Multivariate cosine wavelets

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Abstract. We construct bivariate biorthogonal cosine wavelets on a two-overlapping rectangular grid with bell functions not necessary of tensor product type. The biorthogonal system as well as frame and Riesz basis conditions are given explicitly. Our methods are based on the properties of bivariate total folding and unfolding operators.

1. Introduction

Recently, local trigonometric bases have been investigated by many authors. Let us mention here only Malvar [8], Coifman and Meyer [5], Daubechies, Jaffard and Journee [6], Auscher, Weiss and Wickerhauser [1], Wickerhauser [10], Jawerth and Sweldens [7], Matviyenko [9], Xia and Suter [11], Chui and Shi [3,4], and the literature cited there. In the meantime the idea to use windowed sinusoids as an orthonormal basis for $L^2(\mathbb{R})$ is widely applied, e.g. in image processing to eliminate blocking effects in transform coding. Therefore, not only univariate bases but also bivariate constructions were investigated in more detail.

In this paper we are interested in bivariate biorthogonal cosine wavelet bases generated on arbitrary 'two-overlapping' rectangular grids. In particular, the corresponding bell functions need not have tensor product structure. Our approach is based on univariate results for a nonuniform grid obtained in [3,4]. As a main tool for our purposes we introduce a bivariate folding operator. Such an operator maps a function on $\mathbb{R}^2$ into a function which satisfies certain parity conditions on the gridlines of the underlying rectangular grid. Then, frame and Riesz basis properties of the cosine wavelets could be studied in terms of boundedness of the corresponding total folding and unfolding operators. In particular, it is pointed out that in this context frame and Riesz basis are equivalent conditions. The best possible constants in the inequalities could be given explicitly as minimal and maximal eigenvalue of certain matrices. In a final example we compare a linear bell function of pyramid type with a bilinear bell of tensor product type.

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2. The bivariate folding operator

Let $(a_r)_{r\in\mathbb{Z}}$, $(a_r^+)_{r\in\mathbb{Z}}$, $(a_r^-)_{r\in\mathbb{Z}}$ and $(b_s)_{s\in\mathbb{Z}}$, $(b_s^+)_{s\in\mathbb{Z}}$, $(b_s^-)_{s\in\mathbb{Z}}$ be given sequences with

$$a_r < a_r^+ \le a_{r+1}^- < a_{r+1} \quad \text{and} \quad a_r - a_r^- = a_r^+ - a_r \quad \text{for } r \in \mathbb{Z}$$

and analogous conditions for b_s, so these numbers generate a rectangular grid for $\mathbb{R}^2$ (see **Figure 1**).

We consider now bivariate bell functions $w_{r,s} : \mathbb{R}^2 \to \mathbb{C}$ with support

$$[a_r, a_{r+1}] \times [b_s, b_{s+1}] \subset \operatorname{supp} w_{r,s} \subset [a_r^-, a_{r+1}^+] \times [b_s^-, a_{s+1}^+] . \tag{1}$$

Then

$$\operatorname{supp} w_{r,s} \cap \operatorname{supp} w_{p,q} = \emptyset \quad \text{if } |r-p| > 1 \text{ or } |s-q| > 1 ,$$

which means that at most 4 bell functions overlap. Further we define the functions

$$C^k_{[\alpha,\beta]}(x) := \sqrt{\frac{2}{\beta-\alpha}} \cos\left(\left(k+\tfrac{1}{2}\right)\frac{x-\alpha}{\beta-\alpha}\pi\right) , \quad k,l \in \mathbb{N}_0$$

which form an orthonormal basis of $L^2([\alpha,\beta])$. The main properties of these particular trigonometric polynomials used in the sequel are that $C^k_{[\alpha,\beta]}$ is even with respect to α, i.e.

$$C^k_{[\alpha,\beta]}(\alpha - x) = C^k_{[\alpha,\beta]}(\alpha + x)$$

and odd with respect to β, i.e.

$$C^k_{[\alpha,\beta]}(\beta - x) = -C^k_{[\alpha,\beta]}(\beta + x) .$$

Now we introduce the bivariate cosine wavelets

$$\Psi^{k,l}_{r,s}(x,y) := w_{r,s}(x,y)\, C^k_{[a_r,a_{r+1}]}(x)\, C^l_{[b_s,b_{s+1}]}(y) , \quad r,s \in \mathbb{Z},\ k,l \in \mathbb{N}_0 .$$

To investigate the basis properties of $\Psi^{k,l}_{r,s}$ we define the matrices

$$\boldsymbol{K}_{r,s}(x,y) := \boldsymbol{K}^w_{r,s}(x,y) := \begin{pmatrix} w_{r,s}(x,y) & w_{r,s}(u_r,y) \\ -w_{r-1,s}(x,y) & w_{r-1,s}(u_r,y) \end{pmatrix} ,$$

$$\boldsymbol{L}_{r,s}(x,y) := \boldsymbol{L}^w_{r,s}(x,y) := \begin{pmatrix} w_{r,s}(x,y) & w_{r,s}(x,v_s) \\ -w_{r,s-1}(x,y) & w_{r,s-1}(x,v_s) \end{pmatrix}$$

and

$$\boldsymbol{M}_{r,s}(x,y) := \boldsymbol{M}^w_{r,s}(x,y)$$
$$:= \begin{pmatrix} w_{r,s}(x,y) & w_{r,s}(u_r,y) & w_{r,s}(x,v_s) & w_{r,s}(u_r,v_s) \\ -w_{r-1,s}(x,y) & w_{r-1,s}(u_r,y) & -w_{r-1,s}(x,v_s) & w_{r-1,s}(u_r,v_s) \\ -w_{r,s-1}(x,y) & -w_{r,s-1}(u_r,y) & w_{r,s-1}(x,v_s) & w_{r,s-1}(u_r,v_s) \\ w_{r-1,s-1}(x,y) & -w_{r-1,s-1}(u_r,y) & -w_{r-1,s-1}(x,v_s) & w_{r-1,s-1}(u_r,v_s) \end{pmatrix}$$

where $u_r := 2a_r - x$ and $v_s := 2b_s - y$. For further discussion we consider the partition of $\mathbb{R}^2$ into the disjoint rectangles (see **Figure 1**)

$$\begin{aligned}
R_1 &:= R_1^{r,s} := [a_r^+, a_{r+1}^-) \times [b_s^+, b_{s+1}^-)\,,\\
R_2 &:= R_2^{r,s} := [a_r^-, a_r^+) \times [b_s^+, b_{s+1}^-)\,,\\
R_3 &:= R_3^{r,s} := [a_r^+, a_{r+1}^-) \times [b_s^-, b_s^+)\,,\\
R_4 &:= R_4^{r,s} := [a_r^-, a_r^+) \times [b_s^-, b_s^+)\,,
\end{aligned}$$

i.e.

$$\mathbb{R}^2 = \bigcup_{r,s\in\mathbb{Z}} R_1^{r,s} \cup R_2^{r,s} \cup R_3^{r,s} \cup R_4^{r,s}\,.$$

In this way we obtain that on R_1 only the function $w_{r,s}$ is not zero, on R_2 resp. R_3 we have two non-vanishing functions $w_{r,s}$ and $w_{r-1,s}$ resp. $w_{r,s}$ and $w_{r,s-1}$ and finally on R_4 only the four functions $w_{r,s}$, $w_{r-1,s}$, $w_{r,s-1}$ and $w_{r-1,s-1}$ are not zero.

