ORTHOGONAL DOUBLE COVERS OF COMPLETE GRAPHS BY TREES OF SHORT DIAMETER

UWE LECK AND VOLKER LECK

Abstract. A collection $\mathcal{P}$ of n spanning subgraphs of the complete graph K_n is an orthogonal double cover (ODC) of K_n if every edge of K_n belongs to exactly two members of $\mathcal{P}$, and if every two members of $\mathcal{P}$ share exactly one edge. $\mathcal{P}$ is an ODC of K_n by some graph G if all graphs in $\mathcal{P}$ are isomorphic to G. Gronau, Mullin, and Rosa conjecture that every tree except the path with 4 vertices admits an ODC of the fitting K_n. They proved this to be true for trees of diameter 3. In this paper, we show the correctness of their conjecture for some classes of trees of diameter 4.

1. Introduction

An *orthogonal double cover (ODC)* $\mathcal{G} = \{G_1, \ldots, G_n\}$ of the complete graph on n vertices is a set of spanning subgraphs of K_n such that:

(1) Every edge of K_n belongs to the edge sets of exactly two elements of $\mathcal{G}$.
(2) For $1 \le i \le j \le n$ there is exactly one edge belonging to the edge set of G_i and the edge set of G_j.

Graphs G_i and G_j satisfying (2) are said to be *orthogonal*. The graphs $G_i\,(i = 1, \ldots, n)$ are called *pages* of $\mathcal{G}$, and if all pages $G_i\,(i = 1, \ldots, n)$ are isomorphic to some graph G, then $\mathcal{G}$ is called *ODC of K_n by G*.

The principal question here is: For which graphs G does there exist an ODC of some complete graph by G? This question was investigated first by Demetrovics, Füredi, and Katona [7] who had an application of the existence of ODCs by graphs G of a certain structure. Later results focus on several classes of graphs G as clique graphs [4, 8], graphs of maximum degree 2 [5, 6, 13], graphs consisting of disjoint cycles [10, 15], graphs consisting of short cycles [9, 18], almost hamiltonian cycles [14, 17], graphs on few vertices [20]. The directed analogue of the problem was studied in [10].

Date: August 15, 1997.
Research of the second author supported in part by NATO grant CRG 940085.

A simple double counting immediately provides a necessary condition for an ODC. Namely, if $\mathcal{G} = \{G_1, \ldots, G_n\}$ is an ODC of K_n, then for $i = 1, \ldots, n$ the edge set of G_i contains exactly $n-1$ edges. This fact is a special motivation for investigating ODCs by trees. Unfortunately, to construct an ODC by some tree T turns out to be difficult already for paths, although there are several results in that direction [1, 2, 3, 14, 15, 16, 17]. A theorem in [19] states that for every tree T on at most 13 vertices there is an ODC by T, except in the case $T = P_3$ (the path with 3 edges). The current state of affairs is given in the following theorem.

Theorem 1. *Let $T \neq P_3$ be a tree with $n := |V(T)| \leq 14$. Then there is an ODC of K_n by T.*

The complete list of those ODCs can be found at the WWW–page `ftp://zeus.math.uni-rostock.de/WWW/vleck.html`.

From [12] we know that every tree $T \neq P_3$ of diameter 2 or 3 admits an ODC. In general, the following is conjectured.

Conjecture 1. (Gronau, Mullin, Rosa [12]) *Let T be an arbitrary tree with n vertices, $n \geq 2$, where $T \neq P_3$. Then there exists an ODC of K_n by T.*

In this paper we construct ODCs by T for several classes of trees T of diameter 4.

2. Cyclic orthogonal double covers

An ODC of K_n by some graph G is called *cyclic* if the cyclic group of order n is a subgroup of its automorphism group. For cyclic orthogonal double covers we shall use the abbreviation *CODC*. Every page of a CODC $\mathcal{G}$ is said to be a *starter* for $\mathcal{G}$.

Let the vertex set of K_n be the set $[n] := \{0, 1, \ldots, n-1\}$. Obviously, arranging the elements of $[n]$ equidistantly and in monotone order on a circle, a CODC of K_n can be obtained by taking a suitable starter and all pages arising from rotation of that starter.

To formulate an equivalent condition for the existence of a CODC of K_n by some graph G, we define the *distance* $\ell(x,y)$ of $x, y \in [n]$ to be the length of the shorter one of the two paths from x to y on the circle, i.e.

$$\ell(x,y) := \min\{|x-y|,\, n - |x-y|\}.$$

If there is an edge connecting the vertices x and y, then $\ell(x, y)$ is called the *length* of the edge $\{x, y\}$. Consider two edges $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$ with $x_1, y_1, x_2, y_2 \in [n]$ and $\ell(e_1) = \ell(e_2)$. Their *rotation–index* $r(e_1, e_2)$ is defined to be the length of the shorter one of the two rotations mapping e_1 onto e_2, i.e.

$$r(e_1, e_2) := \begin{cases} \ell(x_1, x_2) & \text{if } \ell(x_1, x_2) = \ell(y_1, y_2) \\ \ell(x_1, y_2) & \text{if } \ell(x_1, y_2) = \ell(x_2, y_1). \end{cases}$$

If for some graph $G = ([n], E)$ the set E contains exactly two edges e_1, e_2 of length k, then $r(e_1, e_2)$ will be abbreviated by $r_G(k)$, or simply by $r(k)$ if there is no danger of ambiguity.

