Bounded quasi-interpolatory polynomial operators

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Abstract

We construct bounded polynomial operators, similar to the classical de la Valleé Poussin operators in Fourier series, which preserve polynomials of a certain degree, but are defined in terms of the values of the function rather than its Fourier coefficients.

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1 Introduction

Let f be a continuous 2π-periodic function, and for integer $m \geq 1$, $s_m^*(f)$ be the m-th partial sum of its trigonometric Fourier series. It is well known (see e.g. [9]) that the de la Valleé Poussin operators

$$v_n^*(f) := \frac{1}{n} \sum_{k=n+1}^{2n} s_k^*(f), \qquad n = 1, 2, \ldots,$$

are linear operators with the following interesting properties, where for integer $n \geq 0$, $I\!H_n$ denotes the class of all trigonometric polynomials of order not exceeding n. For all integer $n \geq 1$, and continuous 2π-periodic functions f,

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$$v_n^*(f) \in I\!H_{2n-1}, \tag{1.1}$$

$$v_n^*(T) = T, \qquad T \in I\!H_n, \tag{1.2}$$

and

$$\max_{-\pi\le x\le\pi} |v_n^*(f,x)| \le c \max_{-\pi\le x\le\pi} |f(x)|. \tag{1.3}$$

These operators are naturally of great importance in the study of trigonometric polynomial approximation. Namely, it follows immediately from (1.2) that

$$\max_{-\pi\le x\le\pi} |f(x) - v_n^*(f,x)| \le cE_n(f),$$

where $E_n(f)$ denotes the best approximation in the sup-norm by trigonometric polynomials of degree at most n. Let us finally mention here a sharper result for strong approximation proved by Leindler (see [8] and the literature cited there)

$$\max_{-\pi\le x\le\pi} \left(\frac{1}{n}\sum_{k=n+1}^{2n} |f(x) - s_k^*(f,x)|\right) \le cE_n(f)\ . \tag{1.4}$$

Operators similar to these are also studied in the context of algebraic polynomial approximation, where, instead of the trigonometric Fourier series, one studies the Fourier series with respect to suitable orthogonal polynomials [2]. Such operators proved to be indispensable in the theory of weighted polynomial approximation (cf. [3], [4], [5], [6], [12]). We observe that the operators v_n^* are defined in terms of the Fourier coefficients of f, which in turn, involve the evaluation of integrals. In many applications (e.g. [13]), it is more desirable to have operators which have properties similar to (1.1), (1.2), and (1.3), but which are defined in terms of the values of the function f.

In the case of periodic functions, such quasi-interpolatory operators were defined by Bernstein [1] where also the boundedness result analogous to (1.3) is proved. In [18], Szabados proved similar results for certain operators based on the zeros of Chebyshev polynomials. In this paper, we generalize the results in two ways. First, we study operators based on the values of the function at the zeros of certain generalized Jacobi polynomials. Second, we study similar operators also in the case of certain Freud-type weight functions, supported on the whole real axis.

In Section 2, we give the preliminary definitions and estimates. These are applied to the case of generalized Jacobi weights in Section 3 and the case of Freud-type weight functions in Section 4.

2 Preliminaries

In this paper, for every real number $x \ge 0$, we denote the class of all algebraic polynomials of degree at most x by Π_x. This somewhat unusual convention will actually simplify our notations later when we need to discuss polynomials of degree not exceeding cn for some constant c and integer n. Let α be a positive Borel measure on $I\!R$ having finite moments; i.e.,

$$\int |t|^r \, d\alpha(t) < \infty, \qquad r = 0, 1, 2, \dots \ .$$

If α has at least N points of increase, then there exists a unique system of polynomials

$$p_n(d\alpha; x) := \gamma_n(d\alpha)x^n + \cdots, \qquad \gamma_n(d\alpha) > 0, \ n = 0, 1, \dots, N-1,$$

such that

$$\int p_m(d\alpha; t)p_k(d\alpha; t)d\alpha(t) = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

If f is Borel measurable function on $I\!R$, we write

$$a_k(d\alpha; f) := \int f(t)p_k(d\alpha; t)d\alpha(t), \qquad k = 0, 1, 2, \dots, N-1,$$

whenever these integrals are well defined. The partial sum of the Fourier-orthonormal expansion of f is then given by

$$s_m(d\alpha; f) := \sum_{k=0}^{m-1} a_k(d\alpha; f)p_k(d\alpha), \qquad m = 1, 2, \dots, N.$$

Using (2.1), we obtain the integral representation

$$s_m(d\alpha; f, x) = \int f(t)K_m(d\alpha; x, t)d\alpha(t), \qquad m = 1, 2, \dots, N,$$

where it is known [2] that the *Christoffel-Darboux kernel* K_m can be expressed as

$$\begin{aligned} K_m(d\alpha; x, t) &:= \sum_{k=0}^{m-1} p_k(d\alpha; x)p_k(d\alpha; t) \\ &= \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \, \frac{p_m(d\alpha; x)p_{m-1}(d\alpha; t) - p_m(d\alpha; t)p_{m-1}(d\alpha; x)}{x - t} \, . \end{aligned} \tag{2.2}$$

It is well known [2] that for each $n = 1, 2, \dots$, the polynomial $p_n(d\alpha)$ has n distinct zeros, all in the smallest interval $S(\alpha)$ containing the support of $d\alpha$. We denote these zeros by $x_{k,n}(d\alpha)$, with the ordering

$$x_{n,n}(d\alpha) < x_{n-1,n}(d\alpha) < \cdots < x_{1,n}(d\alpha) \, .$$

The *Cotes numbers* are defined by

$$\lambda_{k,n}(d\alpha) := \Big\{ K_n(d\alpha; x_{k,n}(d\alpha), x_{k,n}(d\alpha)) \Big\}^{-1}, \qquad k = 1, \dots, n, \ n = 1, 2, \dots, N.$$

