can be written as

$$\psi_{j+1}^p := \frac{2}{d_j} \sum_{r=d_j+1}^{d_{j+1}} \varepsilon_{j+1,r}\, T_r.$$

The vectors $\boldsymbol{\alpha}_{j-1}^p$ ans $\boldsymbol{\beta}_j^p$ needed for decomposition and reconstruction have a simple form

$$\begin{aligned}
\alpha_{j-1,k}^p &= \begin{cases} 2, & \text{for } k = 0, \ldots, d_{j-1} - 1, \\ 1, & \text{for } k = d_{j-1}, \\ 0, & \text{otherwise}, \end{cases} \\
\beta_{j,k}^p &= \begin{cases} 2, & \text{for } k = d_{j-1} + 1, \ldots, d_j, \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}$$

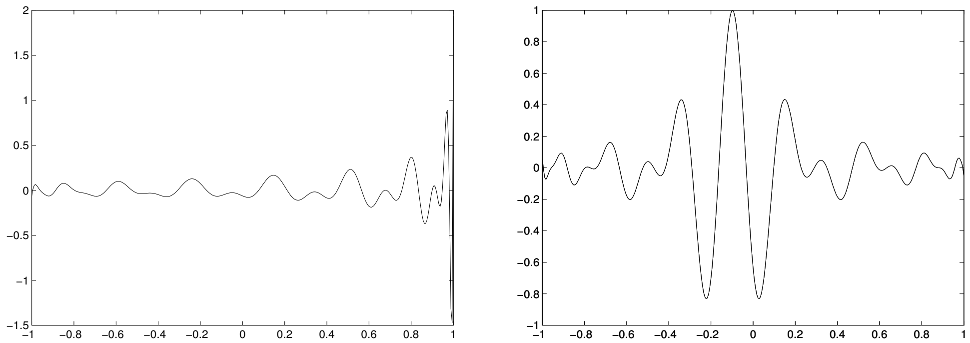

Figure 2: The polynomial wavelet ψ_5^p and the Chebyshev shift $\psi_{5,17}^p$.

As another example one can regard the fundamental spline interpolants constructed from the transformed B-splines of even order in [12].

3 Univariate Decomposition and Reconstruction

In this section, we want to give the matrices for complete decomposition and reconstruction of univariate functions. Functions $f_j \in V_j$ can be represented in the basis of scaling functions as well as in the basis of the wavelets contained in this sample space. If we want to change between these bases of a univariate sample space V_j we need transformations between these bases. The complete decomposition of the scaling functions into wavelets parts in the univariate case means the following. There have to exist coefficients $z_{k,\ell}^j$, such that

$$\phi_{j,k} = \sum_{\nu=0}^{d_0} z_{k,2^j\nu}^j \, \phi_{0,\nu} + \sum_{\mu=0}^{j-1} \sum_{\nu=0}^{d_\mu - 1} z_{k,2^{j-\mu-1}(2\nu+1)}^j \, \psi_{\mu+1,2\nu+1} \, ,$$

where the second index ℓ of the entries of $\boldsymbol{Z}_j := (z_{k,\ell}^j)_{k,\ell=0}^{d_j}$ corresponds to that point $g_{j,\ell}$ in which the function $\phi_{0,\nu}$ or $\psi_{\mu+1,2\nu+1}$, respectively, fulfils the interpolatory condition (2.1) or (2.4) with value 1. On the other hand, there have to be coefficients $r_{k,\ell}^j$, such that the complete reconstruction of the wavelets into terms of scaling functions of a higher level in the univariate case reads as

$$\begin{aligned}\phi_{0,\nu} &= \sum_{\ell=0}^{d_j} r^j_{2^j\nu,\ell}\,\phi_{j,\ell} = \sum_{\ell=0}^{d_j} \varepsilon_{j,\ell}\,\phi_{0,\nu}(g_{j,\ell})\,\phi_{j,\ell},\\ \psi_{\mu+1,2\nu+1} &= \sum_{\ell=0}^{d_j} r^j_{2^{j-\mu-1}(2\nu+1),\ell}\,\phi_{j,\ell} = \sum_{\ell=0}^{d_j} \varepsilon_{j,\ell}\,\psi_{\mu+1,2\nu+1}(g_{j,\ell})\,\phi_{j,\ell}\,,\end{aligned}$$

