The index-λ-closures of the subsets of $\{3, \ldots, 10\}$ including 3

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Abstract

Gronau, Mullin and Pietsch determined the exact closure of index one of all subsets K of $\{3, \ldots, 10\}$ which include 3. We extend their results to obtain the exact closure of such K for all indices.

1 Introduction

Let K be a set of positive integers and λ a positive integer. A *pairwise balanced design (PBD)* of index λ with order v and block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set of cardinality v (the *point-set*) and $\mathcal{B}$ is a familiy of subsets of V (called *blocks*) which satisfy the properties:

1. if $B \in \mathcal{B}$, then $|B| \in K$
2. every pair of distinct elements of V occurs in exactly λ blocks of $\mathcal{B}$

We denote such a design by $PBD(v, K; \lambda)$ and call λ the *index of pairwise balance*. Note that the blocks are not necessarily distinct. A trivial example for this is a $PBD(v, \{v\}; \lambda)$, e. g. every block consists of all points, which clearly exists for all positive integers v and λ.

The notion of PBD–closure is due to Wilson [7]. Let K be a set of positive integers and let $B(K, \lambda) := \{v, \exists PBD(v, K; \lambda)\}$. Then $B(K, \lambda)$ is said to be the *index-λ-PBD–closure* of K. Consider $\mathcal{M}$ as the set of all subsets of $\{3, \ldots, 10\}$ which contain the integer 3. From [3] for each element $M \in \mathcal{M}$ the set $B(M, 1)$ is known. In this paper we determine $B(M, \lambda)$ for all positive integers λ. Note that $B(\{3\}, \lambda)$ is well-known [4].

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2 Closure properties

The following construction called *breaking up blocks* is easily established but turns out to be very powerful in constructing PBDs of higher index from those of lower ones.

Proposition 1 ([2]) *Let K and L be sets of positive integers, and let λ_1, λ_2 be positive integers. Suppose that there exists D, a $PBD(v, L; \lambda_1)$, and for each ℓ which occurs as a block-size of D there exists a $PBD(\ell, K; \lambda_2)$. Then there exists a $PBD(v, K; \lambda_1\lambda_2)$.*

Thus breaking up blocks means to replace blocks by PBDs on their point set. We do so with blocks of length $7, 9$ or 10 using $PBD(7, \{3\}; 1)$, $PBD(9, \{3\}; 1)$ (Steiner-triple systems) and $PBD(10, \{3, 4\}; 1)$ (the affine plane of order 3 with a new point added to each block of a fixed parallel class) and get $B(K, 1) = B(K \cup \{7\}, 1) = B(K \cup \{9\}, 1)$ if $3 \in K$ and $B(K, 1) = B(K \cup \{10\}, 1)$ if $\{3, 4\} \subseteq K$. If we break up every block B of a $PBD(v, K; \lambda_1)$ using the trivial design $PBD(|B|, \{|B|\}; \lambda_2)$ with a fixed positive integer λ_2, e. g. taking every block λ_2 times, we derive a $PBD(v, K; \lambda_1\lambda_2)$. So $B(K, \lambda_1) \subseteq B(K, \lambda_1\lambda_2)$ for every positive integer λ_2, in particular $B(K, 1) \subseteq B(K, \lambda_2)$. Therefore when determining $B(M, \lambda)$ for any λ and $3 \in M$ we need not consider subsets M containing 10 if $\{3, 4\} \subseteq M$ and subsets containing 7 or 9 at all.

Adding the block sets of a $PBD(v, K; \lambda_1)$ and a $PBD(v, K'; \lambda_2)$ clearly gives a $PBD(v, K \cup K'; \lambda_1 + \lambda_2)$. So $B(K, \lambda_1) \cap B(K', \lambda_2) \subseteq B(K \cup K', \lambda_1 + \lambda_2)$, in particular for $K' = K$ it holds $B(K, \lambda_1) \cap B(K, \lambda_2) \subseteq B(K, \lambda_1 + \lambda_2)$.

