# PROBABILITIES AND LARGE QUANTILES OF NONCENTRAL GENERALIZED CHI-SQUARE DISTRIBUTIONS

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**ABSTRACT. Exact values of probability integrals for noncentral generalized chi-square distributions are numerically evaluated based upon new geometric representation formulas for these distributions. Using standard numerical methods exact quantiles can be calculated then. Explicit quantile approximation formulas and algorithms are deduced from an asymptotic expansion for related probabilities of large deviations. Several numerical studies compare the present results with results of other authors.**

*Key words:* spherical samples, geometric representation formulas, generalized chi-square distributions, exact distributions, exact quantiles, large quantiles, approximations, asymptotic expansions, large deviation quantile approximation

# 1 Introduction

The well known family of noncentral chi-square distributions has been considered by many authors. Several methods have been developed for approximating probability integrals and quantiles of these distributions. We mention here the methods of reduction to Bessel or Incomplete Gamma functions, asymptotic normal approximations, normal approximations after suitable transformations, series representations using central distributions, Edgeworth expansions, Cornish-Fisher expansions and continued fractions. For more details we refer to the results in Fisher (1928), Patnaik (1954), Abdel-Aty (1954), Sankaran (1959, 1963), Haynam et al. (1982), Farebrother (1987), Ashour and Abdel-Samad (1990) and Wang and Kennedy (1994).
The family of noncentral generalized chi-square distributions has been studied first in Cacoullos and Koutras (1984) and then in Hsu (1990). The noncentral $g$-generalized chi-square distribution function with $k \geq 2$ degrees of freedom (d.f.) and noncentrality parameter (n.c.p.) $\delta^2 \geq 0$ can be defined as

$$CQ(k, \delta^2; g)(x) = P(\|X + \mu^*\|^2 < x), \ x \in \mathbb{R}$$

for arbitrary $\mu^* \in \mathbb{R}^k$ satisfying $\|\mu^*\|^2 = \delta^2$. The random vector $X$ follows a $k$-dimensional spherically symmetric distribution having the density

$$p(x; g) = C(k; g)\, g(\|x\|^2), x \in \mathbb{R}^k$$

with the density generating function $g$ satisfying

$$0 < I_{k,g} < \infty, \tag{1}$$

where

$$I_{k,g} = \int_0^\infty r^{k-1} g(r^2)\, dr. \tag{2}$$

The norming constant $C(k, g)$ is

$$C(k; g) = (\omega_k I_{k,g})^{-1}$$

with

$$\omega_k = 2\pi^{k/2}/\Gamma(k/2)$$

denoting the surface area of the unit sphere $S_k(1)$ in $\mathbb{R}^k$.
Spherically symmetric distributions, or more general elliptically contoured distributions have been studied by many authors beginning with Schoenberg (1938) and Kelker (1970). Anderson and Fang (1982) studied central distributions of quadratic forms for elliptically contoured random vectors. For the general theory of elliptically contoured distributions and their applications to statistics we refer to the monographs of Watson (1983), Fisher et al. (1987), Johnson (1987), Fang, Kotz and Ng (1990), Fang and Anderson (1990), Fang and Zhang (1990) as well as Gupta and Varga (1993). A geometric approach to this class of distributions has been developed in Richter (1991) and Richter (1995).

The aim of the present paper is to demonstrate new methods for evaluating probability integrals and of approximating large quantiles of noncentral generalized chi-square distributions. The basic idea behind approximating probability integrals is to use new geometric measure representation formulas. Such formulas will be derived in Section 2 and exploited in Section 3. The basic idea behind approximating large quantiles of a probability distribution is to exploit suitable theorems on probabilities of large deviations in a suitable way. Namely, one can use asymptotic expansions for large deviations to derive asymptotic approximations for large quantiles of probability distributions. This general method has been developed recently in Richter (in preparation). It will be used in Sections 4 and 5 for deriving asymptotic quantile formulas and algorithms for the concrete distributions considered in this paper. A suitable asymptotic expansion for large deviations of noncentral $g$-generalized chi-square distributions has been proved recently in Richter and Schumacher (in preparation).
Let $\chi_q^2 = \chi_q^2(k, \delta^2; g)$ denote the $100q$-percentage point of the noncentral $g$-generalized chi-square distribution with $k$ d.f. and n.c.p. $\delta^2$, i.e. the solution of the quantile equation

$$CQ(k, \delta^2; g)\,(\chi_q^2) = q,\; q \in (0,1).$$

Further, let $\chi_{q,N}^2$ be a corresponding solution of the $N$-th order approximative quantile equation