Let us introduce the notion of an *orthogonal labelling (OL)*. A labelling of a graph $G = (V, E)$ is simply a bijection $\phi : V \longrightarrow [n]$. A labelling of a graph $G = (V, E)$ with $|V| = n$ and $|E| = n - 1$ is called orthogonal if (with respect to ϕ) the following conditions are satisfied:

(a) G contains exactly two edges of length k for $k = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$.
(b) The rotation–indices form a permutation of $\{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, that means $\{r_G(k) \mid k = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} = \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$.

The next theorem relates CODCs and OLs.

Theorem 2. [9, 12] *Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = n - 1$. There exists a CODC of K_n by G iff there is an OL of G.*

CODCs are of special interest since they are often suitable starting points for recursive constructions of ODCs. We will use CODCs as starting points for the adding–construction which is described in the next section.

Obviously, every labelling of the n–star (the tree of diameter 2 with n edges) is an OL. Therefore, the following lemma holds.

Lemma 1. *There is a CODC of K_{n+1} by the n–star.*

For our constructions of ODCs by trees of diameter 4, sometimes CODCs by trees of diameter 3 will be needed. As mentioned in the introduction, there are ODCs by all trees of diameter 3, except P_3. It is also known which trees of diameter 3 admit CODCs. By $(p, q; 3)$–tree we denote the tree of diameter 3 whose vertex set contains two adjacent vertices of degrees $p + 1$ and $q + 1$, respectively (figure 1).

Theorem 3. [19] *Let T be the $(p, q; 3)$–tree on $n = p+q+2$ vertices. There is a CODC of K_n by T iff n and pq are not relatively prime.*

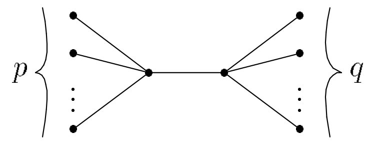

FIGURE 1. The $(p, q; 3)$–tree.

3. THE ADDING–CONSTRUCTION

There are several constructions for ODCs which make use of relations between ODCs and other structures like pairwise balanced designs [8, 12], orthogonal latin squares [5, 12], quasi–groups [9], or rings satisfying certain conditions [8]. So far there are only few applications of these constructions to trees; pairwise balanced designs were used to construct ODCs by trees which arise from amalgamating in a certain way 3–stars [12] or 4–stars [19], or by trees obtained from a star by replacing each edge by a path of length 4 [12].

A problem of the mentioned constructions is that the existence of the used structures often turns out to be a restrictive condition. The *adding–construction*, which we will make use of in the next sections, "adds" two ODCs of K_m and K_n, respectively, and provides an ODC of K_{m+n}. The authors of [12] applied an initial version of the adding–construction to obtain from an ODC of K_n by some graph G an ODC of K_{2n} by G^*, where G^* is obtained from G by joining every vertex with a new vertex by a new edge.

The adding–construction works as follows. Consider the complete graphs K_m and K_n on the vertex sets $\{1, \ldots, m\}$ and $\{\bar{1}, \ldots, \bar{n}\}$, respectively. Let $\mathcal{G}_1$ be an ODC of K_m and $\mathcal{G}_2$ an ODC of K_n. Add the vertices $\bar{1}, \ldots, \bar{n}$ to the vertex set of every page of $\mathcal{G}_1$ and the vertices $1, \ldots, m$ to the vertex sets of all pages of $\mathcal{G}_2$. Let $\mathcal{G}'$ be the collection of all the resulting graphs. To make $\mathcal{G}'$ an ODC $\mathcal{G}$ of K_{m+n}, we have to allot the edges $\{i, \bar{j}\}$ $(i = 1, \ldots, m; j = \bar{1}, \ldots, \bar{n})$ to the elements of $\mathcal{G}'$ in such a way that each of these edges appears in exactly one page generated from $\mathcal{G}_1$ and in exactly one page generated from $\mathcal{G}_2$. Trivially, such allotments exist, and every such allotment gives an ODC $\mathcal{G}$ of K_{m+n}. The problem is to find an allotment providing an ODC with all its pages mutually isomorphic.

The next lemma describes a situation where the desired allotment can be given. To formulate the lemma, we need two more notions.

Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = n - 1$. For a bijection $\rho : V \longrightarrow V(K_n)$ we define $\rho(E)$ to be the set of edges

$\{\rho(x), \rho(y)\}$ such that $\{x, y\} \in E$. Finally, let $\rho(G) := (V(K_n), \rho(E))$. Consider an ODC $\mathcal{G} = \{\rho_1(G), \ldots, \rho_n(G)\}$ of K_n by G. We call $x \in V$ a *rotating vertex* of $\mathcal{G}$ if $\{\rho_1(x), \ldots, \rho_n(x)\} = V(K_n)$. By definition, every vertex of a CODC is a rotating vertex.