where the first index k of the entries of $\boldsymbol{R}_j := (r^j_{k,\ell})^{d_j}_{k,\ell=0}$ corresponds to that point $g_{j,k}$ of the fine grid in which the function $\phi_{0,\nu}$ or $\psi_{\mu+1,2\nu+1}$, respectively, satisfies (2.1) or (2.4) with function value 1.

Of course, one can describe the matrices $\boldsymbol{Z}_j$ of complete decomposition and $\boldsymbol{R}_j$ of complete reconstruction by a stepwise application of the matrices $\boldsymbol{D}_r$ and $\boldsymbol{C}_r$ from Theorem 2.1, respectively. Here, we want to use discrete cosine transform in our decomposition and reconstruction algorithms. Therefore, we need some notations. We restrict ourselves to a special case. Assume here and in the sequel that $d = d_0$ is a power of 2, i.e. $d = 2^{t_d}$. Otherwise things can be done in a similar way but will become somewhat more complicated. Let $k \in \{0, \ldots, d_j - 1\}$ have the representation

$$k = \sum_{s=0}^{t} k_s\, 2^s\,, \qquad k_s \in \{0,1\},\ t < t_d + j.$$

Then, the binary representation mirrored at $j + t_d - 1$ (or in bit reversed order) is given by

$$M_j(k) := \sum_{s=0}^{t} k_s\, 2^{j+t_d-1-s}\,.$$

Additionally, we define

$$M_j(d_j) := -1 \qquad \text{and} \qquad M_j(-1) := d_j\,.$$

Now, we want to give factorizations of the matrices for complete univariate decomposition $\boldsymbol{Z}_j$ and reconstruction $\boldsymbol{R}_j$.

Theorem 3.1. *Let $j \in \mathbb{N}$ and $\boldsymbol{Q}_s$ be the permutation matrices*

$$\boldsymbol{Q}_s := \left(\delta_{k,M_s(l)}\right)_{k=-1,\ell=0}^{d_s-1,\,d_s}, \qquad \tilde{\boldsymbol{Q}}_s := \left(\delta_{k,M_s(l)}\right)_{k,\ell=0}^{d_s-1}.$$

All occuring matrix products are taken from left to right.

(i) *The matrix* $\boldsymbol{R}_j$ *for complete univariate reconstruction can be factorized as*

$$\boldsymbol{Q}_j \boldsymbol{R}_j \boldsymbol{Q}_j^T = \tfrac{1}{d_{j-1}} \left(\boldsymbol{Q}_0 (\mathbf{C}_0^{\mathrm{I}})^T \boxplus \mathop{\boxplus}_{r=0}^{j-1} \tilde{\boldsymbol{Q}}_r (\mathbf{C}_r^{\mathrm{II}})^T \right) \boldsymbol{R}_{j,\lambda} \left((\mathbf{C}_j^{\mathrm{I}})^T \boldsymbol{Q}_j^T \right)$$

with the inner sparse matrix

$$\boldsymbol{R}_{j,\lambda} := \prod_{r=1}^{j} \boldsymbol{C}_{r,\lambda} \boxplus \boldsymbol{I}_{d_j - d_r} \, .$$