3 Necessary conditions

There are two well-known necessary conditions for the existence of a PBD with fixed parameters.

Let $D = (V, \mathcal{B})$ be a $PBD(v, K; \lambda)$ and let β_i count the number of blocks of length $k_i, i = 1, \ldots, n$. Then $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ is called *block type* of D. By double counting the point pairs in D we get:

Lemma 2

$$\sum_{i=1}^{n} \beta_i k_i (k_i - 1) = \lambda v (v - 1)$$

Now fix a point $x \in V$ and let $\gamma_i(x)$ count the number of blocks B of length k_i with $x \in B$, $i = 1, \ldots, n$. Then $\vec{\gamma}(x) = (\gamma_1(x), \ldots, \gamma_n(x))$ is the *point type* of x in D. Double counting the point pairs which include x gives:

Lemma 3

$$\sum_{i=1}^{n} \gamma_i (k_i - 1) = \lambda(v-1)$$

These lemmas lead to the following proposition.

Proposition 4 (necessary conditions) *Let* $K = \{k_1, \ldots, k_n\}$, $k^* := \gcd(k_i(k_i-1), i = 1, \ldots, n)$ *and* $k^{'} := \gcd(k_i - 1, i = 1, \ldots, n)$. *If there exists a* $PBD(v, \{k_1, \ldots, k_n\}; \lambda)$, *then*

$$\lambda v(v-1) \equiv 0 \pmod{k^*}, \tag{1}$$

$$\lambda(v-1) \equiv 0 \pmod{k^{'}}. \tag{2}$$

Wilson [8] proved these conditions to be asymptotically sufficient. Moreover, Hanani showed that they are sufficent for triple systems, e. g. when $K = \{3\}$ [4]. The necessary conditions for all 3–including subsets of $\{3, \ldots, 10\}$ are listed in appendix A.

4 Pre-structures and Hill climbing

Let K be a set of positive integers with $3 \in K$. The *pre-structure* $\mathcal{P}$ of a $PBD(v, K; \lambda)$ is the family of all blocks of this PBD with block sizes different from 3. The lemmas above give necessary conditions for such pre-structures to exist, particularly for the number of blocks modulo 6 and the number of point types modulo 2. We call a family of sets fulfilling these conditions a *candidate* for the pre-structure.

Hill climbing has turned out to be very efficient in constructing triple systems [1],[6]. Analogously to [3] we use it to complete a given pre-structure candidate with blocks of length 3. The advance is that our algorithm handles also higher indices.

Let $F = \{\{x, y\} : \exists P \in \mathcal{P}$ such that $\{x, y\} \in P\}$ denote the family of all pairs occuring in a pre-structure candidate $\mathcal{P}$ and let $n_{\mathcal{P}}(\{x, y\})$ be the number of ocurrences of a pair $\{x, y\}$ in F. Further let T denote the set of triples found so far and let $n_T(\{x, y\})$ count the number of occurences of $\{x, y\}$ in T. The algorithm now searches for a decomposition of the graph $G = \lambda K_v \backslash F$ into triangles.

Input $v, \lambda, n_{\mathcal{P}}$

step 0

$T := \emptyset$, $G = \lambda K_v \backslash F$

step 1

pick a random $p \in V$ with $d_G(p) > 0$
if $d_G(p) = 0$ for all p GOTO Output

step 2

pick randomly two edges $\{x,p\},\{y,p\} \in E(G)$ adjacent to p
if $n_{\mathcal{P}}(\{x,y\}) = \lambda$ GOTO step 1

step 3

(a) if $n_T(\{x,y\}) < \lambda - n_{\mathcal{P}}(\{x,y\})$
$T := T \cup \{x,y,p\}$, $E(G) := E(G) \backslash \{\{x,p\},\{y,p\},\{x,y\}\}$
GOTO step 1