One of the most important properties of these is the (Gaussian) *quadrature formula*:

$$\sum_{k=1}^{n} \lambda_{k,n}(d\alpha)P(x_{k,n}(d\alpha)) = \int P(t)d\alpha(t), \qquad P \in \Pi_{2n-1},\ n = 0, 1, \ldots, N. \tag{2.3}$$

In this paper, we sometimes find it convenient to express the left hand side of (2.3) as a Stieltjes integral. Thus, let α_n be the measure that associates the mass $\lambda_{k,n}(d\alpha)$ with $x_{k,n}(d\alpha)$, $k = 1, \ldots, n$. Then (2.3) can be written in the form

$$\int P(t)d\alpha_n(t) = \int P(t)d\alpha(t), \qquad P \in \Pi_{2n-1},\ n = 0, \ldots, N. \tag{2.4}$$

In the remainder of this section, we assume that α is a *mass distribution;* i.e., it is a positive Borel measure, all of whose moments are finite, and α has infinitely many points of increase. Thus, the orthonormal polynomials $p_k(d\alpha)$ are defined for all non-negative integers k.

We now proceed to define the operators which will be the discrete analogues of the de la Valleé Poussin means. For any integer $m \geq 1$, and Borel measurable function f defined on the smallest interval containing the support of α, we define the discrete Fourier coefficients of f by

$$\begin{aligned} a_{k,m}(d\alpha; f) &:= \sum_{j=1}^{m} \lambda_{j,m}(d\alpha)f(x_{j,m}(d\alpha))p_k(d\alpha; x_{j,m}(d\alpha)) \\ &= \int f(t)p_k(d\alpha; t)d\alpha_m(t), \qquad k = 0, 1, \ldots\ . \end{aligned}$$

The operators analogous to the de la Valleé Poussin operators are now defined by

$$\begin{aligned} \tau_{n,m}(d\alpha; f) &:= \sum_{k=0}^{n} a_{k,m}(d\alpha; f)p_k(d\alpha) \\ &+ \sum_{k=n+1}^{2n-1} \left(2 - \frac{k}{n}\right) a_{k,m}(d\alpha; f)p_k(d\alpha), \qquad n, m = 1, 2, \ldots\ . \end{aligned}$$

The following Theorem 2.1 lists some basic properties of the operators $\tau_{n,m}$. In the sequel, if f is a function defined on a Borel set $A \subseteq I\!R$, we write

$$\|f\|_{\infty,A} := \sup_{t \in A} |f(t)|.$$

During the proof of Theorem 2.1, we will point out that

$$\tau_{n,m}(d\alpha; f) = \frac{1}{n} \sum_{k=1}^{2n} s_k(d\alpha_m; f).$$

Motivated by (1.4), we define the sublinear operator

$$\tau^{\#}_{n,m}(d\alpha; f) := \frac{1}{n} \sum_{k=n+1}^{2n} |s_k(d\alpha_m; f)|. \tag{2.5}$$

Theorem 2.1 *Let α be a mass distribution and $n \geq 1$, $m \geq 2n$ be integers, and Z_m be the set of zeros $\{x_{k,m}(d\alpha)\}_{k=1}^{m}$. Then*

$$\tau_{n,m}(d\alpha; P) = P, \qquad P \in \Pi_n.$$

If $f : Z_m \to I\!R$, then $\tau_{n,m}(f) \in \Pi_{2n-1}$. Let $G : Z_m \to [0, \infty)$, $x \in S(\alpha)$, $I \subseteq S(\alpha)$ be an interval containing x, and J be an interval such that $f(x_{k,m}(d\alpha)) = 0$ if $x_{k,m}(d\alpha) \in J$. Then the following estimate holds

$$\begin{aligned} |\tau_{n,m}(d\alpha; f, x)| \leq \tau^{\#}_{n,m}(d\alpha; f, x) \leq \|fG\|_{\infty, Z_m \setminus J} \sqrt{K_{2n}(d\alpha; x, x)} \\ \times \left\{ \sqrt{\int_{I \setminus J} \frac{1}{G^2(t)} d\alpha_m(t)} + \frac{2\Gamma_{2n}(d\alpha)}{n} \sqrt{\int_{S(\alpha)\setminus(I \cup J)} \frac{1}{G^2(t)(x-t)^2} d\alpha_m(t)} \right\}, \end{aligned} \tag{2.6}$$

where

$$\Gamma_{2n}(d\alpha) := \max_{1 \leq j \leq 2n} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}.$$

Proof. Let $\tilde{p}_k(d\alpha) := \gamma_k^{-1}(d\alpha) p_k(d\alpha)$. From the orthogonality relations, we see that for some polynomial $P \in \Pi_{k-1}$, $k < m$,

$$\begin{aligned} \int \tilde{p}_k(d\alpha_m) \tilde{p}_k(d\alpha) d\alpha &= \gamma_k(d\alpha)^{-2} \int \gamma_k(d\alpha) \tilde{p}_k(d\alpha_m) p_k(d\alpha) d\alpha \\ &= \gamma_k(d\alpha)^{-2} \int \{p_k(d\alpha) + P\} p_k(d\alpha) d\alpha = \gamma_k(d\alpha)^{-2}. \end{aligned}$$

Similary,

$$\int \tilde{p}_k(d\alpha_m) \tilde{p}_k(d\alpha) d\alpha_m = \gamma_k(d\alpha_m)^{-2}.$$