(ii) *The matrix* $\boldsymbol{Z}_j$ *for complete univariate decomposition can be factorized as*

$$\boldsymbol{Q}_j \boldsymbol{Z}_j \boldsymbol{Q}_j^T = \tfrac{2}{d_{j-1}} \left(\boldsymbol{Q}_j (\mathbf{C}_j^{\mathrm{I}})^T \right) \boldsymbol{Z}_{j,\lambda} \left((\mathbf{C}_0^{\mathrm{I}})^T \boldsymbol{Q}_0^T \boxplus \mathop{\boxplus}_{r=0}^{j-1} (\mathbf{C}_r^{\mathrm{III}})^T \tilde{\boldsymbol{Q}}_r^T \right)$$

with the inner sparse matrix

$$\boldsymbol{Z}_{j,\lambda} := \prod_{r=1}^{j} \boldsymbol{D}_{r,\lambda} \boxplus \boldsymbol{I}_{d_j - d_r} \, .$$

We skip the proof which uses the properties of the DCT-I and the orthogonality of the matrices $\boldsymbol{Q}_r$. As we can see by this theorem univariate complete decomposition and reconstruction, respectively, for the level j cost some discrete cosine transforms of length at most d_{j-1} and a multiplication with a matrix with $\mathcal{O}(j2^j)$ non-zero entries, i.e., altogether $\mathcal{O}(j2^j)$ essential operations.

4 Sample and Wavelet Spaces for Sparse Grids

Starting from the univariate interpolation, we want to investigate bivariate interpolation on a sparse grid. The j-th order Boolean sum interpolation operator is defined by

$$B_j := \bigoplus_{r=0}^{j} L_r \otimes L_{j-r},$$

where $A \oplus B := A + B - AB$ and the representation of B_j in terms of ordinary sums is known (see [1, 6]) to be

$$B_j = \sum_{r=0}^{j} L_r \otimes L_{j-r} - \sum_{r=0}^{j-1} L_r \otimes L_{j-r-1}.$$

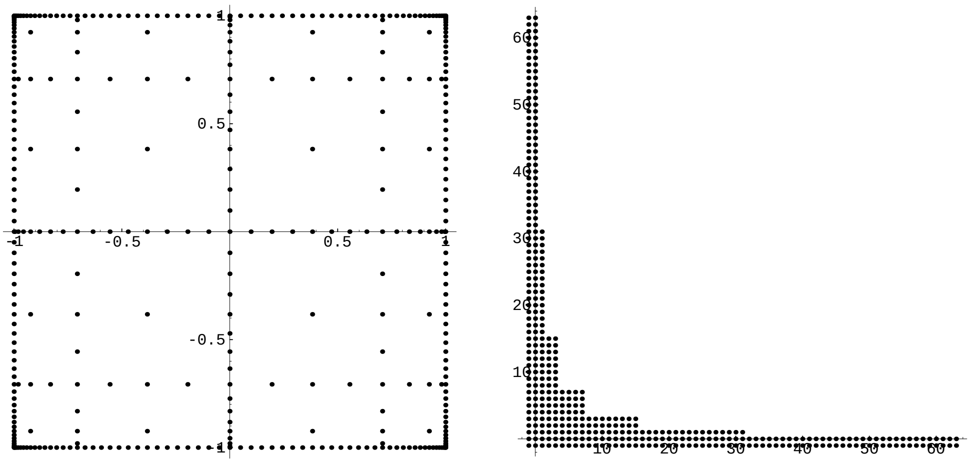

Figure 3: Sparse Gauß–Chebyshev grid (left) and the index set $M_j(\mathcal{J}_j)$ for $d = 1, j = 6$ (right).

The range of this interpolation operator is given by

$$\text{Im } B_j = \sum_{r=0}^{j} \text{Im } L_r \otimes \text{Im } L_{j-r} = \sum_{r=0}^{j} V_r \otimes V_{j-r} \quad \subset \ \text{Im } B_{j+1} \,. \tag{4.1}$$

It interpolates on the sparse grid

$$\mathcal{G}_j^B = \bigcup_{r=0}^{j} \mathcal{G}_r \times \mathcal{G}_{j-r} \quad \subset \ \mathcal{G}_{j+1}^B \,.$$