$$CQ_N(k, \delta^2; g)\,(\chi_{q,N}^2) = q,$$

where $1 - CQ_N(k, \delta^2; g)\,(\chi^2)$ denotes a finite sum including $N+1$ terms of an asymptotic expansion for the large deviation probabilities

$$1 - CQ(k, \delta^2; g)\,(c^2) \text{ as } c \to \infty.$$

It is known from the large deviation theory that the relative approximation error

$$\frac{|1 - CQ(k, \delta^2; g)\,(\chi_{1-\alpha,N}^2) - \alpha|}{\alpha} = r(\alpha) = r(\alpha; k, \delta^2, g, N)$$

tends to zero when $\alpha$ itself approaches zero. This means that asymptotic quantile approximations become better the larger the quantiles itselfs are.
Typical practical applications of this asymptotic theory, however, do not concern the asymptotic quantile behaviour but fixed quantile quantities. A suitable method for controling the number of approximating terms in $1 - CQ_N$ is then to minimize the relative error $r(\alpha; k, \delta^2, g, N)$ with respect to $N$. For doing this one needs to know the distribution function $CQ(k, \delta^2; g)(x), x \in \mathbb{R}$ at least in a neighbourhood of $x = \chi_{1-\alpha}^2$.
We shall prove in Section 2 a new representation formula for the distribution function $CQ(k, \delta^2; g)(x), x \in \mathbb{R}$ for arbitrary $k \geq 2$, $\delta \geq 0$ and $g$ satisfying assumption (1). This formula will be the starting point as well for our numerical studies in Section 3 concerning the evaluation of the noncentral $g$-generalized chi-square distribution functions for several concrete spherical sample distributions as for controling the relative approximation errors of our quantile approximations in Sections 4 and 5. Our geometric representation formula for the noncentral $g$-generalized chi-square distribution function in Section 2 is new both for the Gaussian case, i.e. for the case when the density generating function is

$$g_G(r) = e^{-\frac{r}{2}},\; r > 0$$

and for the general non Gaussian case. Our geometric representation formula for the noncentral $g$-generalized chi-square distribution functions is new in the central case, i.e. for $\delta = 0$, too. We shall make in Section 3 numerical studies of comparisons between our and other representations, respectively approximation formulas. As a result, it will be seen from our studies in Section 3 that the geometric representation formula from Section 2 is comparable with the best of other methods and suitable to be used for controling the relative approximation errors in Sections 4 and 5.

# 2 Geometric representation formulas for noncentral $g$-generalized chi-square distribution functions

Let $\Phi(A; g),\ A \in \mathfrak{B}^k$ denote the probability measure corresponding to the density function $p(x; g),\ x \in \mathbb{R}^k$. Then

$$CQ(k, \delta^2; g)(c^2) = \Phi(A(c); g),\ c > 0$$

where

$$A(c) = \{y\ \in \mathbb{R}^k : \parallel y + \mu^* \parallel^2 < c^2\},\ c > 0.$$

The geometric measure representation formula for elliptically contoured distributions in Richter (1991) says that

$$\Phi(A(c); g) = \frac{1}{I_{k,g}} \int\limits_0^\infty \mathcal{F}(A(c), \nu)\nu^{k-1} g(\nu^2)\, d\nu \tag{3}$$

where $I_{k,g}$ is assumed to satisfy condition (1) and the intersection-percentage function $\mathcal{F}(A, \nu)$, $\nu > 0$ is defined as

$$\mathcal{F}(A, \nu) = U_k(\nu^{-1} A \cap S_k(1)),\ \nu > 0$$

with $U_k$ being the uniform probability distribution on the unit sphere $S_k(1)$. Evaluating $\mathcal{F}(A(c), \nu),\ \nu > 0$, we obtain the main result of this section. We shall make use of the notations

$$\alpha^*(\nu) = \arctan \left( \frac{4\delta^2\nu^2}{(\nu^2 + \delta^2 - c^2)^2} - 1 \right)^{\frac{1}{2}} \text{ for } \nu \geq 0,$$

and

$$f(\nu) = \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{1}{2}} \Gamma(\frac{k-1}{2})} \int\limits_0^{\alpha^*(\nu)} (\sin\alpha)^{k-2} d\alpha.$$

**Theorem 1**
*If assumption* (1) *is satisfied then*

$$CQ(k, \delta^2; g)(c^2) = I_{k,g}^{-1} \int\limits_0^\infty \mathcal{F}(A(c), \nu)\nu^{k-1} g(\nu^2)\, d\nu,\ c > 0 \tag{4}$$