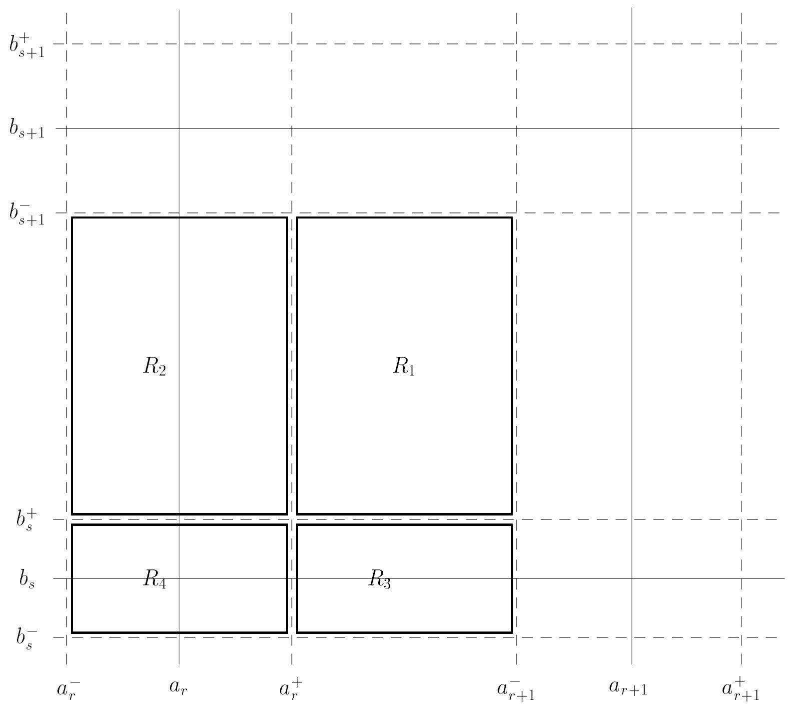

Fig. 1. Grid for the support of the bell function $w_{r,s}$.

With the help of these matrices we introduce the total folding operator $\mathcal{T}_w$. For a given measurable function f we define $\mathcal{T}_w f$ separately in every of the rectangles

$R_i^{r,s}$, $i = 1,2,3,4$; $r,s \in \mathbb{Z}$ in the following way

$$
\begin{aligned}
\mathcal{T}_w f(x,y) \quad &:= \quad w_{r,s}(x,y)\, f(x,y) && \text{for } (x,y) \in R_1^{r,s}\,, \\
\begin{pmatrix} \mathcal{T}_w f(x,y) \\ \mathcal{T}_w f(u_r,y) \end{pmatrix} \quad &:= \quad \boldsymbol{K}_{r,s}(x,y) \begin{pmatrix} f(x,y) \\ f(u_r,y) \end{pmatrix} && \text{for } \begin{matrix} a_r < x < a_r^+\,, \\ b_s^+ < y < b_{s+1}^-\,, \end{matrix} \\
\begin{pmatrix} \mathcal{T}_w f(x,y) \\ \mathcal{T}_w f(x,v_s) \end{pmatrix} \quad &:= \quad \boldsymbol{L}_{r,s}(x,y) \begin{pmatrix} f(x,y) \\ f(x,v_s) \end{pmatrix} && \text{for } \begin{matrix} a_r^+ < x < a_{r+1}^-\,, \\ b_s < y < b_s^+\,, \end{matrix} \\
\begin{pmatrix} \mathcal{T}_w f(x,y) \\ \mathcal{T}_w f(u_r,y) \\ \mathcal{T}_w f(x,v_s) \\ \mathcal{T}_w f(u_r,v_s) \end{pmatrix} \quad &:= \quad \boldsymbol{M}_{r,s}(x,y) \begin{pmatrix} f(x,y) \\ f(u_r,y) \\ f(x,v_s) \\ f(u_r,v_s) \end{pmatrix} && \text{for } \begin{matrix} a_r < x < a_r^+\,, \\ b_s < y < b_s^+\,. \end{matrix}
\end{aligned}
\tag{2}
$$

Note that this definition determines $\mathcal{T}_w f$ uniquely a.e. on $\mathbb{R}^2$.

We want to emphasize that the introduction of $\mathcal{T}_w$ is motivated by the following observation. Since $C^k_{[\alpha,\beta]}$ is even with respect to α and odd with respect to β one can 'fold' $\Psi^{k,l}_{r,s}$ into $[a_r, a_{r+1}] \times [b_s, b_{s+1}]$ to obtain

$$
\int_{\mathbb{R}^2} \Psi^{k,l}_{r,s}(x,y)\, f(x,y)\, dx\, dy = \int_{a_r}^{a_{r+1}} \int_{b_s}^{b_{s+1}} C^k_{[a_r,a_{r+1}]}(x)\, C^l_{[b_s,b_{s+1}]}(y)\, \mathcal{T}_w f(x,y)\, dx\, dy \tag{3}
$$

if the integrals are well-defined. Only for simplicity we have restricted ourselves to the particular cosine functions $C^k_{[\alpha,\beta]}$ which then determine the structure of the folding operator $\mathcal{T}_w f$. Other trigonometric bases would work as well but lead to modified folding operators.