Now we define new sample spaces of bivariate functions by

$$V_j^B := \text{Im } B_j = \sum_{r=0}^{j} V_r \otimes V_{j-r} \,.$$

From that definition and (4.1) we obtain immediately

$$V_j^B \subset V_{j+1}^B \,.$$

In order to characterize bivariate scaling functions again by their interpolatory properties we observe the relation $\mathcal{G}_j^B \subset \mathcal{G}_j \times \mathcal{G}_j$ between the sparse and the regular interpolation grid. Thus, we can define an index set $\mathcal{J}_j$ by

$$\mathcal{J}_j := \{(k, \ell) \in \mathbb{N}_0^2 \ ; \ (g_{j,k}, g_{j,\ell}) \in \mathcal{G}_j^B\}$$

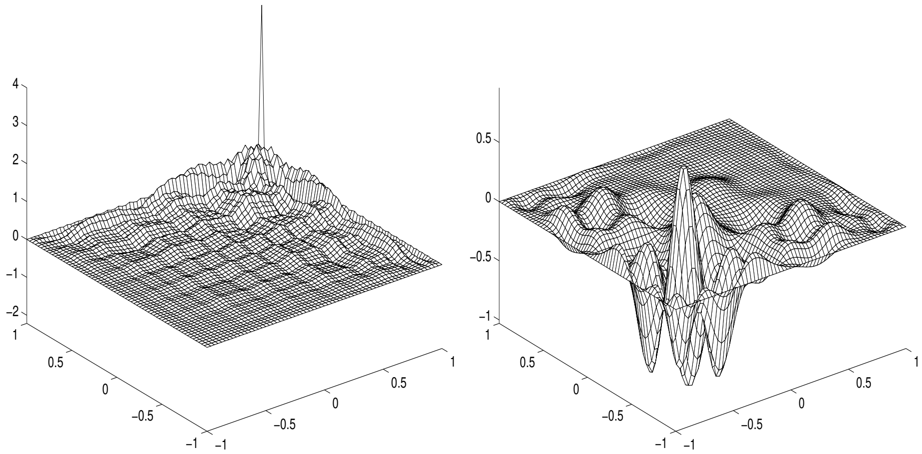

Figure 4: The polynomial scaling functions $\Phi^p_{6,0,0}$ (left) and $\Phi^p_{6,48,48}$ (right).

and choose scaling functions $\Phi_{j,k,\ell}$ as modified fundamental interpolants in V_j^B on the sparse grid $\mathcal{G}_j^B$, i.e., with the property,

$$\Phi_{j,k,\ell}(g_{j,m}, g_{j,n}) = \frac{1}{\varepsilon_{j,k}\varepsilon_{j,\ell}}\delta_{k,m}\delta_{\ell,n}, \qquad (k,\ell),(m,n) \in \mathcal{J}_j.$$

For describing these functions in detail, we first take a finer splitting of the index set $\mathcal{J}_j$

$$\mathcal{J}_j = 2\,\mathcal{J}_{j-1} \cup (\mathcal{J}_j \backslash 2\,\mathcal{J}_{j-1}), \qquad \mathcal{J}_j \backslash 2\,\mathcal{J}_{j-1} = \bigcup_{r=0}^{j} \mathcal{K}_{j,p}$$

with

$$\mathcal{K}_{0,0} = \{(\eta,\kappa)\,;\ \eta = 0,\ldots,d_0,\ \kappa = 0,\ldots,d_0\}$$

and for $j \geq 1$

$$\begin{aligned}
\mathcal{K}_{j,p} &= \{(M_j(\eta), M_j(\kappa))\,;\ \eta = d_{p-1},\ldots,d_p - 1, \\
&\qquad \kappa = d_{j-p-1},\ldots,d_{j-p} - 1\},\ p = 1,\ldots,j-1 \\
\mathcal{K}_{j,0} &= \{(M_j(\eta), M_j(\kappa))\,;\ \eta = -1,\ldots,d_0 - 1,\ \kappa = d_{j-1},\ldots,d_j - 1\} \\
\mathcal{K}_{j,j} &= \{(M_j(\eta), M_j(\kappa))\,;\ \eta = d_{j-1},\ldots,d_j - 1,\ \kappa = -1,\ldots,d_0 - 1\}\,.
\end{aligned}$$