Lemma 2 uses an operation called *adding a star*. Given a graph G, we add the n–star at $x \in V(G)$ simply by putting n new vertices into the vertex set of G and inserting all edges joining x and one of the new vertices.

Lemma 2. *Let G_1 and G_2 be graphs with $|V(G_1)| = m$ and $|V(G_2)| = n$ such that there are ODCs $\mathcal{G}_1$ and $\mathcal{G}_2$ of K_m and K_n by G_1 and G_2, respectively. Furthermore, let G be a graph on $m+n$ vertices which can be obtained from G_1 by adding an n–star at an rotating vertex of $\mathcal{G}_1$. If G can be obtained from G_2 by adding stars at some rotating vertices of $\mathcal{G}_2$, then there is an ODC of K_{m+n} by G.*

Proof. We use the adding–construction with $\mathcal{G}_1$ and $\mathcal{G}_2$ to generate an ODC of K_{m+n} by G.

Let $\mathcal{G}_1 = \{\rho_1(G_1), \ldots, \rho_m(G_1)\}$ and $\mathcal{G}_2 = \{\sigma_1(G_2), \ldots, \sigma_n(G_2)\}$. Furthermore, let $x \in V(G_1)$ be a rotating vertex of $\mathcal{G}_1$ such that G can be obtained from G_1 by adding an n–star at x. For $i = 1, \ldots, m$ add the n–star, whose vertices of degree 1 are the vertices from $V(K_n)$, to $\rho_i(G_1)$ at $\rho_i(x)$. Denote by $\mathcal{G}_1'$ the collection of the m graphs obtained that way.

Let $y_1, \ldots, y_s \in V(G_2)$ be rotating vertices of $\mathcal{G}_2$ such that G can be obtained from G_2 by adding an m_j–star at y_j for $j = 1, \ldots, s$. Then, clearly, $\sum_{j=1}^{s} m_j = m$. Further let $V_1 \cup \cdots \cup V_s$ be a partition of $V(K_m)$ with $|V_j| = m_j$ for $j = 1, \ldots, s$. For $i = 1, \ldots, n$ and $j = 1, \ldots, s$ add the m_j–star, whose vertices of degree 1 are the elements of V_j, to $\sigma_i(G_2)$ at $\sigma_i(y_j)$. Let $\mathcal{G}_2'$ denote the collection of the n graphs obtained by this procedure.

Put $\mathcal{G} = \mathcal{G}_1' \cup \mathcal{G}_2'$. By construction, $\mathcal{G}$ consists of $m+n$ spanning subgraphs of the K_{m+n} on $V(K_m) \cup V(K_n)$ which are isomorphic to G. Every edge $\{x, y\}$ with $x, y \in V(K_m)$ (resp. $x, y \in V(K_n)$) belongs to the edge sets of exactly two elements of $\mathcal{G}_1'$ (resp. $\mathcal{G}_2'$). For $H_1, H_2 \in \mathcal{G}_1'$ (resp. $H_1, H_2 \in \mathcal{G}_2'$) there is exactly one edge contained in $E(H_1) \cap E(H_2)$, both endpoints of this edge are in $V(K_m)$ (resp. $V(K_n)$). Every edge $\{x, y\}$ with $x \in V(K_m)$ and $y \in V(K_n)$ appears in exactly one element of $\mathcal{G}_i'$, $i = 1, 2$. For $H_1 \in \mathcal{G}_1'$ and $H_2 \in \mathcal{G}_2'$, the condition $|E(H_1) \cap E(H_2)| = 1$ is fulfilled, where the common edge has one endpoint in $V(K_m)$ and the other in $V(K_n)$. Consequently, $\mathcal{G}$ is an ODC of K_{m+n} by G. $\square$

In section 6 we will add ODCs $\mathcal{G}_1, \dots, \mathcal{G}_s$ of $K_{n_1}, \dots, K_{n_s}$, respectively, to obtain an ODC of $K_{n_1+\cdots+n_s}$. For that purpose we will add the vertex sets of all K_j's with $j \neq i$ to every page of $\mathcal{G}_i$. Then we will allot every edge $\{u, v\}$ with $u \in V(K_{n_i})$ and $v \in V(K_{n_j})$ to exactly one graph resulted from a page of $\mathcal{G}_i$ and to exactly one graph resulted from a page of $\mathcal{G}_j$. It is not hard to see that this is the same as applying the adding–construction iteratively: first add $\mathcal{G}_1$ and $\mathcal{G}_2$, then add the resulting ODC and $\mathcal{G}_3$, and so on. The advantage of adding more than two ODCs *once* instead of adding two ODCs several times is that we do not need to consider the "in–between" ODCs whose pages might not be mutually isomorphic.