Applying formula (3) we will deduce basis properties of the $\Psi^{k,l}_{r,s}$ from norm estimates of $\mathcal{T}_w$. In particular, we are interested in the L^2-boundedness of $\mathcal{T}_w$ and in the existence and boundedness of $\mathcal{T}_w^{-1}$. For this we consider the following functions

$$
\begin{aligned}
\Delta_K(x,y) &:= |w_{r,s}(x,y)|^2 + |w_{r,s}(u_r,y)|^2 + |w_{r-1,s}(x,y)|^2 + |w_{r-1,s}(u_r,y)|^2\,, \\
\Delta_L(x,y) &:= |w_{r,s}(x,y)|^2 + |w_{r,s}(x,v_s)|^2 + |w_{r,s-1}(x,y)|^2 + |w_{r,s-1}(x,v_s)|^2
\end{aligned}
$$

and constants

$$\begin{aligned}
B_1^{r,s} &:= \operatorname*{ess\,sup}_{(x,y)\in R_1^{r,s}} |w_{r,s}(x,y)|^2 \,, \\
B_2^{r,s} &:= \operatorname*{ess\,sup}_{\substack{a_r<x<a_r^+ \\ b_s^+<y<b_{s+1}^-}} \|\boldsymbol{K}_{r,s}(x,y)\|_2^2 \\
&= \operatorname*{ess\,sup}_{\substack{a_r<x<a_r^+ \\ b_s^+<y<b_{s+1}^-}} \frac{\Delta_K(x,y)}{2} + \sqrt{\frac{\Delta_K^2(x,y)}{4} - |\det \boldsymbol{K}_{r,s}(x,y)|^2} \,, \\
B_3^{r,s} &:= \operatorname*{ess\,sup}_{\substack{a_r^+<x<a_{r+1}^- \\ b_s<y<b_s^+}} \|\boldsymbol{L}_{r,s}(x,y)\|_2^2 \\
&= \operatorname*{ess\,sup}_{\substack{a_r^+<x<a_{r+1}^- \\ b_s<y<b_s^+}} \frac{\Delta_L(x,y)}{2} + \sqrt{\frac{\Delta_L^2(x,y)}{4} - |\det \boldsymbol{L}_{r,s}(x,y)|^2} \,, \\
B_4^{r,s} &:= \operatorname*{ess\,sup}_{\substack{a_r<x<a_r^+ \\ b_s<y<b_s^+}} \|\boldsymbol{M}_{r,s}(x,y)\|_2^2 \,.
\end{aligned}$$

Here

$$\|\boldsymbol{A}\|_2 := \sup_{|\boldsymbol{x}|=1} |\boldsymbol{A}\boldsymbol{x}| = \sqrt{\rho(\boldsymbol{A}^H\boldsymbol{A})}$$

is the spectral norm of a matrix $\boldsymbol{A}$.

Now we can establish the following assertion.

Lemma 1. *Let $\{w_{r,s}\}$ be measurable functions such that*

$$B_0 := \sup_{r,s\in\mathbb{Z}} \max\{B_1^{r,s}, B_2^{r,s}, B_3^{r,s}, B_4^{r,s}\} < \infty \,. \tag{4}$$

Then $\mathcal{T}_w$ is a bounded operator from $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with

$$\|\mathcal{T}_w\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}^2 = B_0 \,.$$

Proof: For arbitrary $f \in L^2(\mathbb{R}^2)$, we have

$$\|\mathcal{T}_w f\|_{L^2(\mathbb{R}^2)}^2 = \sum_{r,s\in\mathbb{Z}} \|\mathcal{T}_w f\|_{L^2(R_1^{r,s})}^2 + \|\mathcal{T}_w f\|_{L^2(R_2^{r,s})}^2 + \|\mathcal{T}_w f\|_{L^2(R_3^{r,s})}^2 + \|\mathcal{T}_w f\|_{L^2(R_4^{r,s})}^2 \,.$$

We consider now every term separately. For R_1 we obtain immediately, by Hölder inequality

$$\|\mathcal{T}_w f\|_{L^2(R_1^{r,s})}^2 = \iint\limits_{R_1^{r,s}} |w_{r,s}(x,y)|^2\, |f(x,y)|^2\, dx\, dy \le B_1^{r,s}\ \|f\|_{L^2(R_1^{r,s})}^2 \,,$$

where $B_1^{r,s}$ is the best possible constant. For R_2 we have

$$\begin{aligned}\|\mathcal{T}_w f\|^2_{L^2(R_2^{r,s})} &= \int_{b_s^+}^{b_{s+1}^-}\int_{a_r}^{a_r^+} \left|\Big(\mathcal{T}_w f(x,y), \mathcal{T}_w f(u_r,y)\Big)^T\right|^2 \, dx\, dy \\ &\leq \int_{b_s^+}^{b_{s+1}^-}\int_{a_r}^{a_r^+} \|\boldsymbol{K}_{r,s}(x,y)\|_2^2 \, \left|\Big(f(x,y), f(u_r,y)\Big)^T\right|^2 \, dx\, dy\,.\end{aligned}$$

In particular, equality is attained if $f \in L^2(R_2^{r,s})$ is given by

$$\begin{pmatrix} f(x,y) \\ f(u_r,y)\end{pmatrix} = a(x,y)\begin{pmatrix} g_1(x,y) \\ g_2(u_r,y)\end{pmatrix}, \quad (x,y) \in (a_r, a_r^+) \times (b_s^+, b_{s+1}^-)$$

with an arbitrary function $a \in L^2\left((a_r, a_r^+) \times (b_s^+, b_{s+1}^-)\right)$ and a normalized eigenvector $((g_1(x,y), g_2(x,y))^T$ corresponding to the maximal eigenvalue of $\boldsymbol{K}_{r,s}(x,y)^T \boldsymbol{K}_{r,s}(x,y)$. Again $B_2^{r,s}$ is the best possible constant in the Hölder inequality

$$\|\mathcal{T}_w f\|^2_{L^2(R_2^{r,s})} \leq B_2^{r,s} \, \|a\|^2_{L^2\left((a_r,a_r^+)\times(b_s^+,b_{s+1}^-)\right)} = B_2^{r,s} \, \|f\|^2_{L^2(R_2^{r,s})}\,.$$

Analogously we obtain

$$\|\mathcal{T}_w f\|^2_{L^2(R_3^{r,s})} \leq B_3^{r,s} \, \|f\|^2_{L^2(R_3^{r,s})}$$

and

$$\|\mathcal{T}_w f\|^2_{L^2(R_4^{r,s})} \leq B_4^{r,s} \, \|f\|^2_{L^2(R_4^{r,s})}$$

with best possible constants $B_3^{r,s}$ and $B_4^{r,s}$. Therefore,

$$\begin{aligned}&\|\mathcal{T}_w f\|^2_{L^2(\mathbb{R}^2)} \\ &\quad \leq \sum_{r,s\in\mathbb{Z}} B_1^{r,s} \, \|f\|^2_{L^2(R_1^{r,s})} + B_2^{r,s} \, \|f\|^2_{L^2(R_2^{r,s})} + B_3^{r,s} \, \|f\|^2_{L^2(R_3^{r,s})} + B_4^{r,s} \, \|f\|^2_{L^2(R_4^{r,s})} \\ &\quad \leq \sup_{r,s\in\mathbb{Z}} \max\{B_1^{r,s}, B_2^{r,s}, B_3^{r,s}, B_4^{r,s}\} \, \|f\|^2_{L^2(\mathbb{R}^2)}\,.\end{aligned}$$