Hence,

$$\mathcal{J}_j = \bigcup_{t=0}^{j} 2^{j-t} \bigcup_{r=0}^{t} \mathcal{K}_{t,r} \,. \tag{4.2}$$

For indices $(k,\ell) \in 2^{j-t}\mathcal{K}_{t,p}, 0 \le p \le t \le j$, we can describe the modified fundamental interpolants $\Phi_{j,k,\ell}$ in V_j^B by

$$\begin{aligned}\Phi_{j,k,\ell} &= \sum_{s=p}^{j+p-t} \phi_{s,2^{s-j}k} \otimes \phi_{j-s,2^{-s}\ell} - \sum_{s=p}^{j+p-t-1} \phi_{s,2^{s-j}k} \otimes \phi_{j-s-1,2^{-s-1}\ell} \qquad (4.3)\\ &= \sum_{s=p}^{j+p-t} \phi_{s,M_s(M_j(k))} \otimes \phi_{j-s,M_{j-s}(M_j(\ell))} \\ &\quad - \sum_{s=p}^{j+p-t-1} \phi_{s,M_s(M_j(k))} \otimes \phi_{j-s-1,M_{j-s-1}(M_j(\ell))} \,.\end{aligned}$$

The wavelet space W_{j+1}^B is defined as the orthogonal complement of the sample space on the lower level j relative to the sample space on the higher level $j+1$, i.e.,

$$V_{j+1}^B = V_j^B \oplus^\perp W_{j+1}^B$$

with respect to the inner product in $L_w^2(I^2)$

$$\langle f,g\rangle_{I^2} := \frac{4}{\pi^2} \iint\limits_{I^2} f(x,y)g(x,y)w(x)w(y)\,dxdy\,.$$

The wavelet spaces can be characterized in terms of tensor products of the univariate wavelet spaces.

Theorem 4.1. *For $j \in \mathbb{N}_0$, it holds that*

$$W_{j+1}^B = \bigoplus_{r=0}^{j+1}{}^{\perp}\, W_r \otimes W_{j+1-r}$$

with the notation $W_0 := V_0$.

The proof can be found in [14].

We define for $(m,n) \in \mathcal{K}_{j+1,p}$ the wavelet function

$$\Psi_{j+1,m,n} := \psi_{p,2^{p-j-1}m} \otimes \psi_{j+1-p,2^{-p}n} \tag{4.4}$$

with $\psi_{0,r} := \phi_{0,r}$. Using the representation of the wavelet space given by Theorem 4.1, we obtain immediately bases of the sample and the wavelet spaces.

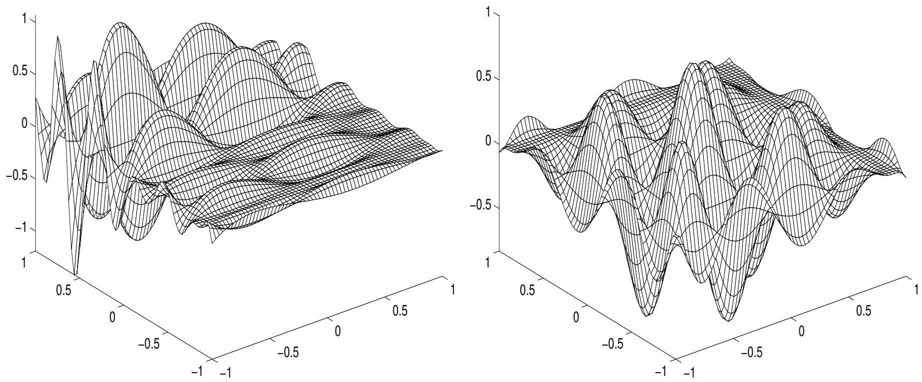

Figure 5: The polynomial wavelet functions $\Psi^p_{6,20,48}$ (left) and $\Psi^p_{6,40,40}$ (right).