In view of the fact that $m \geq 2n$, the quadrature formula (2.4) implies that

$$\gamma_k(d\alpha_m) = \gamma_k(d\alpha), \qquad k = 0, \ldots, 2n-1.$$

Using the quadrature formula again, we obtain for $k = 0, \ldots, 2n-1$

$$\begin{aligned} \int (\tilde{p}_k(d\alpha_m) - \tilde{p}_k(d\alpha))^2 d\alpha &= \int \tilde{p}_k(d\alpha_m)^2 d\alpha + \int \tilde{p}_k(d\alpha)^2 d\alpha \\ &\quad - 2 \int \tilde{p}_k(d\alpha_m) \tilde{p}_k(d\alpha) d\alpha \\ &= \int \tilde{p}_k(d\alpha_m)^2 d\alpha_m - \gamma_k^{-2}(d\alpha) = 0. \end{aligned}$$

We have thus shown that

$$p_k(d\alpha) = p_k(d\alpha_m), \qquad k = 0, \dots, 2n-1 \ .$$

Therefore,

$$a_{k,m}(d\alpha; f) = a_k(d\alpha_m; f) \ .$$

A simple computation then leads to

$$\tau_{n,m}(d\alpha; f) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(d\alpha_m; f) \ .$$

The operator $\tau_{n,m}$ is thus a discretization of the de la Valleé Poussin-type operator for the orthonormal polynomial expansions. The first inequality in (2.6) is now clear. Since $s_k(d\alpha_m; P) = P$ for every $P \in \Pi_n$, and $k = n+1, \dots, 2n$, we obtain that $\tau_{n,m}(d\alpha; P) = P$ for all $P \in \Pi_n$. Also, it is clear that for any function f defined on Z_m we have $\tau_{n,m}(d\alpha; f) \in \Pi_{2n-1}$.

The estimate (2.6) is obtained using an argument similar to the one in [3], [16]. Let I be a neighbourhood of x. Then we define

$$A_x f(t) := \begin{cases} f(t), & \text{if} \quad t \in I \ , \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_x f(t) := \begin{cases} 0, & \text{if} \quad t \in I \ , \\ \dfrac{f(t) - A_x f(t)}{x - t}, & \text{otherwise,} \end{cases}$$

to obtain the splitting

$$f(t) = A_x f(t) + (x - t) B_x f(t) \ .$$

For $n < k \leq 2n$ we have

$$\begin{aligned} |s_k(d\alpha_m; A_x f, x)|^2 &= \left| \int_{I \setminus J} f(t) K_k(d\alpha; x, t) \, d\alpha_m(t) \right|^2 \\ &\leq \left\{ \int |K_k(d\alpha; x, t)|^2 \, d\alpha_m(t) \right\} \left\{ \int_I |f(t)|^2 \, d\alpha_m(t) \right\} \\ &\leq K_k(d\alpha; x, x) \, \|f \cdot G\|^2_{\infty, Z_m \setminus J} \int_{I \setminus J} \frac{1}{G^2(t)} \, d\alpha_m(t) \ . \end{aligned}$$

Hence,

$$\tau^{\#}_{n,m}(d\alpha; A_x f, x) \leq \sqrt{K_{2n}(d\alpha_m; x, x)} \sqrt{\int_{I \setminus J} \frac{1}{G^2(t)} \, d\alpha_m(t)} \quad \|f \cdot G\|_{\infty, Z_m \setminus J} \ . \tag{2.7}$$

Next, applying the Christoffel-Darboux-formula (2.2), we write

$$
\begin{aligned}
& s_k(d\alpha_m; f - A_x f, x) = s_k(d\alpha_m; (x-\circ)B_x f, x) \\
= \;& \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} \int \frac{p_{k-1}(d\alpha;t)p_k(d\alpha;x) - p_{k-1}(d\alpha;x)p_k(d\alpha;t)}{x-t}(x-t)B_x f(t)\, d\alpha_m(t) \\
= \;& \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} \left(p_k(d\alpha;x)a_{k-1}(d\alpha_m, B_x f) - p_{k-1}(d\alpha;x)a_k(d\alpha_m, B_x f)\right) .
\end{aligned}
$$

Since

$$
|s_k(d\alpha_m; f - A_x f, x)| \le \Gamma_{2n}(d\alpha) \left(|p_k(d\alpha;x)a_{k-1}(d\alpha_m, B_x f)| + |p_{k-1}(d\alpha;x)a_k(d\alpha_m, B_x f)|\right) ,
$$

we get using Bessel's inequality that

$$
\begin{aligned}
& \tau^{\#}_{n,m}(d\alpha; f - A_x f, x) \\
= \;& \frac{1}{n} \sum_{k=n+1}^{2n} |s_k(d\alpha_m; f - A_x f, x)| \\
\le \;& \frac{2\Gamma_{2n}(d\alpha)}{n} \sqrt{K_{2n}(d\alpha;x,x)} \sqrt{\sum_{k=n}^{2n} |a_k(d\alpha_m; B_x f)|^2} \\
\le \;& \frac{2\Gamma_{2n}(d\alpha)}{n} \sqrt{K_{2n}(d\alpha;x,x)} \sqrt{\int |B_x f(t)|^2\, d\alpha_m(t)} \\
\le \;& \frac{2\Gamma_{2n}(d\alpha)}{n} \sqrt{K_{2n}(d\alpha;x,x)} \sqrt{\int_{Z_m \setminus (I \cup J)} \frac{1}{G^2(t)(x-t)^2}\, d\alpha_m(t)}\; \|f \cdot G\|_{\infty, Z_m \setminus (I \cup J)} .
\end{aligned}
\tag{2.8}
$$