Hence, $\|\mathcal{T}_w\|^2_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = B_0$. ■

Remark 2. *Note that* $B_0 < \infty$ *iff* $\sup_{r,s\in\mathbb{Z}} \|w_{r,s}\|_{L^\infty(\mathbb{R}^2)} < \infty$, *which is obviously equivalent to*

$$\Big\| \sum_{r,s\in\mathbb{Z}} |w_{r,s}|^2 \Big\|_{L^\infty(\mathbb{R}^2)} < \infty.$$

Let us now assume that

$$w_{r,s} \neq 0 \text{ a.e. on } R_1^{r,s}\,, \tag{5a}$$
$$\det \boldsymbol{K}_{r,s} \neq 0 \text{ a.e. on } R_2^{r,s}\,, \tag{5b}$$
$$\det \boldsymbol{L}_{r,s} \neq 0 \text{ a.e. on } R_3^{r,s}\,, \tag{5c}$$
$$\det \boldsymbol{M}_{r,s} \neq 0 \text{ a.e. on } R_4^{r,s}\,. \tag{5d}$$

In the same way as in (2) we define an operator $\mathcal{V}_w$ by replacing $w_{r,s}$, $\boldsymbol{K}^w_{r,s}$, $\boldsymbol{L}^w_{r,s}$ and $\boldsymbol{M}^w_{r,s}$ by $w_{r,s}^{-1}$, $(\boldsymbol{K}^w_{r,s})^{-1}$, $(\boldsymbol{L}^w_{r,s})^{-1}$ and $(\boldsymbol{M}^w_{r,s})^{-1}$, respectively. If the conditions (5a)-(5d) are satisfied the operator $\mathcal{V}_w$ is well–defined a.e. and we obtain by simple inspection that

$$\mathcal{V}_w \mathcal{T}_w f = \mathcal{T}_w \mathcal{V}_w f = f \quad \text{a.e.}\,.$$

Analogously to **Lemma 1** we introduce the following constants

$$A_1^{r,s} := \operatorname*{ess\,inf}_{(x,y)\in R_1^{r,s}} |w_{r,s}(x,y)|^2 \tag{6}$$

$$A_2^{r,s} := \operatorname*{ess\,inf}_{\substack{a_r<x<a_r^+\\ b_s^+<y<b_{s+1}^-}} \|\boldsymbol{K}_{r,s}^{-1}(x,y)\|_2^{-2}$$

$$= \operatorname*{ess\,inf}_{\substack{a_r<x<a_r^+\\ b_s^+<y<b_{s+1}^-}} \frac{\Delta_K(x,y)}{2} - \sqrt{\frac{\Delta_K^2(x,y)}{4} - |\det \boldsymbol{K}_{r,s}(x,y)|^2} \tag{7}$$

$$A_3^{r,s} := \operatorname*{ess\,inf}_{\substack{a_r^+<x<a_{r+1}^-\\ b_s<y<b_s^+}} \|\boldsymbol{L}_{r,s}^{-1}(x,y)\|_2^{-2}$$

$$= \operatorname*{ess\,inf}_{\substack{a_r^+<x<a_{r+1}^-\\ b_s<y<b_s^+}} \frac{\Delta_L(x,y)}{2} - \sqrt{\frac{\Delta_L^2(x,y)}{4} - |\det \boldsymbol{L}_{r,s}(x,y)|^2} \tag{8}$$

$$A_4^{r,s} := \operatorname*{ess\,inf}_{\substack{a_r<x<a_r^+\\ b_s<y<b_s^+}} \|\boldsymbol{M}_{r,s}^{-1}(x,y)\|_2^{-2}\,.$$

Note that for an invertible matrix it holds $\|\boldsymbol{A}^{-1}\|_2^{-2} = \min \sigma(\boldsymbol{A}^H\boldsymbol{A})$.

Lemma 3. *Let* $\{w_{r,s}\}$ *be measurable functions and the conditions* (5a)-(5d) *be satisfied. If*

$$A_0 := \inf_{r,s\in\mathbb{Z}} \min\{A_1^{r,s}, A_2^{r,s}, A_3^{r,s}, A_4^{r,s}\} > 0\,, \tag{9}$$

then $\mathcal{V}_w$ *is a bounded operator from* $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *with*

$$\|\mathcal{V}_w\|^2_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = A_0^{-1}$$

and $\mathcal{T}_w^{-1} = \mathcal{V}_w$.

The proof is analogous to the proof of **Lemma 1**.

Let us mention here that in the bivariate setting the condition $A_0 > 0$ could not be rewritten as easy as $B_0 < \infty$ in **Remark 2**. To explain this let us restrict ourselves to nonnegative bell functions $w_{r,s}$. As in the univariate setting and analogous to **Remark 2** it follows from

$$\operatorname*{ess\,inf}_{x,y\in\mathbb{R}} \sum_{r,s\in\mathbb{Z}} w_{r,s}(x,y) > 0 \tag{10}$$

that

$$\inf_{r,s\in\mathbb{Z}} \min\{A_1^{r,s}, A_2^{r,s}, A_3^{r,s}\} > 0\,.$$

This can be easily derived from (6)-(8). However, if the bell functions are not tensor products of univariate functions, then $A_0 > 0$ is a stronger assumption than (10). This can be seen by the following example.

Let the bell functions in $R_4^{r,s}$ be uniquely defined by setting them piecewise constant ξ or 1 for all $(x,y) \in (a_r, a_r^+) \times (b_s, b_s^+)$ and such that

$$h(\xi) := \det \boldsymbol{M}^w_{r,s}(x,y) = \det \begin{pmatrix} 1 & \xi & \xi & \xi \\ -\xi & \xi & -1 & \xi \\ -\xi & -1 & \xi & \xi \\ \xi & -\xi & -\xi & 1 \end{pmatrix}\,.$$

Then we obtain $h(\xi) = (\xi+1)^3(3\xi-1)$, i.e., $h(1/3) = 0$. In other words, this yields an example that all four bell functions supported in R_4 are greater or equal $1/3$ but $A_4^{r,s}$ is unbounded.