(b) if $n_T(\{x,y\}) = \lambda - n_{\mathcal{P}}(\{x,y\})$
delete randomly one of the triangles containing $\{x,y\}$,e.g. $\{x,y,z\}$
$T := T \backslash \{x,y,z\} \cup \{x,y,p\}$
GOTO step 1

Output T

5 The closures

In this chapter, we show that the necessary conditions for the closures are sufficient up to the trivial and very few nontrivial exceptions. An integer $v \in \mathbb{N}$ is a *trivial exception* of $B(K,\lambda)$, if it fulfills the necessary conditions to be a member of $B(K,\lambda)$ but it doesn't fulfill the necessary conditions for $B(K',\lambda)$, where K' consists of all $k \in K$ with $k \leq v$.

Let K denote a subset of $\{3,\ldots,10\}$ which include 3. Moreover let $\mathbb{N}\ x_1,\ldots,x_n(y)$ denote the set of all nonnegative integers greater than 2 that are $x_1 \pmod y$ or $\ldots$ or $x_n \pmod y$, especially let $\mathbb{N}$ be the set of all nonnegative integers greater than 2. Besides the constructions we use two programs to decide wether an integer belongs to a closure or not, one uses the hill climbing approach and the other, `desy` by Pietsch [5], constructs all PBDs for the given parameters. Clearly only `desy` can show nonexistence. If $v < 17$ the constructed PBDs are normally found by `desy` otherwise by hill climbing. A list of the used pre-structures can be found in appendix D.

5.1 $\lambda = 1$

Gronau, Mullin and Pietsch [3] showed that the necessary conditions are almost sufficient. The exceptions can be found in appendix B.

5.2 $\lambda = 2$

The necessary conditions for the following closures are equal, and they are known to be already sufficient for $B(\{3\},2)$. So $B(\{3\},2) = B(\{3,4\},2) = B(\{3,6\},2) = B(\{3,10\},2) = B(\{3,4,6\},2) = B(\{3,6,10\},2) = \mathbb{N}\ 0,1(3)$.

From $B(\{3,4,5\},1)$ and the existence of $PBD(4,\{3\};2), PBD(6,\{3\};2)$ and $PBD(8,\{3,5\},2)$ we get that $B(\{3,5\},2) = \mathbb{N}$. It follows, that $B(K,2) = \mathbb{N}$ for all K with $\{3,5\} \subseteq K$.

Analogously $B(\{3,4,8\},1)$ gives that $B(\{3,8\},2) \supseteq \mathbb{N}\backslash\{5,11,14,17\}$. Since a PBD on v points can not have blocks of size larger than v and $5 \notin B(\{3\},2)$ it follows that $5 \notin B(\{3,8\},2)$, `desy` shows that this is true also for $11,14,17$, e.g. equality holds.

Clearly $B(\{3,8\},2) \subseteq B(\{3,4,8\},2)$. We found a $PBD(v,\{3,4,8\};2)$ for $v = 11,14,17$. Therefore $B(\{3,4,8\},2) = \mathbb{N}\backslash\{5\}$. In the same way we get $B(\{3,6,8\},2) = \mathbb{N}\backslash\{5,11,14\}$ and $B(\{3,8,10\},2) = \mathbb{N}\backslash\{5,11,14\}$ using `desy` to show nonexistence.

Because of $B(\{3,4,8\},2) \subseteq B(\{3,4,6,8\},2)$ we have that $B(\{3,4,6,8\},2) = \mathbb{N}\backslash\{5\}$. From $B(\{3,6,8\},2)$ we get $B(\{3,6,8,10\},2) \supseteq \mathbb{N}\backslash\{5,11,14\}$, `desy` proves equality.

5.3 $\lambda = 3$

The closure of $B(\{3\},3)$ is wellknown, and because the necessary conditions are the same it holds $B(\{3\},3) = B(\{3,5\},3) = \mathbb{N}\ 1(2)$.