Since

$$
\tau^{\#}_{n,m}(d\alpha; f, x) \le \tau^{\#}_{n,m}(d\alpha; A_x f, x) + \tau^{\#}_{n,m}(d\alpha; f - A_x f, x),
$$

the second estimate in (2.6) is proved in view of (2.7) and (2.8). □

3 Generalized Jacobi weights

A *generalized Jacobi weight* is a function of the form

$$
w(x) := \begin{cases} \prod_{k=1}^{\rho} (x - \xi_k)^{\beta_k}, & x \in [-1,1], \\ 0, & \text{otherwise,} \end{cases}
\tag{3.1}
$$

where $\rho \ge 1$ is an integer, $-1 =: \xi_\rho < \cdots < \xi_1 := 1$, and $\beta_k > -1$ for $k = 1, \ldots, \rho$. The class of generalized Jacobi weights will denoted by GJ; orthonormal polynomials with respect to a weight in GJ will be called GJ polynomials. These polynomials are studied extensively by Nevai in [17]. Following [17], if $w \in GJ$ is of the form (3.1), and $m \geq 1$ is an integer, we write for $x \in [-1,1]$,

$$\overline{w}_m(x) := \left(\sqrt{1-x} + \frac{1}{m}\right)^{2\beta_1+1} \prod_{k=2}^{\rho-1} \left(|x-\xi_k| + \frac{1}{m}\right)^{\beta_k} \left(\sqrt{1+x} + \frac{1}{m}\right)^{2\beta_\rho+1}.$$

If w is the Legendre weight, $\beta_k = 0$, $k = 1, \ldots, \rho$, then it is easy to see that

$$\overline{w}_m(x) \sim \Delta_m(x) := \sqrt{1-x^2} + \frac{1}{m}.$$

In the sequel, we adopt the following convention regarding constants. The letters $c, c_1, \ldots$ will denote positive constants depending only on the weight function and other fixed parameters of the problem, but their value may be different in different occurences, even within the same formula. The notation $A \sim B$ denotes the fact that $cA \leq B \leq c_1 A$.

Theorem 3.1 *Let $w \in GJ$, $d\alpha(x) := w(x)dx$, and*

$$G(x) := (1-x^2)^{\gamma/2}\sqrt{w(x)}, \qquad -1 < \gamma < 1.$$

If $L \geq 2$, $n \geq 1$, $2n \leq m \leq Ln$ are integers, and f is a function defined on Z_m, then

$$||\tau_{n,m}(d\alpha; f)\Delta_{\sqrt{m}}^{\gamma}\sqrt{\overline{w}_m}||_{\infty,[-1,1]} \leq ||\tau_{n,m}^{\#}(d\alpha; f)\Delta_{\sqrt{m}}^{\gamma}\sqrt{\overline{w}_m}||_{\infty,[-1,1]} \leq c||fG||_{\infty,Z_m}. \tag{3.2}$$

PROOF. We recall from [17] a few facts about the GJ polynomials. Writing

$$x_{j,m}(d\alpha) =: \cos\theta_{j,m} =: x_{j,m}, \qquad j = 1, \ldots, m,$$

we have

$$0 < \theta_{j,m} < \pi, \qquad \theta_{j,m} - \theta_{j-1,m} \sim \frac{1}{m}, \qquad j = 1, \ldots, m, \tag{3.3}$$

where $\theta_{0,m} := 0$, $\theta_{m+1,m} := \pi$. Further, $\Gamma_n(d\alpha) \sim 1$,

$$K_{2n}(d\alpha; x, x) \sim \frac{n}{\overline{w}_n(x)}, \qquad x \in [-1,1], \tag{3.4}$$

and

$$\lambda_{j,m} \sim \frac{1}{m} w(x_{j,m})\sqrt{1-x_{j,m}^2}, \qquad j = 1, \ldots, m. \tag{3.5}$$

Theorem 2.1 now implies that for $x \in [-1,1]$, and any interval I containing x,

$$\tau_{n,m}^{\#}(d\alpha; f, x) \leq c||fG||_{\infty,Z_m}\sqrt{\frac{n}{\overline{w}_n(x)}}$$
$$\times\left\{\sqrt{\int_I G^{-2}(t)d\alpha_m(t)} + \frac{1}{n}\sqrt{\int_{[-1,1]\setminus I}(G(t)(x-t))^{-2}d\alpha_m(t)}\right\}.$$

Using (3.5), we obtain for $x \in [-1,1]$

$$\sqrt{\overline{w}_n(x)}\tau_{n,m}^{\#}(d\alpha; f, x) \leq c||fG||_{\infty,Z_m}\left(\sqrt{S_1} + \frac{1}{m}\sqrt{S_2}\right), \tag{3.6}$$

where, with $\delta := 1 - 2\gamma$,

$$S_1 := \sum_{x_{j,m}\in I} (1 - x_{j,m}^2)^{\delta/2},$$
$$S_2 := \sum_{x_{j,m}\in[-1,1]\setminus I} \frac{(1 - x_{j,m}^2)^{\delta/2}}{(x - x_{j,m})^2}. \tag{3.7}$$

We will estimate S_1 and S_2 for $x \geq 0$, the case when $x < 0$ is similar. In the remainder of the proof $x =: \cos\phi$ is a fixed number, with $0 \leq \phi \leq \pi/2$.