3. Biorthogonal bases

We start this section with the definition of a bell function $\tilde{w}_{r,s}$ with

$$\operatorname{supp} \tilde{w}_{r,s} \subset [a_r^-, a_{r+1}^+] \times [b_s^-, b_{s+1}^+]\,.$$

We define $\tilde{w}_{r,s}$ by giving the complex conjugate of $\tilde{w}_{r,s}$ in the following way

$$\overline{\tilde{w}_{r,s}(x,y)} = \begin{cases} 1/w_{r,s}(x,y)\,, & \text{if } (x,y) \in R_1^{r,s}, \\ w_{r-1,s}(u_r,y)/\det \boldsymbol{K}_{r,s}(x,y)\,, & \text{if } (x,y) \in R_2^{r,s}, \\ w_{r+1,s}(u_{r+1},y)/\det \boldsymbol{K}_{r+1,s}(x,y)\,, & \text{if } (x,y) \in R_2^{r+1,s}, \\ w_{r,s-1}(x,v_s)/\det \boldsymbol{L}_{r,s}(x,y)\,, & \text{if } (x,y) \in R_3^{r,s}, \\ w_{r+1,s}(x,v_{s+1})/\det \boldsymbol{L}_{r,s+1}(x,y)\,, & \text{if } (x,y) \in R_3^{r,s+1}, \\ \det \boldsymbol{M}^{1,1}_{r,s}(x,y)/\det \boldsymbol{M}_{r,s}(x,y)\,, & \text{if } (x,y) \in R_4^{r,s}, \\ \det \boldsymbol{M}^{2,1}_{r+1,s}(x,y)/\det \boldsymbol{M}_{r+1,s}(x,y)\,, & \text{if } (x,y) \in R_4^{r+1,s}, \\ \det \boldsymbol{M}^{3,1}_{r,s+1}(x,y)/\det \boldsymbol{M}_{r,s+1}(x,y)\,, & \text{if } (x,y) \in R_4^{r,s+1}, \\ \det \boldsymbol{M}^{4,1}_{r+1,s+1}(x,y)/\det \boldsymbol{M}_{r+1,s+1}(x,y)\,, & \text{if } (x,y) \in R_4^{r+1,s+1}, \\ 0 & \text{otherwise} \end{cases}$$

where $\boldsymbol{M}^{i,j}_{r,s}(x,y)$ is the minor of the matrix $\boldsymbol{M}_{r,s}(x,y)$ canceling the i-th row and the j-th column.

Let us mention here that the continuity of $w_{r,s}$, for all $r, s \in \mathbb{Z}$ implies the continuity of $\tilde{w}_{r,s}$.

Remark 4. *If the bell functions $w_{r,s}$ are tensor products of univariate bells, i.e.*

$$w_{r,s} = w_r^1 \otimes w_s^2$$

we obtain that

$$\begin{aligned} \boldsymbol{M}_{r,s}(x,y) =& \boldsymbol{M}_r^1(x) \otimes \boldsymbol{M}_s^2(y) , \\ \boldsymbol{K}_{r,s}(x,y) =& \boldsymbol{M}_r^1(x) , \\ \boldsymbol{L}_{r,s}(x,y) =& \boldsymbol{M}_s^2(y) . \end{aligned}$$

In this case the bell functions $\tilde{w}_{r,s}$ are also tensor products of univariate bell functions.

With $\tilde{w}_{r,s}$ defined above we can describe biorthogonal functions

$$\tilde{\Psi}_{r,s} := \tilde{w}_{r,s}\, C^k_{[a_r,a_{r+1}]} \otimes C^l_{[b_s,b_{s+1}]} .$$

Lemma 5. *Let $w_{r,s} \in L^2(\mathbb{R}^2)$ for all $r,s \in \mathbb{Z}$. Let the functions $\{\tilde{w}_{r,s}\}$ be well defined, i.e.* (5a)-(5d) *are satisfied. If $\tilde{\Psi}^{k,l}_{r,s} \in L^2(\mathbb{R}^2)$ for all $r,s \in \mathbb{Z},\ k,l \in \mathbb{N}_0$, then the system $\{\tilde{\Psi}^{k,l}_{r,s}\}$ is biorthogonal to $\{\Psi^{k,l}_{r,s}\}$, i.e.*

$$\langle \Psi^{k',l'}_{r',s'} , \tilde{\Psi}^{k,l}_{r,s} \rangle_{L^2(\mathbb{R}^2)} = \delta_{r,r'}\delta_{s,s'}\delta_{k,k'}\delta_{l,l'} . \tag{11}$$

Proof: If $\min(|r-r'|,|s-s'|) \geq 2$, then the condition (11) follows immediately from

$$\operatorname{supp} \tilde{w}_{r,s} \subset [a_r^-, a_{r+1}^+] \times [b_s^-, b_{s+1}^+] .$$

Now we show (11) for $r' = r-1$ and $s' = s-1$. Because $C^{k'}_{[a,b]}\, C^k_{[b,c]}$ is an odd function with respect to b we conclude

$$\begin{aligned} &\langle \Psi^{k',l'}_{r-1,s-1} , \tilde{\Psi}^{k,l}_{r,s} \rangle_{L^2(\mathbb{R}^2)} \\ &= \iint\limits_{R_4^{r,s}} w_{r-1,s-1}(x,y)\, \overline{\tilde{w}_{r,s}(x,y)} \\ &\qquad\qquad \times C^{k'}_{[a_{r-1},a_r]}(x)\, C^k_{[a_r,a_{r+1}]}(x)\, C^{l'}_{[b_{s-1},b_s]}(y)\, C^l_{[b_s,b_{s+1}]}(y)\, dx\, dy \\ &= \int_{a_r}^{a_r^+}\int_{b_s}^{b_s^+} \Big(w_{r-1,s-1}(x,y)\overline{\tilde{w}_{r,s}(x,y)} - w_{r-1,s-1}(u_r,y)\overline{\tilde{w}_{r,s}(u_r,y)} \\ &\qquad - w_{r-1,s-1}(x,v_s)\overline{\tilde{w}_{r,s}(x,v_s)} + w_{r-1,s-1}(u_r,v_s)\overline{\tilde{w}_{r,s}(u_r,v_s)} \Big) \\ &\qquad\qquad \times C^{k'}_{[a_{r-1},a_r]}(x)\, C^k_{[a_r,a_{r+1}]}(x)\, C^{l'}_{[b_{s-1},b_s]}(y)\, C^l_{[b_s,b_{s+1}]}(y)\, dx\, dy . \end{aligned}$$