Taking $B(\{3,4,5\},1)$ and $PBD(5,\{3\};3)$ gives that $B(\{3,4\},3) \subseteq \mathbb{N}\backslash\{6,8\}$. Solutions for 6 and 8 can be found with `desy`. So $B(K,3) = \mathbb{N}$ for all K with $\{3,4\} \in K$. From $B(\{3,5,6\},1)$ and $PBD(5,\{3\};3)$ we have that $B(\{3,6\},3) \subseteq \mathbb{N}\backslash\{4,8,10,12,14,20,22\}$. We found solutions for all exceptions but for 4. It follows that $B(K,3) = \mathbb{N}\backslash\{4\}$ for all K with $\{3,6\} \in K$ and $4 \notin K$. The same way we get $B(\{3,8\},3) = \mathbb{N}\backslash\{4,6,10\}$ and $B(\{3,10\},3) = \mathbb{N}\backslash\{4,6,8,12\}$, because we find that $12,14,16,18,20,26,28,30,34 \in B(\{3,8\},3)$ and $14,16,18,20,22,24,26,32,34,36,38,42,44 \in B(\{3,10\},3)$.

With the help of `desy` we find that $B(\{3,5,8\},3) = \mathbb{N}\backslash\{4,6,10\}$ and $B(\{3,5,10\},3) = \mathbb{N}\backslash\{4,6,8,12\}$.

Clearly $B(\{3,8,10\},3) = B(\{3,5,8,10\},3) = \mathbb{N}\backslash\{4,6\}$.

5.4 $\lambda = 4$

It holds $B(K,4) \supseteq B(K,2)$ for all K, and since the necessary conditions are equal, we only have to investigate the exceptions of the index 2 closures. It turns out that $11,14,17 \in B(K,4)$ for $\{3,8\} \in K$.

5.5 $\lambda = 5$

We get $B(\{3,4\},5) = \mathbb{N}\ 0,1(3)$ from $B(\{3,4\},2) \cap B(\{3,4\},3)$ and from the necessary conditions. Analogously $B(\{3,5\},5) = \mathbb{N}\ 1(2)$, $B(\{3,6\},5) = \mathbb{N}\ 0,1(3)\backslash\{4\}$. Furthermore we obtain $B(\{3,8\},5) \supseteq \mathbb{N}\backslash\{4,5,6,10,11,14,17\}$ and $B(\{3,10\},5) \supseteq \mathbb{N}\backslash\{4,6,12\}$. We find that $B(\{3,8\},5) = \mathbb{N}\backslash\{5,6\}$ and $B(\{3,10\},5) = \mathbb{N}\backslash\{4,6\}$.

The closure of the other subsets is the intersection of the closures of index 2 and 3, with the exceptions $10,11,14,17 \in B(K,5)$ for all K with $\{3,8\} \subseteq K$ and $12 \in B(K,5)$ when $\{3,10\} \subseteq K$.

5.6 $\lambda = 6$

Since $B(\{3\},6) = \mathbb{N}$ we trivially have that $B(K,6) = \mathbb{N}$ if $3 \in K$.

5.7 $\lambda \geq 7$

Since the necessary conditions for the closures are the same for $\lambda' = \lambda$ mod 6 and $B(K,\lambda) \subseteq B(K,\lambda+6)$ we only need to consider the exceptions of the necessary conditions for $\lambda = 1,\ldots,6$. If an exception is smaller than the largest block it is still an exception if we delete this block-size from K. Therefore we need not consider these values.

Clearly $B(K,7) \supseteq B(K,5) \cap B(K,2)$. This solves the case $\lambda = 7$ up to the exceptions $11,14,17$ if $\{3,8\} \subseteq K$. But then $11,14,17 \in B(K,3) \cap B(K,4) \subseteq B(K,7)$.

From $\lambda = 4$ we know that $11,14,17 \in B(K,4) \subseteq B(K,8)$ for $\{3,8\} \in K$. Finally $10 \in B(K,4) \cap B(K,5) \subseteq B(K,9)$ for $\{3,8\} \in K$ and $12 \in B(K,4) \cap B(K,5) \subseteq B(K,9)$ for $\{3,10\} \in K$.