Case 1. $(0 \leq \phi \leq 2/\sqrt{m})$

We write $I^\circ := [0, \phi + 2/\sqrt{m}]$, and

$$I := \{x = \cos\theta \ : \ \theta \in I^\circ\}. \tag{3.8}$$

In view of (3.3) and the fact that $\delta > -1$,

$$\frac{1}{m} S_1 = \frac{1}{m} \sum_{\theta_{j,m}\in I^\circ} \sin^\delta \theta_{j,m} \leq \frac{c}{m} \sum_{\theta_{j,m}\in I^\circ} \theta_{j,m}^\delta \leq c \int_0^{\phi+3/\sqrt{m}} t^\delta dt \leq c(\phi + \frac{1}{\sqrt{m}})^{\delta+1}. \tag{3.9}$$

Since $\phi + \dfrac{1}{\sqrt{m}} \sim \dfrac{1}{\sqrt{m}}$, we deduce that

$$S_1 \leq c(\phi + \frac{1}{\sqrt{m}})^{\delta-1} \leq c(\sin\phi + \frac{1}{\sqrt{m}})^{-2\gamma} = c\Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.10}$$

It is easy to check that $\sin\theta \sim \theta$ if $\theta \in [0, 3\pi/4]$, and hence, that

$$S_2 \leq c \sum_{\theta_{j,m}\in[0,\pi]\setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2}. \tag{3.11}$$

If $\theta_{j,m} \in [\pi/2, \pi]$, then $\theta_{j,m}^2 - \phi^2 \geq c$. Using (3.3), and the fact that $\delta > -1$, we obtain

$$\sum_{\theta_{j,m}\in[\pi/2,\pi]} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} \leq c \sum_{\theta_{j,m}\in[\pi/2,\pi]} \sin^\delta(\pi - \theta_{j,m}) \leq cm \int_0^{3\pi/4} t^\delta dt \leq cm.$$

Since $\delta < 3$ and $\phi + \dfrac{1}{\sqrt{m}} \sim 1/\sqrt{m}$,

$$m \leq m^2 m^{-(\delta-1)/2} \leq cm^2\Big(\phi + \frac{1}{\sqrt{m}}\Big)^{\delta-1}.$$

Hence,

$$\sum_{\theta_{j,m}\in[\pi/2,\pi]} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} \leq cm^2\Big(\phi + \frac{1}{\sqrt{m}}\Big)^{\delta-1}. \tag{3.12}$$

If $\phi+\dfrac{2}{\sqrt{m}} \le \theta_{j,m} \le \pi/2$, then $\theta_{j,m}^2-\phi^2 \ge c\theta_{j,m}^2$ and $\sin\theta_{j,m}\sim\theta_{j,m}$. Hence, using (3.3) and the fact that $\delta<3$, we get

$$\begin{aligned}\sum_{\theta_{j,m}\in[0,\pi/2]\setminus I^\circ} \frac{\sin^\delta\theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} &\le cm\Big\{\frac{1}{m}\sum_{\theta_{j,m}\in[0,\pi/2]\setminus I^\circ}\theta_{j,m}^{\delta-4}\Big\}\\ &\le cm\int_{\phi+\frac{1}{\sqrt{m}}}^{\infty} t^{\delta-4}dt \le cm\Big(\phi+\frac{1}{\sqrt{m}}\Big)^{\delta-3} \le cm^2\Big(\phi+\frac{1}{\sqrt{m}}\Big)^{\delta-1}.\end{aligned}$$

Along with (3.11) and (3.12), this gives

$$S_2 \le cm^2\Big(\phi+\frac{1}{\sqrt{m}}\Big)^{\delta-1} \le cm^2\Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.13}$$

The estimates (3.6), (3.10), (3.13) yield that when $0\le\phi\le 2/\sqrt{m}$, we have

$$\Delta_{\sqrt{m}}^{-\gamma}(x)\sqrt{\overline{w}_n(x)}\tau_{n,m}^{\#}(d\alpha;f,x) \le c\|fG\|_{\infty,Z_m}. \tag{3.14}$$

Case 2. $(2/\sqrt{m}<\phi\le\pi/2)$

In this case, we take $I^\circ := [\phi-1/(m\phi), \phi+1/(m\phi)]$, and $I := \{\cos\theta \ : \ \theta\in I^\circ\}$. The following estimates will be used in the remainder of this proof often, sometimes without an explicit reference:

$$\frac14\Big(\phi+\frac{1}{\sqrt{m}}\Big) \le \frac12\Big(\phi+\frac{1}{m\phi}\Big) \le \phi-\frac{1}{m\phi} \le \phi \le \phi+\frac{1}{m\phi} \le \phi+\frac{1}{\sqrt{m}}. \tag{3.15}$$

Also, in view of (3.3), the number of $\theta_{j,m}$'s in I° is at most $c/\phi \le c\Big(\phi+\dfrac{1}{\sqrt{m}}\Big)^{-1}$. Hence,

$$S_1 \le c\sum_{\theta_{j,m}\in I^\circ}\theta_{j,m}^\delta \le c\Big(\phi+\frac{1}{\sqrt{m}}\Big)^{\delta-1} \le c\Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.16}$$

As in Case 1, we deduce easily that

$$\sum_{\theta_{j,m}\in[3\pi/4,\pi]}\frac{\sin^\delta\theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} \le cm.$$

If $1\le\delta<3$ then

$$\frac{1}{m} \le m^{-(\delta-1)/2} \le c\phi^{\delta-1},$$

and if $-1<\delta<1$ then it is clear that $1/m \le c\phi^{\delta-1}$. Thus, in either case, $m \le m^2\Big(\phi+\dfrac{1}{\sqrt{m}}\Big)^{\delta-1}$, and we get

$$\sum_{\theta_{j,m}\in[3\pi/4,\pi]}\frac{\sin^\delta\theta_{j,m}}{(\theta_{j,m}^2-\phi^2)^2} \le cm^2\Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.17}$$