4. $(p, q, r; 4)$–TREES

Let $p \geq q \geq 1$ and $r \geq 0$ be integers. By $(p, q, r; 4)$*–tree* we denote the tree of diameter 4 consisting of a path with 3 vertices and exactly p vertices of degree 1 adjacent to one endpoint of that path, exactly q vertices of degree 1 adjacent to the other endpoint, and exactly r vertices of degree 1 adjacent to the middle point (figure 2).

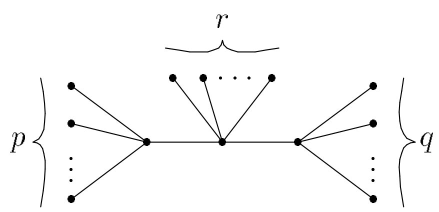

FIGURE 2. The $(p, q, r; 4)$–tree.

For some classes of $(p, q, r; 4)$–trees, ODCs were constructed in [12]. In this section we give ODCs for all such trees.

Lemma 3. *Let T be a $(p, q, r; 4)$–tree on $n = p + q + r + 3$ vertices with $r \leq p - q$. Then there is a CODC of K_n by T.*

Proof. We prove the lemma by explicitly giving an OL ϕ of the T.

Let $x, y, z \in V(T)$ with degrees greater than 1 such that $E(T)$ contains $\{x, y\}$ and $\{y, z\}$, and x is adjacent to exactly p vertices of degree 1. Put

$$\phi(x) = q + r + 1, \quad \phi(y) = 2q + r + 1, \quad \phi(z) = q.$$

Partition the vertices of degree 1 adjacent to x into two classes of $q-1$ and $p-q+1$ vertices and label them with

$$\begin{array}{ll} p+r+1+i, & i=1,\dots,q-1, \\ 2q+r+1+i, & i=1,\dots,p-q+1, \end{array}$$

respectively. Finally, label the r vertices of degree 1 adjacent to y with $q+1,\dots,q+r$ and the q vertices of degree 1 adjacent to z with $0,\dots,q-1$.

It is easy to see that with the above labelling every length $< \left\lfloor \frac{n}{2} \right\rfloor$ appears exactly twice and that

$$\begin{array}{rcll} r(i) & = & r+1+i & \text{for } i=1,\dots,q, \\ r(i) & = & i-q & \text{for } i=q+1,\dots,q+r+1, \\ r(i) & = & i & \text{for } i=q+r+2,\dots,\left\lfloor \frac{n}{2} \right\rfloor. \end{array}$$

Consequently, ϕ is an OL of T. □

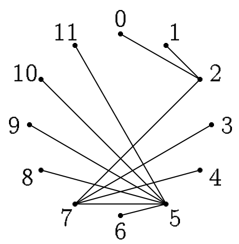

FIGURE 3. An orthogonal labelling of the $(5,2,2;4)$–tree.

Theorem 4. *Let T be the $(p,q,r;4)$–tree on $n=p+q+r+3$ vertices. Then there is an ODC of K_n by T.*

Proof. Without loss of generality we may assume $r>p-q$ because of lemma 3. We take two CODCs and apply lemma 2, i.e. use the adding–construction, to obtain an ODC by the $(p,q,r;4)$–tree. We obtain ODCs by all $(p,q,r;4)$–trees, except some trees on few vertices for which solutions are known.

Case 1. Assume $r\geq 3$. By lemma 3, there is a CODC of K_{n-r} by the $(p,q,0;4)$–tree. Obviously, the $(p,q,r;4)$–tree can be obtained from the $(p,q,0;4)$–tree by adding an r–star. By lemma 1, there is a CODC of K_r by the $(r-1)$–star. The $(p,q,r;4)$–tree can be obtained from the $(r-1)$–star by adding a p–star at a vertex of degree 1 and adding a q–star at another vertex of degree 1. Now lemma 2 with the $(p,q,0;4)$–tree as G_1 and the $(r-1)$–star as G_2 implies the assertion.

Case 2. Assume $r\leq 2$ and $p\geq 4$. Since $r>p-q$ holds, we have $q-(p-3)\geq 2\geq r$. Now by lemma 3, there is a CODC of K_{n-3}

by the $(q, p-3, r; 4)$–tree. The $(p, q, r; 4)$–tree can be obtained from the $(q, p-3, r; 4)$–tree by adding a 2–star. By lemma 1, there is a CODC of K_3 by the 2–star. The $(p, q, r; 4)$–tree can be obtained from the 2–star by adding an r–star at the vertex of degree 2, a p–star at a vertex of degree 1, and a q–star at the other vertex of degree 1. Applying lemma 2 with the $(q, p-3, r; 4)$–tree as G_1 and the 2–star as G_2 yields the assertion.