From the definition of $\tilde{w}_{r,s}(x,y)$ on $R_4^{r,s}$ it follows that

$$\det \boldsymbol{M}_{r,s}(x,y)\Big(w_{r-1,s-1}(x,y)\overline{\tilde{w}_{r,s}(x,y)} - w_{r-1,s-1}(u_r,y)\overline{\tilde{w}_{r,s}(u_r,y)} - w_{r-1,s-1}(x,v_s)\overline{\tilde{w}_{r,s}(x,v_s)} + w_{r-1,s-1}(u_r,v_s)\overline{\tilde{w}_{r,s}(u_r,v_s)}\Big)$$

$$= w_{r-1,s-1}(x,y)\det \boldsymbol{M}_{r,s}^{1,1}(x,y) - w_{r-1,s-1}(u_r,y)\det \boldsymbol{M}_{r,s}^{1,1}(u_r,y) - w_{r-1,s-1}(x,v_s)\det \boldsymbol{M}_{r,s}^{1,1}(x,v_s) + w_{r-1,s-1}(u_r,v_s)\det \boldsymbol{M}_{r,s}^{1,1}(u_r,v_s)$$

$$= w_{r-1,s-1}(x,y)\det \boldsymbol{M}_{r,s}^{1,1}(x,y) + w_{r-1,s-1}(u_r,y)\det \boldsymbol{M}_{r,s}^{1,2}(x,y) - w_{r-1,s-1}(x,v_s)\det \boldsymbol{M}_{r,s}^{1,3}(x,y) - w_{r-1,s-1}(u_r,v_s)\det \boldsymbol{M}_{r,s}^{1,4}(x,y)$$

$$= \det M_{r,s}^{4\to 1}(x,y) = 0\,,$$

where $M_{r,s}^{j\to i}$ is the matrix $M_{r,s}$ with its i-th row replaced by the j-th row. Therefore we obtain

$$\langle \Psi_{r-1,s-1}^{k',l'}, \tilde{\Psi}_{r,s}^{k,l}\rangle_{L^2(\mathbb{R}^2)} = 0\,.$$

Analogously we can show (11) for all other r', s' with $\max(|r-r'|,|s-s'|) = 1$.

Finally, since $C_{[a,b]}^{k'}\, C_{[a,b]}^{k}$ is even with respect to a and b we obtain

$$\begin{aligned}
&\langle \Psi_{r',s'}^{k',l'}, \tilde{\Psi}_{r,s}^{k,l}\rangle_{L^2(\mathbb{R}^2)} \\
&= \int_{a_r}^{a_{r+1}} C_{[a_r,a_{r+1}]}^{k'}(x)\, C_{[a_r,a_{r+1}]}^{k}(x)\,dx \int_{b_s}^{b_{s+1}} C_{[b_s,b_{s+1}]}^{l'}(y)\, C_{[b_s,b_{s+1}]}^{l}(y)\,dy \\
&= \delta_{k,k'}\delta_{l,l'}\,.
\end{aligned}$$

■

We define now the total unfolding operator $\mathcal{U}_w$ by replacing in (2) $w_{r,s}$, $\boldsymbol{K}_{r,s}^w$, $\boldsymbol{L}_{r,s}^w$ and $\boldsymbol{M}_{r,s}^w$ by $\overline{w_{r,s}}$, $(\boldsymbol{K}_{r,s}^w)^H$, $(\boldsymbol{L}_{r,s}^w)^H$ and $(\boldsymbol{M}_{r,s}^w)^H$, respectively. From the definition of $\tilde{w}_{r,s}$ we deduce

$$\left(\boldsymbol{M}_{r,s}^w\right)^{-1} = \left(\boldsymbol{M}_{r,s}^{\tilde{w}}\right)^H\,.$$

Therefore we obtain $\mathcal{T}_w^{-1} = \mathcal{U}_{\tilde{w}}$.

Furthermore, the norm of $\mathcal{U}_w$ can be determined in the following way. For a non-singular matrix $\boldsymbol{A}$ it holds that $\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{A}\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{A}^{-1}$. Hence $\boldsymbol{A}\boldsymbol{A}^H$ and $\boldsymbol{A}^H\boldsymbol{A}$ have the same eigenvalues and thus $\|\boldsymbol{A}^{\pm 1}\|_2 = \|\boldsymbol{A}^{\pm H}\|_2$. Therefore, it follows that $\|\mathcal{U}_w\| = \|T_w\|$ and $\|\mathcal{U}_w^{-1}\| = \|T_w^{-1}\|$ if $\mathcal{T}_w$ and $\mathcal{T}_w^{-1}$ are bounded operators.

For many applications one needs that $\{\Psi_{r,s}^{k,l}\}$ is a frame, i.e. there exist numbers $0 < A \le B < \infty$ such that

$$A\|f\|^2_{L^2(\mathbb{R}^2)} \le \sum_{r,s,k,l} \left|\langle f, \Psi_{r,s}^{k,l}\rangle\right|^2 \le B\|f\|^2_{L^2(\mathbb{R}^2)}$$

or that $\{\Psi_{r,s}^{k,l}\}$ is a Riesz basis, i.e. it is complete in $L^2(\mathbb{R}^2)$ and

$$A \sum_{r,s,k,l} \left|\left\langle f, \tilde{\psi}_{r,s}^{k,l}\right\rangle\right|^2 \le \|f\|^2_{L^2(\mathbb{R}^2)} \le B \sum_{r,s,k,l} \left|\left\langle f, \tilde{\psi}_{r,s}^{k,l}\right\rangle\right|^2$$

with $0 < A \le B < \infty$. For cosine wavelets, it turns out that there is no difference between Riesz bases and frames, as stated in the following.

Theorem 6. *Let $\{w_{r,s}\}$ be measurable functions such that $\Psi_{r,s}^{k,l} \in L^2(\mathbb{R}^2)$. Then the following statements are equivalent.*

(i) *The system $\{\Psi_{r,s}^{k,l}\}$ is a frame of $L^2(\mathbb{R}^2)$ with frame bounds A and B.*

(ii) *The system $\{\Psi_{r,s}^{k,l}\}$ is a Riesz basis of $L^2(\mathbb{R}^2)$ with Riesz bounds A and B.*

(iii) *It holds $0 < A \le A_0 \le B_0 \le B < \infty$ with A_0 and B_0 from* (9) *and* (4).