Next, we write

$$I_{2,1} := [0, \frac{1}{2}(\phi - \frac{1}{m\phi})], \qquad I_{2,2} := [\frac{1}{2}(\phi - \frac{1}{m\phi}), \phi - \frac{1}{m\phi}],$$
$$I_{2,3} := [\phi + \frac{1}{m\phi}, 2(\phi + \frac{1}{m\phi})], \qquad I_{2,4} := [2(\phi + \frac{1}{m\phi}), 3\pi/4],$$
$$S_{2,j} := \sum_{\theta_{j,m}\in I_{2,j}} \frac{\theta_{j,m}^\delta}{(\theta_{j,m}^2 - \phi^2)^2}, \qquad j = 1,2,3,4.$$

Then

$$\sum_{\theta_{j,m}\in[0,3\pi/4]\setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} \le c\sum_{j=1}^4 S_{2,j}. \tag{3.18}$$

We estimate $S_{2,3}$ and $S_{2,4}$; the estimates for $S_{2,2}$ and $S_{2,1}$ are similar. Using (3.3),

$$\begin{aligned} S_{2,3} &\le \frac{c}{\phi^2}\left(\phi + \frac{1}{m\phi}\right)^\delta \sum_{\theta_{j,m}\ge\phi+1/(m\phi)} \frac{1}{(\theta_{j,m}-\phi)^2} \\ &\le c\phi^{\delta-2} m\int_{\phi+1/(m\phi)} (t-\phi)^{-2}dt \le cm^2\phi^{\delta-1} \le cm^2\left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta-1}. \end{aligned} \tag{3.19}$$

If $\theta \in I_{2,4}$, then $\theta - \phi \ge \theta/2$. Using (3.3) and the fact that $\delta < 3$, we get

$$\begin{aligned} S_{2,4} &\le c\sum_{\theta_{j,m}\in I_{2,4}} \theta_{j,m}^{\delta-4} \le cm\int_{\phi+1/(m\phi)} t^{\delta-4}dt \\ &\le cm\left(\phi + \frac{1}{m\phi}\right)^{\delta-3} \le cm^2\left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta-1}. \end{aligned} \tag{3.20}$$

From (3.18), (3.19), (3.20), and similar estimates for $S_{2,1}$ and $S_{2,2}$, we obtain

$$\sum_{\theta_{j,m}\in[0,3\pi/4]\setminus I^\circ} \frac{\sin^\delta \theta_{j,m}}{(\theta_{j,m}^2 - \phi^2)^2} \le cm^2\left(\phi + \frac{1}{\sqrt{m}}\right)^{\delta-1} \le cm^2\Delta_{\sqrt{m}}^{-2\gamma}(x).$$

Along with (3.17), this yields

$$S_2 \le cm^2\Delta_{\sqrt{m}}^{-2\gamma}(x). \tag{3.21}$$

From (3.6), (3.16), and (3.21), we conclude that

$$\Delta_{\sqrt{m}}^{-\gamma}(x)\sqrt{\overline{w}_n(x)}\tau_{n,m}^{\#}(d\alpha; f, x) \le c||fG||_{\infty,Z_m}. \tag{3.22}$$

The estimates (3.14), (3.22), and analogous estimates for $x \in [-1,0)$ yield (3.2). $\square$

We end this section by observing the "continuous analogue" of Theroem 3.1. It is probably not new, but we are unable to locate a precise reference. The proof of the following theorem is similar to that of Theorem 3.1, but simpler.

Theorem 3.2 *Let $w \in GJ$, $d\alpha(x) := w(x)dx$, and G, L, m, n be as in Theorem 3.1. Let $f : [-1,1] \to I\!R$ be a measurable function such that fG is essentially bounded on $[-1,1]$. Then for $x \in [-1,1]$,*

$$\Delta_{\sqrt{m}}^{\gamma}(x)\sqrt{\overline{w}_m(x)}\frac{1}{n}\sum_{k=n+1}^{2n} |s_k(d\alpha; f, x)| \le c||fG||_{\infty,[-1,1]}. \tag{3.23}$$

4 Freud-type weight functions

Let $w : I\!R \to (0,\infty)$, and $Q := \log(1/w)$. The function w is called a Freud-type weight function if each of the following conditions is satisfied. The function Q is an even, convex function on $I\!R$, Q is twice continuously differentiable on $(0,\infty)$ and there are constants c_1 and c_2 such that

$$0 < c_1 \le \frac{xQ''(x)}{Q'(x)} \le c_2 < \infty, \qquad 0 < x < \infty. \tag{4.1}$$

The most commonly discussed examples include $\exp(-|x|^\alpha)$, $\alpha > 1$. In the remainder of this section, w will denote a fixed Freud-type weight function, $\{p_k\}$ will denote the sequence of polynomials orthonormal on $I\!R$ with respect to the measure $w^2(x)dx$. From all notations, we will omit the mention of this measure; thus $x_{k,n}$ will be the k-th zero of p_n, etc.

Associated with the weight function w are two sets of numbers: The Freud-number q_x is the least positive solution of the equation

$$q_x Q'(q_x) = x, \qquad x > 0.$$

The number a_x is the solution of the equation

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x t Q'(a_x t)}{\sqrt{1-t^2}}dt\ .$$

It is not difficult to see that $a_x \sim q_x \sim q_{2x}$, $x > 0$. One of the most important properties of a_x is the following: For every integer $n \ge 1$ and $P \in \Pi_n$, (cf. [15], [12]),

$$\max_{x\in \boldsymbol{R}} |P(x)w(x)| = \max_{|x|\le a_n} |P(x)w(x)|,$$

and, if $0 < p < \infty$, N is the least integer not exceeding $n + 2/p$, then

$$\int_{\boldsymbol{R}} |P(x)w(x)|^p dx \le 2\int_{|x|\le a_N} |P(x)w(x)|^p dx. \tag{4.2}$$

Our main theorem in this section is the following.