Case 3. Assume $r \leq 2$ and $p \leq 3$. In this case we have $n \leq 11$, and the desired ODC exists by theorem 1. □

5. $(3; k_1, k_2, k_3; 4)$–TREES

In the following we describe an arbitrary tree of diameter 4 by its parameters t and $k_1, \ldots, k_t$. Let $t \geq 2$ and $k_1 \geq \cdots \geq k_t \geq 0$ be integers, and let k_1, k_2 be positive. By $(t; k_1, \ldots, k_t; 4)$–tree we denote the tree of diameter 4 which can be obtained from the t–star by adding an k_i–star at the i–th vertex of degree 1 for $i = 1, \ldots, t$ (figure 4). For convenience, the n–star and the $(p, q; 3)$–tree are the same as the $(n; 0, \ldots, 0; 4)$–tree and the $(p+1; q, 0, \ldots, 0; 4)$–tree, respectively.

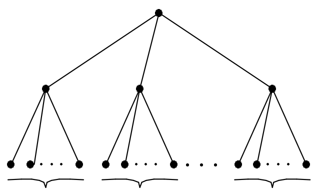

FIGURE 4. The $(t; k_1, \ldots, k_t; 4)$–tree.

Let us call the three vertices of the $(t; k_1, \ldots, k_t; 4)$–tree, which are adjacent to $k_1, \ldots, k_t$ vertices of degree 1, respectively, the *central vertices* and the vertex adjacent to all central vertices the *top* of the $(t; k_1, \ldots, k_t; 4)$–tree.

Theorem 5.[1] *Let T be the $(3; k_1, k_2, k_3; 4)$–tree on $n = k_1 + k_2 + k_3 + 4$ vertices. Then there is an ODC of K_n by T.*

Proof. Let T' be a tree that can be obtained from T by adding a 4–star at some central vertex. To prove the theorem, we shall give CODCs for some small $(3; k_1, k_2, k_3; 4)$–trees and use the following recursion: If there is an ODC $\mathcal{G}$ of K_n by T such that the central

[1] The authors thank H.–D.O.F. Gronau for suggesting the investigation of $(3; k_1, k_2, k_3; 4)$–trees.

vertices of T are rotating vertices of $\mathcal{G}$, then there is an ODC $\mathcal{G}'$ of K_{n+4} by T' such that the central vertices of T' are rotating vertices of $\mathcal{G}'$.

Assume there exists an ODC $\mathcal{G}$ of K_n by T such that the central vertices of T are rotating vertices of $\mathcal{G}$. By lemma 1, there is a CODC $\bar{\mathcal{G}}$ of K_4 by the 3–star. Since T' can be obtained from the 3–star by adding stars at vertices of degree 1, lemma 2 with $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_2 = \bar{\mathcal{G}}$ yields the existence of an ODC $\mathcal{G}'$ of K_{n+4} by T'. Let us go back to the construction of $\mathcal{G}'$ given in the proof of lemma 2. By this construction, if x is a central vertex of T', then in the pages from $\mathcal{G}'_1$ x runs through $V(G)$ and in the pages from $\mathcal{G}'_2$ through the vertex set of the 3–star. Consequently, x is a rotating vertex of $\mathcal{G}'$.

To complete the proof, we need to give OLs for all $(3; k_1, k_2, k_3; 4)$–trees with $k_1 \leq 3$, where $k_1 = 0$ or $k_2 = 0$ is allowed. There are exactly 20 such trees. 5 of them admit OLs by lemma 1, theorem 3, or lemma 3. For another 10, OLs are given in figure 5. For 2 of the remaining 5, namely the $(3; 1, 1, 1; 4)$– and the $(3; 2, 2, 0; 4)$–tree, OLs are given in [12].

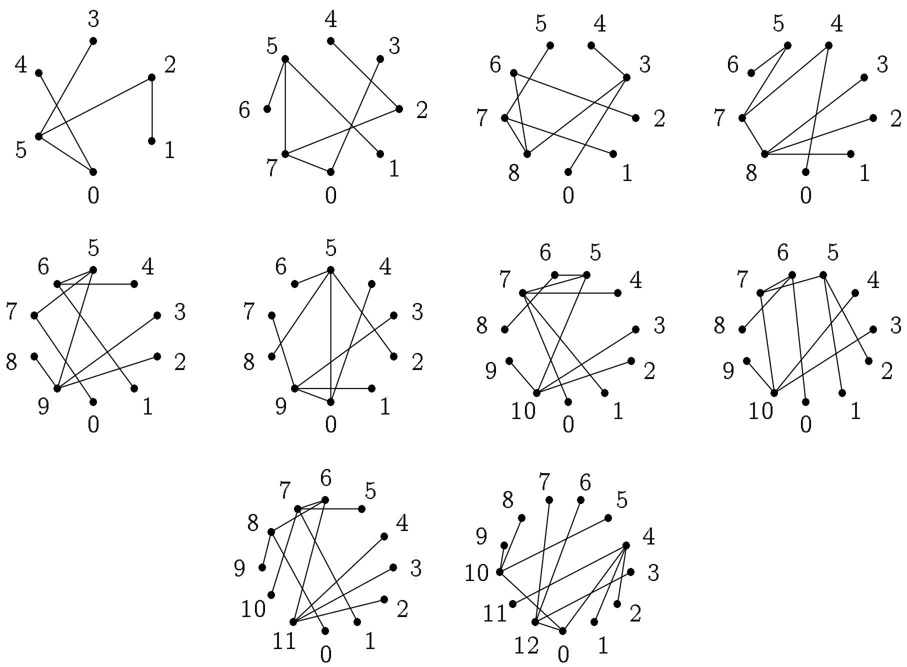

FIGURE 5. OLs for small $(3; k_1, k_2, k_3; 4)$–trees.