Proof: First we show (i) $\Longleftrightarrow$ (iii). It is known that the functions $C^k_{[\alpha,\beta]}$, $k \in \mathbb{N}_0$ form an orthonormal basis of $L^2([\alpha,\beta])$. Therefore with (3) we obtain

$$\sum_{r,s\in\mathbb{Z}} \sum_{k,l\in\mathbb{N}_0} \left|\langle f, \Psi_{r,s}^{k,l}\rangle\right|^2 = \|\mathcal{T}_w f\|^2_{L^2(\mathbb{R}^2)} .$$

Hence, $\{\Psi_{r,s}^{k,l}\}$ satisfies the frame inequality with the best possible frame bounds $A_0 = \|\mathcal{T}_w^{-1}\|^{-2}$ and $B_0 = \|\mathcal{T}_w\|^2$.

Now we show that (i, iii) $\Longleftrightarrow$ (ii). In particular, (i) follows from (ii) immediately. Furthermore, if the implication

$$\sum_{r,s,k,l} a_{r,s}^{k,l}\Psi_{r,s}^{k,l} = 0 \;\Rightarrow\; a_{r,s}^{k,l} = 0,\ \forall r,s,k,l \tag{12}$$

is proved, the statement (ii) follows from (i) (see e.g. [2] Theorem 2.1). To show (12) let us notice that

$$\Psi_{r,s}^{k,l} = \mathcal{U}_w\left(\chi_{[a_r,a_{r+1}]\times[b_s,b_{s+1}]}\, C^k_{[a_r,a_{r+1}]} \otimes C^l_{[b_s,b_{s+1}]}\right) ,$$

where $\chi_{[a_r,a_{r+1}]\times[b_s,b_{s+1}]}$ is the characteristic function of $[a_r,a_{r+1}]\times[b_s,b_{s+1}]$. From (iii) it follows, that $\mathcal{U}_w^{-1}$ is a bounded operator. Therefore we obtain that

$$\sum_{r,s\in\mathbb{Z}} \sum_{k,l\in\mathbb{N}_0} a_{r,s}^{k,l}\Psi_{r,s}^{k,l} = 0$$

implies

$$
\begin{aligned}
0 &= \mathcal{U}_w^{-1} \sum_{r,s\in\mathbb{Z}} \sum_{k,l\in\mathbb{N}_0} a_{r,s}^{k,l}\, \mathcal{U}_w \left(\chi_{[a_r,a_{r+1}]\times[b_s,b_{s+1}]}\, C_{[a_r,a_{r+1}]}^k \otimes C_{[b_s,b_{s+1}]}^l \right) \\
&= \sum_{r,s\in\mathbb{Z}} \chi_{[a_r,a_{r+1}]\times[b_s,b_{s+1}]} \sum_{k,l\in\mathbb{N}_0} a_{r,s}^{k,l}\, C_{[a_r,a_{r+1}]}^k \otimes C_{[b_s,b_{s+1}]}^l \,.
\end{aligned}
$$

This means that for all $r,s\in\mathbb{Z}$ it holds that

$$
\sum_{k,l\in\mathbb{N}_0} a_{r,s}^{k,l}\, C_{[a_r,a_{r+1}]}^k \otimes C_{[b_s,b_{s+1}]}^l = 0 \quad \text{a.e on } [a_r,a_{r+1}]\times[b_s,b_{s+1}]\,.
$$

Since $\{C_{[a_r,a_{r+1}]}^k \otimes C_{[b_s,b_{s+1}]}^l : \ k,l\in\mathbb{N}_0\}$ is a basis of $L^2([a_r,a_{r+1}]\times[b_s,b_{s+1}])$ it follows that $a_{r,s}^{k,l}=0$ for all $r,s\in\mathbb{Z}$ and $k,l\in\mathbb{N}_0$. Hence, (12) is proved. ∎

Finally we present conditions on the bells to obtain orthonormal bases. Obviously the $\Psi_{r,s}^{k,l}$ constitute an orthonormal basis iff

$$
w_{r,s} = \tilde{w}_{r,s} \quad \text{for all } r,s\in\mathbb{Z}\,.
$$

Because $\boldsymbol{M}_{r,s}^{\tilde{w}} = (\boldsymbol{M}_{r,s}^{w})^{-H}$ this is equivalent to

$$
\begin{aligned}
|w_{r,s}| &= 1 \quad \text{a.e on } R_1^{r,s}\,,\\
\boldsymbol{K}_{r,s}^H \boldsymbol{K}_{r,s} &= \boldsymbol{I} \quad \text{a.e on } R_2^{r,s}\,,\\
\boldsymbol{L}_{r,s}^H \boldsymbol{L}_{r,s} &= \boldsymbol{I} \quad \text{a.e on } R_3^{r,s}\,,\\
\boldsymbol{M}_{r,s}^H \boldsymbol{M}_{r,s} &= \boldsymbol{I} \quad \text{a.e on } R_4^{r,s}\,.
\end{aligned}
$$

It is easy to see, that the last three equations can be written as

$$
\begin{aligned}
\boldsymbol{K}_{r,s}^H \begin{pmatrix} w_{r,s} \\ w_{r-1,s} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{a.e on } R_2^{r,s}\,,\\
\boldsymbol{L}_{r,s}^H \begin{pmatrix} w_{r,s} \\ w_{r,s-1} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{a.e on } R_3^{r,s}\,,\\
\boldsymbol{M}_{r,s}^H \begin{pmatrix} w_{r,s} \\ w_{r-1,s} \\ w_{r,s-1} \\ w_{r-1,s-1} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{a.e on } R_4^{r,s}
\end{aligned}
$$

for all $r,s\in\mathbb{Z}$. Similar conditions can be found also in [11].

4. Examples

First we want to consider a continuous, piecewise linear non-tensor product bell function in more detail. We restrict ourselves to the simple rectangular pyramid $w^1_{r,s} = w^1_{0,0}(\cdot - r, \cdot - s)$ with

$$w^1_{0,0}(x,y) = \begin{cases} y, & \text{if } 0 \le x \le 2 \text{ and } 0 \le y \le \min(x, 2-x), \\ 1-y, & \text{if } 0 \le x \le 2 \text{ and } \max(x, 2-x) \le y \le 2, \\ x, & \text{if } 0 \le y \le 2 \text{ and } 0 \le x \le \min(y, 2-y), \\ 1-x, & \text{if } 0 \le y \le 2 \text{ and } \max(y, 2-y) \le y \le 2, \\ 0 & \text{otherwise} \end{cases}$$

for the grid $a_r = r + \frac{1}{2}$, $b_s = s + \frac{1}{2}$, $a^+_{r-1} = a^-_r = r$ and $b^+_{s-1} = b^-_s = s$ (see **Fig. 2**).