It follows that for $\lambda \geq 4$ the necessary conditions are sufficient up to the trivial exceptions.

Conclusion

The exact index-λ-closures of all subsets of $\{3,\ldots,10\}$ which contain 3 have been determined for all λ. The results are summarized in appendix C.

A Necessary Conditions

$\mathbb{N}$	set of all nonnegative integers greater than 2
$\mathbb{N}\ x_1, \ldots, x_n(y)$	set of all integers which are $x_1, \ldots, x_n$ modulo y

Subset	λ mod 6					
	1	2	3	4	5	0
3	$\mathbb{N}\ 1,3(6)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 1(2)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 1,3(6)$	$\mathbb{N}$
$3,4$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	N	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$
$3,5$	$\mathbb{N}\ 1(2)$	$\mathbb{N}$	$\mathbb{N}\ 1(2)$	$\mathbb{N}$	$\mathbb{N}\ 1(2)$	$\mathbb{N}$
$3,6$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$
$3,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,10$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$
$3,4,5$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,4,6$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$
$3,4,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,6$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,6,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,6,10$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}\ 0,1(3)$	$\mathbb{N}$
$3,8,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,4,5,6$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,4,5,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,4,6,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,6,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,6,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,8,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,6,8,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,4,5,6,8$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$
$3,5,6,8,10$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$

B The Closures with Index 1

subset	exceptions
3	-
$3, 4$	6
$3, 5$	-
$3, 6$	$4, 10, 12, 22$
$3, 8$	$4, 5, 6, 10, 11, 12, 14, 16, 17, 18, 20, 23, 26, 28, 29, 30, 34, 35, 36, 38$
$3, 10$	$4, 6, 12, 16, 18, 22, 24, 34, 36, 42$
$3, 4, 5$	$6, 8$
$3, 4, 6$	-
$3, 4, 8$	$5, 6, 11, 14, 17$
$3, 5, 6$	$4, 8, 10, 12, 14, 20, 22$
$3, 5, 8$	$4, 6, 10, 12, 14, 16, 18, 20, 26, 28, 30, 34$
$3, 5, 10$	$4, 6, 8, 12, 14, 16, 18, 20, 22, 24, 26, 32, 34, 36, 38, 42, 44$
$3, 6, 8$	$4, 5, 10, 11, 12, 14, 17, 20, 23$
$3, 6, 10$	$4, 12, 22$
$3, 8, 10$	$4, 5, 6, 11, 12, 14, 16, 17, 18, 20, 23, 26, 29, 35, 36$
$3, 4, 5, 6$	8
$3, 4, 5, 8$	6
$3, 4, 6, 8$	$5, 11, 14, 17$
$3, 5, 6, 8$	$4, 10, 12, 14, 20$
$3, 5, 6, 10$	$4, 8, 12, 14, 20, 22$
$3, 5, 8, 10$	$4, 6, 12, 14, 16, 18, 20, 26$
$3, 6, 8, 10$	$4, 5, 11, 12, 14, 17, 20, 23$
$3, 4, 5, 6, 8$	-
$3, 5, 6, 8, 10$	$4, 12, 14, 20$