The 3 trees left are the $(3; 2, 2, 2; 4)$–, $(3; 1, 0, 0; 4)$–, and $(3; 3, 0, 0; 4)$–trees. These trees admit ODCs by theorem 1, but unfortunately they do not admit CODCs. Since all $(3;k_1,0,0;4)$–trees are of diameter 3, for our recursion it suffices to have CODCs of the $(3;4,1,0;4)$– and $(3;4,3,0;4)$–trees instead of the $(3;1,0,0;4)$– and $(3;3,0,0;4)$–trees, respectively. These CODCs exist by lemma 3. To complete the proof, we must find an ODC of K_{10} by the $(3;2,2,2;4)$–tree such that the central vertices are rotating vertices. Such an ODC of the K_{10} on $[9]$ is given in table 1, where also the top rotates. Every row in table 1 describes a labelling of the $(3;2,2,2;4)$–tree, and the 10 labelled graphs are the pages. □

a	b	c	d	x_b	y_b	x_c	y_c	x_d	y_d
0	1	2	3	4	7	5	8	6	9
1	2	3	0	4	7	5	8	6	9
2	3	0	1	4	7	5	8	6	9
3	0	1	2	4	7	5	8	6	9
4	5	8	7	0	1	2	3	6	9
5	6	9	8	0	1	2	3	7	4
6	7	4	9	0	1	2	3	8	5
7	8	5	4	0	1	2	3	9	6
8	9	6	5	0	1	2	3	4	7
9	4	7	6	0	1	2	3	5	8

TABLE 1. An ODC of K_{10} by the $(3;2,2,2;4)$–tree with the top a and the central vertices c,d,e rotating and with x_i, y_i adjacent to $i \in \{b,c,d\}$.

Theorem 5, lemma 1, and lemma 2 enable us to construct ODCs for another class of trees.

Corollary 1. *Let T be a $(t;k_1,\ldots,k_t;4)$–tree on $n = t+1+\sum_{i=1}^{t} k_i$ vertices such that $t \geq 7$ and $k_i = 0$ for $4 \leq i \leq t$. Then there is an ODC of K_n by T.*

Proof. In the proof of theorem 5, we constructed ODCs for all $(3;k_1,k_2,k_3;4)$–trees such that the central vertices are rotating vertices. One immediately observes that also the tops are rotating vertices of the constructed ODCs. Starting with such an ODC of $K_{4+k_1+k_2+k_3}$ by the $(3;k_1,k_2,k_3;4)$–tree and a CODC of K_{t-3} by the $(t-4)$–tree, lemma 2 yields the desired ODC of K_n by T. □

6. Other trees of diameter 4

In theorem 6 below, we give a class of ODCs by trees of diameter 4 which can be generated applying the adding–construction with a number of CODCs by the t–star. That is why we have the strong condition (i) in the theorem. But this condition can be weakened considerably using the existence results for CODCs from the previous sections, as shown in corollary 2.

Theorem 6. *Let T be a $(t; k_1, \ldots, k_t; 4)$–tree on $n = t+1+\sum_{h=1}^{t} k_h$ vertices, and let $k_h = (t+1)\ell_h + r_h$ with $r_h \in [t+1]$ for $h = 1, \ldots, t$. Furthermore, let the two following conditions be satisfied:*

(i) $n \equiv 0 \pmod{t+1}$,

(ii) $(t+1) \cdot \sum_{h=1}^{t} \ell_h \geq \sum_{h=1}^{t} r_h$.

Then there is an ODC of K_n by T. Moreover, the top and the central vertices of T are rotating vertices of this ODC.

Proof. Put $s := \frac{n}{t+1}$ and $\ell := \sum_{h=1}^{t} \ell_h$. Now (ii) is equivalent to $\ell \geq \frac{s-1}{2}$. Let S denote the t–star, and let $\{x_1, \ldots, x_t\} \subset V(S)$ be the set of vertices which have degree 1 in S and x_0 the vertex of degree t.

We use the adding–construction to generate an ODC of K_n by T from s different CODCs of K_{t+1} by the S.

Let $V(K_n) = V_0 \cup \cdots \cup V_{s-1}$ be a partition of $V(K_n)$ into pairwise disjoint sets of cardinality $t+1$. For $i \in [s]$ and $j \in [t+1]$ let $\rho_{ij} : V(S) \longrightarrow V_i$ be bijections such that for $i \in [s]$ the set $\mathcal{G}_i = \{\rho_{i0}(S), \ldots, \rho_{it}(S)\}$ is a CODC of the complete graph on V_i by S. The $\mathcal{G}_i$'s, and therefore the ρ_{ij}'s, exist by lemma 1. Throughout this proof, the indices of the V's and the first indices of the ρ's will be considered as the elements of the cyclic group over $[s]$.