Corollary 4.2. *Let $j \in \mathbb{N}_0$, then,*

(i) *the basis and the dimension of the sample space are given by*

$$\begin{aligned} \mathcal{B}(V_j^B) &= \{\Phi_{j,k,\ell}\,;\ (k,\ell) \in \mathcal{J}_j\}, \\ \dim V_j^B &= d^2((j-1)2^{j-1} + 2^{j+2} + 1), \end{aligned}$$

(ii) *the basis and the dimension of the wavelet space are given by*

$$\begin{aligned} \mathcal{B}(W_{j+1}^B) &= \{\Psi_{j+1,m,n}\,;\ (m,n) \in \mathcal{K}_{j+1,p},\ p = 0,\ldots,j+1\}, \\ \dim W_{j+1}^B &= d^2(j2^{j-1} + 2^{j+2}). \end{aligned}$$

Hence, the dimension of the constructed bivariate sample space is growing as $\mathcal{O}(j2^j)$ while the dimension of ordinary tensor products of univariate sample spaces grows as $\mathcal{O}(2^{2j})$.

5 Complete Decomposition and Reconstruction

We want to establish the relations for the complete decomposition of the fundamental interpolants on the sparse grid, i.e., the scaling functions in the bivariate case, into wavelets parts in matrix form

$$\boldsymbol{\Phi}_j = \boldsymbol{Z}_j^B \, \boldsymbol{\Psi}_j$$

and for complete reconstruction of the wavelet parts into the basis of fundamental interpolants

$$\boldsymbol{\Psi}_j = \boldsymbol{R}_j^B \, \boldsymbol{\Phi}_j$$

with

$$\boldsymbol{\Phi}_j = (\Phi_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j} ,$$

where the indices follow the ordering

$$2^j\mathcal{K}_{0,0}, \; 2^{j-1}\mathcal{K}_{1,0}, \; 2^{j-1}\mathcal{K}_{1,1}, \; \ldots, \; \mathcal{K}_{j,j}$$

and

$$\boldsymbol{\Psi}_j = (\Psi_{u,m,n})_{u=0,\ldots,j,\, q=0,\ldots,u,\, (m,n)\in\mathcal{K}_{u,q}} .$$

We want to describe the matrices $\boldsymbol{Z}_j^B$ and $\boldsymbol{R}_j^B$ in terms of matrices for the univariate relations.

Theorem 5.1. *With the permutation matrices*

$$\begin{aligned}
\boldsymbol{S}_{k,s} &:= \left(\delta_{\eta,\lfloor(d_{k-s}+1)^{-1}r\rfloor-1}\,\delta_{\kappa,r \bmod (d_{k-s}+1)-1}\right)_{(\eta,\kappa)\in M_j(\mathcal{J}_j), r=0,\ldots,(d_s+1)(d_{k-s}+1)} , \\
\boldsymbol{T}_j &:= \left(\delta_{M_j(k),m}\,\delta_{M_j(\ell),n}\right)_{(k,\ell)\in\mathcal{J}_j,\,(m,n)\in M_j(\mathcal{J}_j)} ,
\end{aligned}$$

and $\boldsymbol{Q}_s$ *as in Theorem* 3.1*,*

(i) *the matrix* $\boldsymbol{Z}_j^B$ *for complete decomposition has the form*

$$\begin{aligned}
\boldsymbol{Z}_j^B &= \boldsymbol{T}_j \left(\sum_{s=0}^{j} \boldsymbol{S}_{j,s} \left((\boldsymbol{Q}_s \boldsymbol{Z}_s \boldsymbol{Q}_s^T) \otimes (\boldsymbol{Q}_{j-s} \boldsymbol{Z}_{j-s} \boldsymbol{Q}_{j-s}^T) \right) \boldsymbol{S}_{j,s}^T \right. \\
&\quad \left. - \sum_{s=0}^{j-1} \boldsymbol{S}_{j-1,s} \left((\boldsymbol{Q}_s \boldsymbol{Z}_s \boldsymbol{Q}_s^T) \otimes (\boldsymbol{Q}_{j-s-1} \boldsymbol{Z}_{j-s-1} \boldsymbol{Q}_{j-s-1}^T) \right) \boldsymbol{S}_{j-1,s}^T \right) \boldsymbol{T}_j^T ,
\end{aligned}$$