*with*

$$\mathcal{F}(A(c),\nu) = I_{(\delta-c,\delta+c)}(\nu) f(\nu)$$

*if* $c \le \delta$ *and*

$$\mathcal{F}(A(c),\nu) = \begin{cases} 1 & : 0 \le \nu \le c-\delta, \\ 1-f(\nu) & : c-\delta < \nu \le (c^2-\delta^2)^{1/2}, \\ f(\nu) & : (c^2-\delta^2)^{1/2} < \nu < c+\delta, \\ 0 & : c+\delta \le \nu \end{cases}$$

*if* $c > \delta$.

**Remark 1:** Using a recurrence relation in Zeitler (1996) the integral in the function $f(\nu)$ can be simplified for $k \ge 4$ :

$$f(\nu) = \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{k-1}{2})} \left( \left[ -\frac{(\sin\alpha)^{k-3}\cos\alpha}{k-2} \right]_0^{\alpha^*(\nu)} + \frac{k-3}{k-2} \int\limits_0^{\alpha^*(\nu)} (\sin\alpha)^{k-4} d\alpha \right).$$

Note that $\mathcal{F}(A(c),\nu) = 0$ for $\nu > \delta + c$ so that the integration in (4) is restricted to the finite intervall $(0, c+\delta)$.

*Proof of Theorem 1:* Without loss of generality, we put $\mu^* = -\mu \mathbb{1}_k$, $\mathbb{1}_k^T = (1,\ldots,1) \in \mathbb{R}^k$, $\mu > 0$. Define the half spaces

$$H_k(e,r) = \{x \in \mathbb{R}^k : \; \Pi_e x = \lambda e, \; \lambda > r\}, \; r > 0$$

where $\Pi_e$ denotes the orthogonal projection onto $e$, $e \in \mathbb{R}^k$. Let the $k$-dimensional sphere with radius $r$ and centre $0 \in \mathbb{R}^k$ be

$$S_k(r) = rS_k(1), \; r > 0.$$

*Case 1:* $c \le \delta$. Following the notation in Richter (1995) for $\delta - c < \nu < \delta + c$ we define the so called direction type function

$$e_{A(c)}(\nu) = -\mathbb{1}_k$$

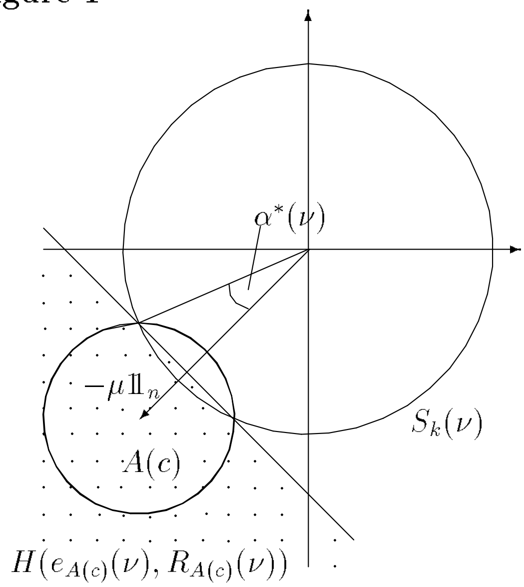
and the so called distance type function

$$R_{A(c)}(\nu) = \frac{|\nu^2 + \delta^2 - c^2|}{2\delta}.$$

Then, see Figure 1,

$$A(c) \cap S_k(\nu) = H(e_{A(c)}(\nu), R_{A(c)}(\nu)) \cap S_k(\nu). \tag{5}$$

**Figure 1**




It has been proved in Richter (1995) that from (5) it follows that

$$\mathcal{F}(A(c), \nu) = \frac{\omega_{k-1}}{\omega_k} \int\limits_0^{\alpha^*(\nu)} (\sin \alpha)^{k-2} \, d\alpha,$$

where

$$\begin{aligned} \alpha^*(\nu) &= \arctan \left( \left( \frac{\nu}{R_{A(c)}(\nu)} \right)^2 - 1 \right)^{1/2} \\ &= \arctan \left( \frac{4\delta^2 \nu^2}{(\nu^2 + \delta^2 - c^2)^2} - 1 \right)^{1/2}. \end{aligned} \tag{6}$$