We shall describe now how to generate from the page $\rho_{ij}(S)$ of $\mathcal{G}_i$ the page G_{ij} of a new ODC $\mathcal{G} = \{G_{ij} \mid i \in [s], j \in [t+1]\}$ of K_n by T. Firstly, for all $h \in \{1, \ldots, t\}$ with $\ell_h > 0$, add to $\rho_{ij}(S)$ at the vertex $\rho_{ij}(x_h)$ the $(\ell_h(t+1))$–star whose vertices of degree 1 are those from

$$\Big\{x \in V_{i+q} \mid \textstyle\sum_{p=1}^{h-1} \ell_p < q \leq \sum_{p=1}^{h} \ell_p\Big\}.$$

Secondly, for every $i \in [s]$ put $U_i := \{x \in V_{i+q} \mid \ell < q < s-\ell\}$, and let $U_i = U_{i1} \cup \cdots \cup U_{it}$ be a partition with $|U_{ih}| = r_h$ for $h = 1, \ldots, t$. Now for $h = 1, \ldots, t$ add at $\rho_{ij}(x_h)$ the r_h–star whose vertices of degree 1 are those from U_{ih}. (Add nothing if $r_h = 0$.) The resulting graph is G_{ij}.

By the above construction, all graphs G_{ij} are isomorphic to T. Consider two different graphs G_{ij} and $G_{i'j'}$.

Case 1. Assume $i = i'$. Then, by the above construction, G_{ij} and $G_{i'j'}$ share exactly one edge, and this edge is $\{\rho_{ij}(x_0), \rho_{ij'}(x_0)\}$.

Case 2. Assume $i \neq i'$. Without loss of generality, we may assume $i + q = i'$ with $q \leq \frac{s}{2}$. Now (ii) implies $q \leq \ell$. Let $h \in \{1, \ldots, t\}$ such that $\sum_{p=1}^{h-1} \ell_p < q \leq \sum_{p=1}^{h} \ell_p$. By construction, in G_{ij} all vertices from $V_{i'}$ are adjacent to $\rho_{ij}(x_h)$. Let $i' + d = i$.

Case 2.1. Assume $d \leq \ell$. Then in $G_{i'j'}$ all vertices from V_i are adjacent to $\rho_{i'j'}(x_g)$, where $g \in \{1, \ldots, t\}$ such that $\sum_{p=1}^{g-1} \ell_p < d \leq \sum_{p=1}^{g} \ell_p$. Consequently, G_{ij} and $G_{i'j'}$ share exactly the edge $\{\rho_{ij}(x_h), \rho_{i'j'}(x_g)\}$.

Case 2.2. Assume $d > \ell$. Then ρ_{ij} is adjacent to $\rho_{i'j'}(x_g)$, where $g \in \{1, \ldots, t\}$ such that $\rho_{ij}(x_h) \in U_{i'g}$. Therefore G_{ij} and $G_{i'j'}$ share exactly the edge $\{\rho_{ij}(x_h), \rho_{i'j'}(x_g)\}$.

Note that in both cases 2.1. and 2.2. the numbers h, g are independent of j, j'. Since $\mathcal{G}_0, \ldots, \mathcal{G}_{s-1}$ are cyclic, for all $i \in [s]$ every set $\{\rho_{ij}(x_h) \mid j \in [t+1]\}$ is equal to V_i. Consequently, every edge $\{x, y\}$ with $x, y \in V_i$ appears in two graphs G_{ij} and $G_{ij'}$, and every edge $\{x, y\}$ with $x \in V_i, y \in V_{i'}$ appears in two graphs G_{ij} and $G_{i'j'}$. A simple double counting shows that then every edge from $E(K_n)$ must appear in *exactly* two of the constructed graphs. Therefore $\mathcal{G}$ is an ODC of K_n by T. The fact that the top and the central vertices of T are rotating vertices of $\mathcal{G}$ is easily verified. $\square$

Let us consider the interaction between the pages of two ODCs $\mathcal{G}_i$ and $\mathcal{G}_j$, if we construct an ODC by some graph G adding s ODCs $\mathcal{G}_1, \ldots, \mathcal{G}_s$ by graphs $G_1, \ldots, G_s$, respectively, like in the proof of theorem 6. To make the construction work, we always add *one* star from the one side and are allowed to add one or more stars from the other side. In detail, we add one star to every page of $\mathcal{G}_i$ such that the new vertices are all $x \in V(G_j)$, where to every page from $\mathcal{G}_j$ stars are added such that the set of the new vertices is $V(G_i)$. In average, we have to add to G_i just one star for at least half of the other G_j's. Therefore and because G must be obtainable from every G_i by adding stars, choosing the t–star as G_i for all i and presuming (ii) is in a sence the best we can do, if we want to use our construction to generate ODCs by $(t; k_1, \ldots, k_t; 4)$–stars.

The following corollary shows a way to avoid condition (i).