(ii) *the matrix* $\boldsymbol{R}_j^B$ *for complete reconstruction has the form*

$$\begin{aligned}
\boldsymbol{R}_j^B &= \boldsymbol{T}_j \left(\sum_{s=0}^{j} \boldsymbol{S}_{j,s} \left((\boldsymbol{Q}_s \boldsymbol{R}_s \boldsymbol{Q}_s^T) \otimes (\boldsymbol{Q}_{j-s} \boldsymbol{R}_{j-s} \boldsymbol{Q}_{j-s}^T) \right) \boldsymbol{S}_{j,s}^T \right. \\
&\quad \left. - \sum_{s=0}^{j-1} \boldsymbol{S}_{j-1,s} \left((\boldsymbol{Q}_s \boldsymbol{R}_s \boldsymbol{Q}_s^T) \otimes (\boldsymbol{Q}_{j-s-1} \boldsymbol{R}_{j-s-1} \boldsymbol{Q}_{j-s-1}^T) \right) \boldsymbol{S}_{j-1,s}^T \right) \boldsymbol{T}_j^T .
\end{aligned}$$

For the proof, we refer the reader to [15] where it has been carried out in the periodic case (only some slight modifications are necessary).

Now, we want to present the algorithms for complete decomposition and reconstruction, respectively, of a given function for our Boolean sum approach. We project a bivariate continuous (or piecewise continuous) function f into the sample space V_j. We obtain the projection f_j of f by interpolation

$$f_j := B_j f = \sum_{(k,\ell)\in\mathcal{J}_j} \varepsilon_{j,k}\,\varepsilon_{j,\ell}\, f(t_{j,k}, t_{j,\ell})\, \Phi_{j,k,\ell}$$

on the sparse Gauß–Chebyshev grid. Writing this in vector form, we obtain $f_j = \boldsymbol{v}_j^T \boldsymbol{\Phi}_j$ with the coefficient vector $\boldsymbol{v}_j$ consisting of the function values $v_{j,k,\ell} = \varepsilon_{j,k}\,\varepsilon_{j,\ell}\, f(t_{j,k}, t_{j,\ell})$, $(k,\ell) \in \mathcal{J}_j$ sampled from the sparse grid. In terms of the wavelet basis of V_j, we get another representation of the interpolant as $f_j = \boldsymbol{w}_j^T \boldsymbol{\Psi}_j$ with a coefficient vector $\boldsymbol{w}_j$. Then, we use the relations of complete decomposition and reconstruction for the basis vectors

$$\boldsymbol{\Phi}_j = \boldsymbol{Z}_j^B\, \boldsymbol{\Psi}_j \qquad \text{and} \qquad \boldsymbol{\Psi}_j = \boldsymbol{R}_j^B\, \boldsymbol{\Phi}_j$$

and obtain the relations of complete decomposition and reconstruction, respectively, for the coefficient vectors

$$\boldsymbol{w}_j = (\boldsymbol{Z}_j^B)^T\, \boldsymbol{v}_j \qquad \text{and} \qquad \boldsymbol{v}_j = (\boldsymbol{R}_j^B)^T\, \boldsymbol{w}_j. \tag{5.1}$$

The matrices involved in the Equations (5.1) are of the same structure. We have

$$\boldsymbol{y}_j = (\boldsymbol{A}_j^B)^T\, \boldsymbol{x}_j$$

with the letter $\boldsymbol{A}$ standing for $\boldsymbol{Z}$ in the complete decomposition relation and for $\boldsymbol{R}$ in the reconstruction relation, respectively. By Theorem 5.1, the multiplication with the matrix $(\boldsymbol{A}_j^B)^T$ reduces to the stepwise application of the algorithms in the univariate case.