Theorem 4.1 *Let w be a Freud-type weight function, and $Q := \log(1/w)$ satisfy the following Lipschitz condition:*

$$|Q'(a_y\cos\theta) - Q'(a_y\cos\phi)| \le c\frac{y}{a_y}|\theta-\phi|^\lambda, \qquad y > 0,$$

where c and λ are positive constants independent of y. If $f : Z_m \to I\!R$, and $L, \delta > 0$, then for each integer $n \ge 1$ and $(2+\delta)n \le m \le Ln$,

$$\|\tau_{n,m}(f)w\|_{\infty,\boldsymbol{R}} \le \|\tau^{\#}_{n,m}(f)w\|_{\infty,\boldsymbol{R}} \le c\|wf\|_{\infty,Z_m}\ , \tag{4.3}$$

where $Z_m = \{x_{k,m}\}_{k=1}^m$, and c is a positive constant depending only on w, L, and δ.

In order to prove this theorem, we need the following lemma, where we summarize some of the important, relevant facts regarding Freud polynomials.

Lemma 4.1 *Let $n \geq 1$ be an integer.*
(a) *For $x \in I\!R$,*

$$K_n(x,x) \leq c\frac{n}{q_n}w^{-2}(x)\ . \tag{4.4}$$

(b) *We have*

$$\Gamma_n \sim q_n\ . \tag{4.5}$$

(c) *For any $\eta > 0$ and $|x| \leq (1-\eta)a_n$,*

$$K_n(x,x) \geq c(\eta)\frac{n}{q_n}w^{-2}(x). \tag{4.6}$$

(d) *For any $\eta > 0$ and $x_{k,n}, x_{k-1,n}, x_{k+1,n}$ in $[-(1-\eta)a_n, (1-\eta)a_n]$,*

$$x_{k-1,n} - x_{k,n} \sim x_{k,n} - x_{k+1,n} \sim \frac{q_n}{n}.$$

(e) *If $|x-t| \leq cq_n/n$, $|x|, |t| \leq c_1q_n$, then $w(x) \sim w(t)$.*
(f) *Let $b \in I\!R$ and $0 \leq p \leq 2$. Then*

$$\begin{aligned} c\int_0^{c_1q_n} w^{2-p}(x)(1+x^2)^b dx &\leq \sum_{j=1}^{n} \lambda_{j,n}w^{-p}(x_{j,n})(1+x_{j,n}^2)^b \\ &\leq c_2\int_0^{c_3q_n} w^{2-p}(x)(1+x^2)^b dx\ . \end{aligned} \tag{4.7}$$

Proof. Parts (a) and (b) are proved in [6]. Using (4.1), it is not difficult to verify (cf. [6]) that $Q'(Aq_n) \sim n/q_n$ for any $A > 0$. Part (e) is then a simple application of the mean value theorem. Part (f) was proved by Knopmacher and Lubinsky [7] under slightly different conditions on the weight function. As stated, the result is proved in [12] (Theorem 8.2.7) using their ideas. Parts (c) and (d) were proved by Lubinsky and Saff [10] under somewhat stronger conditions on the weight function. We are unable to find a proof of these facts under the stated conditions. Therefore, we provide a brief proof, utilizing the ideas in [20], [12]. It is proved in [12] (Lemma 10.3.3(b)) that there exists a sequence of real, nonnegative polynomials $R_k^* \in \Pi_k$, such that

$$w(a_kx)R_k^*(x)\begin{cases} \leq c, & \text{if } x \in I\!R, \\ \geq c_1(1-x^2+n^{-2/3}), & \text{if } |x| \leq 1. \end{cases}$$

Writing $x := a_ny/a_k$, we obtain

$$w(a_ny)R_k^*(a_ny/a_k)\begin{cases} \leq c, & \text{if } x \in I\!R, \\ \geq c_1(1-(a_ny/a_k)^2+n^{-2/3}), & \text{if } |y| \leq a_k/a_n. \end{cases}$$

In particular, for any $\eta > 0$, there exists a sequence of real, nonnegative polynomials $R_k := R_{k,\eta}$ such that

$$w(a_n y)R_n(y)\begin{cases} \le c, & \text{if } y \in I\!R, \\ \ge 1, & \text{if } |y| \le (1-\eta/2)a_k/a_n. \end{cases} \tag{4.8}$$

Next, we recall [11], [12] (Lemma 8.3.2) that for any $A > 0$ and $k < n < Ak$,

$$c_1\Big(\frac{n}{k} - 1\Big) \le \frac{a_n}{a_k} - 1 \le c_2\Big(\frac{n}{k} - 1\Big).$$

Hence, for any $\eta > 0$, we may choose an $\epsilon > 0$ such that with k being the integer part of $(1+\epsilon)^{-1}n$,

$$(1-\eta)a_n \le (1 - \frac{\eta}{2})a_k \le (1 - \frac{\eta}{2})a_n. \tag{4.9}$$

Now, let $\eta > 0$ and $|x| \le (1-\eta)a_n$ be fixed, and $\xi := x/a_n$. We choose the integer k so as to satisfy (4.9). Using (4.2), and a well known extremal property of the Christoffel functions $K_n(x,x)$, (cf. [2]), we obtain by making obvious substitutions,

$$\begin{aligned} K_n(x,x)^{-1} &= \inf_{P\in\Pi_{n-1}} P(x)^{-2}\int_{I\!R} |P(t)w(t)|^2 dt \\ &\le 2\inf_{P\in\Pi_{n-1}} P(x)^{-2}\int_{|t|\le a_n} |P(t)w(t)|^2 dt \\ &= 2a_n \inf_{P\in\Pi_{n-1}} P(\xi)^{-2}\int_{|u|\le 1} |P(u)w(a_n u)|^2 du \\ &\le 2a_n \inf_{P\in\Pi_{n-k-1}} [P(\xi)R(\xi)]^{-2}\int_{|u|\le 1} |P(u)R(u)w(a_n u)|^2 du\ . \end{aligned}$$