Corollary 2. *Let T and n be like in theorem 6, and let $m \geq t + 1$ be an integer. Furthermore, let T' be a tree which can be obtained from T by adding an m–star at the top or at some central vertex.*

If T' has a subtree U on m vertices such that the t–star is a subtree of U and there is a CODC of K_m by U, then there is an ODC of K_{m+n} by T'.

Proof. By theorem 6 and the existence of a CODC by U, we may apply lemma 2 with $G_1 = T$ and $G_2 = U$ to obtain an ODC of K_{m+n} by T'. $\square$

Let us conclude with another corollary which is weaker than the results before but probably more handy and easier to keep in mind.

Corollary 3. *Let V be a $(t; k_1, \ldots, k_t; 4)$–tree on n vertices with $k_1 \geq 4t + 3$ and $k_h \geq t + 1$ for $h = 2, \ldots, t$. Then there is an ODC of K_n by V.*

Proof. Let $n = (t+1)s+r$ with $r \in [t+1]$. We apply corollary 2 with the $(t; k_1 - 2t - 2 - r, k_2, \ldots, k_t; 4)$–tree as T and the $(t + r, 1, t - 2; 4)$–tree as U. Obviously, the t–star is a subtree of U, and by lemma 3, U admits a CODC. T satisfies condition (i) in theorem 6. By $r \leq t$, $k_1 \geq 4t + 3$, and $k_h \geq t + 1$ for $h = 2, \ldots, t$, the left hand side of (ii) is at least $t^2 + t$, where the right hand side is at most t^2. $\square$

References

[1] B. Alspach, K. Heinrich, and G. Liu, *Orthogonal factorizations in graphs*, Cont. Design Theory: A Collection of Surveys (J.H. Dinitz and D.R. Stinson eds.), J. Wiley, N.Y., 1992, 13–40.

[2] B.A. Anderson, *A doubling construction for certain self–orthogonal 2–sequencings*, to appear.

[3] B.A. Anderson and P.A. Leonard, *A class of self–orthogonal 2–sequencings*, Des. Codes Crypt. **1** (1991), 149–181.

[4] F.E. Bennett and L. Wu, *On minimum matrix representation of closure operations*, Discr. Appl. Math. **26** (1990), 25–40.

[5] D.E. Bryant and A. Khodkar, *On orthogonal double covers of graphs*, manuscript, 1996.

[6] M.S. Chung and D.B. West, *The p–intersection number of a complete bipartite graph and orthogonal double coverings of a clique*, Combinatorica **14** (1994), 453–461.

[7] J. Demetrovics, Z. Füredi, and G.O.H. Katona, *Minimum matrix representation of closure operations*, Discr. Appl. Math. **11** (1985), 115–128.

[8] B. Ganter and H.–D.O.F. Gronau, *On two conjectures of Demetrovics, Füredi and Katona on partitions*, Discr. Math. **88** (1991), 149–155.

[9] B. Ganter, H.–D.O.F. Gronau, and R.C. Mullin, *On orthogonal double covers of K_n*, Ars Combinatoria **37** (1994), 209–221.

[10] A. Granville, H.–D.O.F. Gronau, and R.C. Mullin, *On a problem of Hering concerning orthogonal double covers of K_n*, Jour. Comb. Theory **A 72**(2) (1995), 345–350.

[11] H.–D.O.F. Gronau, personal communication, 1996.

[12] H.–D.O.F. Gronau, R.C. Mullin, and A. Rosa, *Orthogonal double covers of complete graphs by trees*, Graphs and Combinatorics, to appear.

[13] H.–D.O.F. Gronau, R.C. Mullin, and P.J. Schellenberg, *On orthogonal double covers and a conjecture of Chung and West*, Jour. Comb. Designs **3** (1995), 213–231.

[14] K. Heinrich and G. Nonay, *Path and cycle decompositions of complete multigraphs*, Ann. Discr. Math. **27** (1985), 275–286.

[15] F. Hering, *Balanced pairs*, Ann. Discr. Math. **20** (1984), 177–182.

[16] F. Hering, *Block designs with cyclic block structure*, Ann. Discr. Math. **6** (1980), 201–214.

[17] J.D. Horton and G. Nonay, *Self–orthogonal Hamiltonian path decompositions*, Discr. Math. **97** (1991), 251–264.

[18] U. Leck and V. Leck, *There is no ODC with all pages isomorphic to* $C_4 \dot\cup C_3 \dot\cup C_3 \dot\cup v$, Utilitas Mathematica **49** (1996), 185–189.

[19] U. Leck and V. Leck, *On orthogonal double covers by trees*, Jour. Comb. Designs, to appear.

[20] V. Leck, *Orthogonale Doppelüberdeckungen des K_n*, Diplomarbeit, Universität Rostock, 1996.

Universität Rostock, Fachbereich Mathematik, 18051 Rostock, Germany

E-mail address, Uwe Leck: `uleck@zeus.math.uni-rostock.de`

E-mail address, Volker Leck: `vleck@zeus.math.uni-rostock.de`