Algorithm. (Complete decomposition or reconstruction)

Input vector: $\boldsymbol{x}_j = (x_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Output vector: $\boldsymbol{y}_j = (y_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Help vector: $\boldsymbol{z}_j = (z_{j,k,\ell})_{(k,\ell)\in\mathcal{J}_j}$

Notation: $\boldsymbol{z}_{j,k} = (z_{j,k,\ell})_{\ell\in\mathcal{L}}$, $\boldsymbol{z}_{j,\ell} = (z_{j,k,\ell})_{k,\in\mathcal{K}}$, with an appropriate choice of the index sets $\mathcal{K}, \mathcal{L}$ (the ranges for k, ℓ as given in the algorithm)

Initialization: $y_{j,k,\ell} := 0$ for all $(k,\ell) \in \mathcal{J}_j$

For all $s = 0, \ldots, j$ do

$$
\begin{array}{ll}
z_{j,k,\ell} := x_{j,M_j(k),M_j(\ell)} & \text{for } k = -1, \ldots, d_s - 1, \\
& \quad \ell = -1, \ldots, d_{j-s} - 1 \\
\boldsymbol{z}_{j,k} := \boldsymbol{Q}_{j-s}\, \boldsymbol{A}_{j-s}^T\, \boldsymbol{Q}_{j-s}^T\, \boldsymbol{z}_{j,k} & \text{for } k = -1, \ldots, d_s - 1 \\
\boldsymbol{z}_{j,\ell} := \boldsymbol{Q}_s\, \boldsymbol{A}_s^T\, \boldsymbol{Q}_s^T\, \boldsymbol{z}_{j,\ell} & \text{for } \ell = -1, \ldots, d_{j-s} - 1 \\
y_{j,M_j(k),M_j(\ell)} := y_{j,M_j(k),M_j(\ell)} + z_{j,k,\ell} & \text{for } k = -1, \ldots, d_s - 1, \\
& \quad \ell = -1, \ldots, d_{j-s} - 1
\end{array}
$$

For all $s = 0, \ldots, j-1$ do

$$
\begin{array}{ll}
z_{j,k,\ell} := x_{j,M_j(k),M_j(\ell)} & \text{for } k = -1, \ldots, d_s - 1, \\
& \quad \ell = -1, \ldots, d_{j-s-1} - 1 \\
\boldsymbol{z}_{j,k} := \boldsymbol{Q}_{j-s-1}\, \boldsymbol{A}_{j-s-1}^T\, \boldsymbol{Q}_{j-s-1}^T\, \boldsymbol{z}_{j,k} & \text{for } k = -1, \ldots, d_s - 1 \\
\boldsymbol{z}_{j,\ell} := \boldsymbol{Q}_s\, \boldsymbol{A}_s^T\, \boldsymbol{Q}_s^T\, \boldsymbol{z}_{j,\ell} & \text{for } \ell = -1, \ldots, d_{j-s-1} - 1 \\
y_{j,M_j(k),M_j(\ell)} := y_{j,M_j(k),M_j(\ell)} - z_{j,k,\ell} & \text{for } k = -1, \ldots, d_s - 1, \\
& \quad \ell = -1, \ldots, d_{j-s-1} - 1
\end{array}
$$

The products $\boldsymbol{Q}_s \boldsymbol{A}_s^T \boldsymbol{Q}_s^T$ are of the structure given in Theorem 3.1. That's why they will be realized by fast cosine transforms and multiplication with a sparse matrix.

We count the univariate complete decompositions and the univariate complete reconstructions, respectively, for the levels occuring in the algorithm. In the end we need altogether $\mathcal{O}(j^2\, 2^j)$ essential operations for the bivariate complete decomposition and the complete reconstruction, respectively, for the level j if we use the Boolean sum approach.

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