C Exceptions of the sufficiency of the necessary conditions

Subset	Exceptions for λ modulo 6, $\lambda \geq 2$					
	1	2	3	4	5	0
3	-	-	-	-	-	-
$3,4$	-	-	-	-	-	-
$3,5$	-	-	-	-	-	-
$3,6$	4	-	4	-	4	-
$3,8$	$4,5,6$	$5,11(2),14(2),17(2)$	$4,6,10(3)$	5	$4,5,6$	-
$3,10$	$4,6$	-	$4,6,8,12(3)$	-	$4,6$	-
$3,4,5$	-	-	-	-	-	-
$3,4,6$	-	-	-	-	-	-
$3,4,8$	5	5	-	5	5	-
$3,5,6$	4	-	4	-	4	-
$3,5,8$	$4,6$	-	$4,6,10(3)$	-	$4,6$	-
$3,5,10$	$4,6,8$	-	$4,6,8,12(3)$	-	$4,6,8$	-
$3,6,8$	$4,5$	$5,11(2),14(2)$	4	5	$4,5$	-
$3,6,10$	4	-	4	-	4	-
$3,8,10$	$4,5,6$	$5,11(2),14(2)$	$4,6$	5	$4,5,6$	-
$3,4,5,6$	-	-	-	-	-	-
$3,4,5,8$	-	-	-	-	-	-
$3,4,6,8$	5	5	-	5	5	-
$3,5,6,8$	4	-	4	-	4	-
$3,5,6,10$	4	-	4	-	4	-
$3,5,8,10$	$4,6$	-	$4,6$	-	$4,6$	-
$3,6,8,10$	$4,5$	$5,11(2),14(2)$	4	5	$4,5$	-
$3,4,5,6,8$	-	-	-	-	-	-
$3,5,6,8,10$	4	-	4	-	4	-

If we write $x(y)$, then x is not in the closure only for this special $\lambda = y$.

D The Pre-structures

D.1 Pre-structures for $\lambda = 2$

$PBD(17, \{3,4,8\}; 2)$

$V = \{0, \ldots, 16\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 ), (8 9 10 11 12 13 14 15)`
`(0 1 8 9), (2 3 10 11), (4 5 12 13), (6 7 14 15)`

$PBD(17, \{3,8,10\}; 2)$

$V = \{0, \ldots, 16\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 8 9)`
`(0 1 2 3 4 10 11 12), ( 5 6 7 8 9 10 11 12)`

D.2 Pre-structures for $\lambda = 3$

$PBD(18, \{3,8\}; 3)$

$V = \{0, \ldots, 17\}$
$\mathcal{B} =$ `( 0 1 2 3 4 5 6 7), (0 1 2 8 9 10 11 12),`
`(0 1 2 13 14 15 16 17)`

$PBD(20, \{3,8\}; 3)$

$V = \{0, \ldots, 19\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7), (0 1 8 9 10 11 12),`
`(0 1 2 13 14 15 16 17)`

$PBD(26, \{3,8\}; 3)$

$V = \{0, \ldots, 25\}$
$\mathcal{B} =$ `(0 1 3 4 6 8 9 10), (0 1 3 4 6 8 9 11),`
`(0 1 2 5 6 7 12 13), (0 2 4 5 7 9 14 15),`
`(0 2 3 5 7 8 16 17), (18 19 20 21 22 23 24 25)`

$PBD(28, \{3,8\}; 3)$

$V = \{0, \ldots, 27\}$
$\mathcal{B} =$ `(0 1 2 4 6 8 9 10), (0 1 3 4 6 8 11 12),`
`(0 1 3 5 6 7 13 14), (0 2 3 5 7 8 15 16),`
`(0 2 4 5 7 17 18 19), (20 21 22 23 24 25 26 27)`

$PBD(30, \{3,8\}; 3)$

$V = \{0, \ldots, 29\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7), (0 1 2 3 4 5 6 7),`
`(0 1 2 3 4 5 6 7), (0 8 9 10 11 12 13 14),`
`(0 15 16 17 18 19 20 21), (22 23 24 25 26 27 28 29)`

$PBD(34, \{3,8\}; 3)$

$V = \{0, \ldots, 33\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7), (0 1 2 3 4 5 6 8),`
`(0 1 2 3 4 5 6 9), (10 11 12 13 14 15 16 17),`
`(18 19 20 21 22 23 24 25), (26 27 28 29 30 31 32 33)`