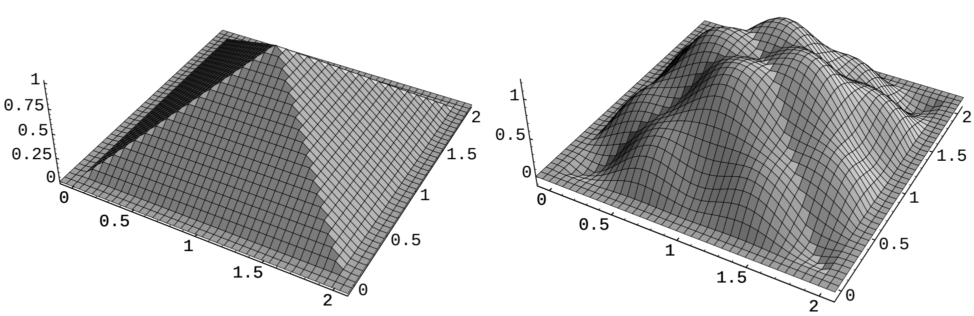

Fig. 2. The piecewise linear bell $w^1_{0,0}$ and the dual bell $\tilde{w}^1_{0,0}$.

As dual bell $\tilde{w}_{r,s}$ we obtain a piecewise rational, continuous function (see **Fig. 2**) with

$$\tilde{w}^1_{0,0}(x,y) = \frac{y\left(2y^2 - 2x^2 + 4x - 1\right)}{4x^4 - 8x^3 + 8x^2 - 4x + 8xy^2 - 8x^2y^2 + 4y^4 + 1}$$

for $0 \le x \le 1$ and $0 \le y \le \min(x, 1-x)$ and

$$\tilde{w}^1_{0,0}(x,y) = \frac{2x^2 - 4x - 2x^2y + 4xy - y - 2y^2 + 2y^3 + 2}{4y^4 - 8y^3 + 4y + 16xy^2 - 8x^2y^2 + 8x^2y - 16xy - 16x + 24x^2 - 16x^3 + 4x^4 + 5}$$

for $\frac{1}{2} \le x \le 1$ and $1 - x \le y \le x$. The remaining values of the function can be determined by symmetry arguments as follows

$$\begin{aligned} \tilde{w}^1_{0,0}(x,y) &= \tilde{w}^1_{0,0}(y,x)\ , && \text{if } 0 \le x \le 1 \text{ and } x \le y \le 1, \\ \tilde{w}^1_{0,0}(x,y) &= \tilde{w}^1_{0,0}(-x,y), && \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le 1, \\ \tilde{w}^1_{0,0}(x,y) &= \tilde{w}^1_{0,0}(x,-y), && \text{if } 0 \le x \le 2 \text{ and } 1 \le y \le 2. \end{aligned}$$

To compute the best possible Riesz bounds for this bell function following **Theorem 6** we have to consider the eigenvalues of the corresponding matrix $(\boldsymbol{M}_{0,0}^{w^1})^T \boldsymbol{M}_{0,0}^{w^1}$. By the above mentioned symmetry we only have to deal with x, y from the triangle $0 \le x \le 1$, $0 \le y \le \min(x, 1-x)$. In this case the eigenvalues are

$$\begin{aligned} \lambda_1(x,y) = \lambda_2(x,y) &= 2(x-y-\tfrac{1}{2})^2 + \tfrac{1}{2}\,, \\ \lambda_3(x,y) = \lambda_4(x,y) &= 2(x+y-\tfrac{1}{2})^2 + \tfrac{1}{2}\,. \end{aligned}$$

By easy calculations we obtain the Riesz bounds $A_0 = \frac{1}{2}$ and $B_0 = 1$.

The second example is the tensor product

$$w_{0,0}^2(x,y) := h(x)h(y)$$

of the linear B-spline

$$h(x) := \begin{cases} x, & \text{if } 0 \le x < 1, \\ 2-x, & \text{if } 1 \le x < 2, \\ 0 & \text{otherwise} \end{cases}$$

(see **Fig. 3**).

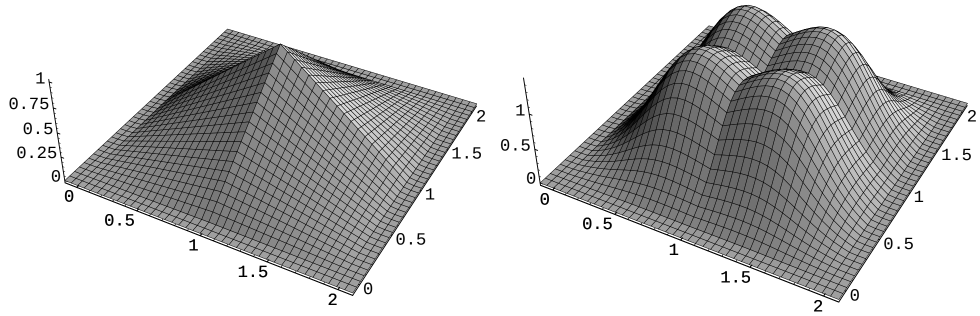

Fig. 3. The piecewise bilinear bell $w_{0,0}^2$ and the dual bell $\tilde{w}_{0,0}^2$.

The corresponding dual bell function is given by

$$\tilde{w}_{0,0}^2(x,y) = \frac{x\,y}{(2x^2-2x+1)\,(2y^2-2y+1)}$$

for $0 \le x, y \le 1$. The remaining values can be determined by the same symmetry conditions as in the first example.

In the tensor product case the four eigenvalues of $\boldsymbol{M}_{0,0}^T \boldsymbol{M}_{0,0}$ are products of the eigenvalues of $(\boldsymbol{M}_0^1)^T \boldsymbol{M}_0^1$ and $(\boldsymbol{M}_0^2)^T \boldsymbol{M}_0^2$ (see **Remark 4**). Easily one computes

$$\lambda_1(x,y) = \lambda_2(x,y) = \lambda_3(x,y) = \lambda_4(x,y) = \left(2(x-\tfrac{1}{2})^2 + \tfrac{1}{2}\right)\left(2(y-\tfrac{1}{2})^2 + \tfrac{1}{2}\right)\,.$$

Here we obtain the best possible Riesz bounds $A_0 = \frac{1}{4}$ and $B_0 = 1$.

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