Since $|x| \le (1-\eta)a_n$, the choice of k shows that $|\xi| \le (1-\eta/2)a_k/a_n$. Then (4.8) implies that

$$\begin{aligned} K_n(x,x)^{-1} &\le c a_n w^2(a_n\xi) \inf_{P\in\Pi_{n-k-1}} P(\xi)^{-2}\int_{|u|\le 1} |P(u)|^2 du \\ &= c a_n w^2(x) \inf_{P\in\Pi_{n-k-1}} P(\xi)^{-2}\int_{|u|\le 1} |P(u)|^2 du\ . \end{aligned}$$

The estimate (4.6) now follows from well known estimates for the Christoffel functions for Legendre weights (cf. (3.4)) and the fact that $k \sim n$. This proves part (c) of the lemma. Part (d) follows by using ideas as in [4], [5], [12]. $\square$

We are now in a position to prove Theorem 4.1.

PROOF OF THEOREM 4.1. In this proof, let $x \in [-a_{2n}, a_{2n}]$ be a fixed number, and ν be an integer such that $x \in [x_{\nu+1,m}, x_{\nu,m})$. The constants in this proof will generally depend upon δ and L. Necessarily, there exists $\eta > 0$ such that $|x| \le (1-\eta)a_m$. We use the estimate (2.6) with $I = [x - q_m/m, x + q_m/m]$, with J equal to the empty interval, and $G = w$. The set $S(\alpha)$ in this case is $I\!R$. In view of (4.4) and (4.5), we obtain the estimate

$$
\begin{aligned}
|w(x)\tau_{n,m}(f,x)| \;&\le\; c\|fw\|_{\infty,Z_m} \\
&\times \left\{ \sqrt{\frac{n}{q_n}\int\limits_I \frac{1}{w^2(t)}\,d\alpha_m(t)} + \sqrt{\frac{q_n}{n}\int\limits_{\boldsymbol{R}\setminus I} \frac{1}{w^2(t)(x-t)^2}\,d\alpha_m(t)} \right\}. \qquad (4.10)
\end{aligned}
$$

The number of points $x_{j,m}$ in I is bounded from above, independently of n and m. For each such $x_{j,m} \in I$, (4.6) (with m in place of n and a different value of η) shows that

$$
\lambda_{j,m} w^{-2}(x_{j,m}) \le c\frac{q_m}{m}\ .
$$

Hence,

$$
\int_I w^{-2}(t)d\alpha_m(t) \le c\frac{q_m}{m} \le c\frac{q_n}{n}. \tag{4.11}
$$

Let

$$
I_1 := [-(1-\frac{\eta}{2})a_m, (1+\frac{\eta}{2})a_m] \setminus I
$$

and

$$
I_2 := (-\infty, -(1-\frac{\eta}{2})a_m) \cup ((1-\frac{\eta}{2})a_m, \infty).
$$

In view of (4.7) with $b=0$ and $p=2$,

$$
\sum_{x_{j,m}\in I_2} \frac{w^{-2}(x_{j,m})\lambda_{j,m}}{(x-x_{j,m})^2} \le c a_m^{-2} \sum_{j=1}^{n} w^{-2}(x_{j,m})\lambda_{j,m} \le c q_m^{-1}. \tag{4.12}
$$

If $x_{j,m} \in I_1$, then

$$
|x - x_{j,m}| \sim |x - x_{j-1,m}| \sim |x - x_{j+1,m}|
$$

and

$$
w^{-2}(x_{j,m})\lambda_{j,m} \sim \frac{q_m}{m} \sim x_{j,m} - x_{j+1,m}\ .
$$

Therefore, taking into account the fact that $m \sim n$, we obtain

$$
\begin{aligned}
\sum_{x_{j,m}\in I_1} \frac{w^{-2}(x_{j,m})\lambda_{j,m}}{(x-x_{j,m})^2} \;&\le\; \sum_{x_{j,m}\in I_1} \frac{x_{j,m}-x_{j+1,m}}{(x-x_{j,m})^2} \\
&\le\; \int_{\boldsymbol{R}\setminus I} \frac{dt}{(x-t)^2} \le c\frac{m}{q_m} \le c\frac{n}{q_n}\ . \qquad (4.13)
\end{aligned}
$$

Substituting the estimates (4.13), (4.12) and (4.11) into (4.10), we arrive at (4.3). □

We end this section by observing the following amusing inequality, obtained by using Theorem 4.1 with a polynomial of degree n in place of f, and observing that $\tau_{n,m}(P) = P$ for all $P \in \Pi_n$.

Corollary 4.1 *Let w, L, δ, n, m, Z_m be as in Theorem 4.1. Then for any $P \in \Pi_n$,*

$$c_1 \|wP\|_{\infty,\mathbb{R}} \le \|wP\|_{\infty,Z_m} \le \|wP\|_{\infty,\mathbb{R}}. \tag{4.14}$$

In [14], we have shown how estimates such as (4.14) together with the uniform boundedness of the de la Vallée Poussin means of the Freud polynomial expansions lead to estimates similar to (4.14) with an L^p norm instead of the supremum norm. In turn, it is well known that such estimates are important in the study of interpolatory processes at the zeros of Freud polynomials.

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