$PBD(18, \{3,10\}; 3)$

$V = \{0, \ldots, 17\}$
$\mathcal{B} =$ `(0 1 2 4 6 7 9 11 12 14), (0 1 3 4 6 8 9 11 13 14),`
`(0 1 3 5 6 8 10 11 13 15), (0 2 3 5 7 8 10 12 13 16),`
`(0 2 4 5 7 9 10 12 14 17)`

$PBD(20, \{3,10\}; 3)$

$V = \{0, \ldots, 19\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 8 9), (10 11 12 13 14 15 16 17 18 19)`

$PBD(22, \{3,10\}; 3)$

$V = \{0, \ldots, 21\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 8 9), (0 1 2 3 10 11 12 13 14 15),`
`(0 1 2 3 16 17 18 19 20 21)`

$PBD(24, \{3,10\}; 3)$

$V = \{0, \ldots, 23\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 8 9), (0 1 2 10 11 12 13 14 15 16),`
`(0 1 2 17 18 19 20 21 22 23)`

$PBD(26, \{3,10\}; 3)$

$V = \{0, \ldots, 25\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7 8 9), (0 1 10 11 12 13 14 15 16 17),`
`(0 1 18 19 20 21 22 23 24 25)`

$PBD(32, \{3, 10\}; 3)$

$V = \{0, \ldots, 31\}$
$\mathcal{B} =$ (0 1 2 4 5 6 7 8 9 10), (0 1 3 11 12 13 14 15 16 17),
(0 2 3 18 19 20 21 22 23 24), (1 2 3 25 26 27 28 29 30 31)

$PBD(34, \{3, 10\}; 3)$

$V = \{0, \ldots, 33\}$
$\mathcal{B} =$ (0 1 2 3 4 5 6 7 8 9), (0 1 10 11 12 13 14 15 16 17),
(0 2 18 19 20 21 22 23 24 25), (1 2 26 27 28 29 30 31 32 33)

$PBD(36, \{3, 10\}; 3)$

$V = \{0, \ldots, 35\}$
$\mathcal{B} =$ (0 1 2 3 4 5 6 7 8 9), (0 1 10 11 12 13 14 15 16 17),
(0 1 18 19 20 21 22 23 24 25), (26 27 28 29 30 31 32 33 34 35)

$PBD(38, \{3, 10\}; 3)$

$V = \{0, \ldots, 37\}$
$\mathcal{B} =$ (0 1 2 3 4 5 6 7 8 9), (0 10 11 12 13 14 15 16 17 18),
(0 19 20 21 22 23 24 25 26 27),
(28 29 30 31 32 33 34 35 36 37)

$PBD(42, \{3, 10\}; 3)$

$V = \{0, \ldots, 41\}$
$\mathcal{B} =$ (0 1 2 3 4 5 6 7 8 9), (0 1 2 3 4 5 6 7 8 10),
(0 1 2 3 4 5 6 7 8 11), (12 13 14 15 16 17 18 19 20 21),
(22 23 24 25 26 27 28 29 30 31), (32 33 34 35 36 37 38 39 40 41) can also be derived from a $TD(3, 14)$ and a $PBD(14, \{3, 10\}; 3)$

$PBD(44, \{3, 10\}; 3)$

$V = \{0, \ldots, 43\}$
$\mathcal{B} =$ (0 1 2 3 4 5 6 7 8 9), (0 1 10 11 12 13 14 15 16 17),
(0 1 18 19 20 21 22 23 24 25), (0 26 27 28 29 30 31 32 33 34),
(0 35 36 37 38 39 40 41 42 43)

D.3 Pre-structures for $\lambda = 5$

$PBD(14, \{3,8\}; 5)$

$V = \{0, \ldots, 13\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7), (0 1 2 3 4 5 6 7),`
`(0 1 2 3 4 5 6 7), (0 1 2 3 4 8 9 10),`
`(0 1 2 3 4 11 12 13)`

$PBD(17, \{3,8\}, 5)$

$V = \{0, \ldots, 16\}$
$\mathcal{B} =$ `(0 1 2 3 4 5 6 7), (0 1 2 3 4 5 6 7)`

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