Further, we have for the remaining values of $\nu$

$$\mathcal{F}(A(c), \nu) = 0 \text{ if } \{\nu < \delta - c\} \text{ or } \{\nu > \delta + c\}.$$

*Case 2:* $c > \delta$. We consider the complement of the set $A(c)$, $A(c)^C$, for $c - \delta < \nu \leq (c^2 - \delta^2)^{1/2}$ and the set $A(c)$ for $(c^2 - \delta^2)^{1/2} < \nu < c + \delta$. We define the direction type functions

$$e_{A(c)^C}(\nu) = 1\!\!1_k \text{ and } e_{A(c)}(\nu) = -1\!\!1_k$$

and the distance type functions

$$R_{A(c)^C}(\nu) = R_{A(c)}(\nu) = \frac{|\nu^2 + \delta^2 - c^2|}{2\delta}.$$

Then holds

$$A(c)^C \cap S_k(\nu) = H(e_{A(c)^C}(\nu), R_{A(c)^C}(\nu)) \cap S_k(\nu)$$

and

$$A(c) \cap S_k(\nu) = H(e_{A(c)}(\nu), R_{A(c)}(\nu)) \cap S_k(\nu),$$

in the respective intervals for $\nu$. Analogously to Case 1, we have

$$\mathcal{F}(A(c), \nu) = 1 - \mathcal{F}(A(c)^C, \nu) = 1 - \frac{\omega_{k-1}}{\omega_k} \int\limits_0^{\alpha^*(\nu)} (\sin \alpha)^{k-2} \, d\alpha$$

for $c - \delta < \nu \leq (c^2 - \delta^2)^{1/2}$ and

$$\mathcal{F}(A(c), \nu) = \frac{\omega_{k-1}}{\omega_k} \int_0^{\alpha^*(\nu)} (\sin \alpha)^{k-2} \, d\alpha$$

for $(c^2 - \delta^2)^{1/2} < \nu < c + \delta$ with $\alpha^*(\nu)$ from (6).
Further we have for the remaining values of $\nu$

$$\mathcal{F}(A(c), \nu) = \begin{cases} 1 & : 0 \leq \nu \leq c - \delta, \\ 0 & : \nu \geq c + \delta. \end{cases}$$

The assertion of Theorem 1 follows now from (3). $\square$

**Remark 2:** We have essentially proved that, with suitable definitions of the directional type and the distance type functions for the above mentioned "remaining values of $\nu$", the Borel sets $A(c)$ satisfy the conditions ($\mathfrak{A}1$) and ($\mathfrak{A}2$) in Richter (1995) and belong therefore to the therein defined system of Borel sets $\mathfrak{A}(dir, dist)$.

**Corollary 1**
*If the $k$-dimensional sample distribution has a Kotz-type density generator*

$$g_K(r) = r^{M-1} e^{-\beta r^\gamma}, \;\; r > 0$$

*for certain constants $\beta > 0$, $\gamma > 0$, $2M + k > 2$ then*

$$CQ(k, \delta^2; g_K)(c^2) = \frac{2\gamma\beta^{\frac{2M+k-2}{2\gamma}}}{\Gamma\left(\frac{2M+k-2}{2\gamma}\right)} \int\limits_0^\infty \mathcal{F}(A(c), \nu) \nu^{2M+k-3} e^{-\beta\nu^{2\gamma}} \, d\nu.$$

*If the $k$-dimensional sample distribution has a Pearson-VII-type density generator*

$$g_P(r) = (1 + r/m)^{-M}, \;\; r > 0$$

*for certain constants $M > k/2, m > 0$ then*

$$CQ(k, \delta^2; g_P)(c^2) = \frac{2\Gamma(M)}{m^{k/2}\Gamma(M - \frac{k}{2})\Gamma(\frac{k}{2})} \int\limits_0^\infty \mathcal{F}(A(c), \nu) \frac{\nu^{k-1}}{(1 + \frac{\nu^2}{m})^M} \, d\nu$$

*where $\mathcal{F}(A(c), \nu)$, $\nu > 0$ is both times precisely the same function as in Theorem 1.*

**Remark 3:** The norming constants which correspond to the density generating functions $g_G, g_K$ and $g_P$ are

$$\begin{aligned} C_G(k,g) &= \frac{2^{1-k/2}}{\Gamma(k/2)}, \\ C_K(k,g) &= \frac{\gamma\beta^{\frac{2M+k-2}{2\gamma}}\Gamma(k/2)}{\pi^{k/2}\Gamma\left(\frac{2M+k-2}{2\gamma}\right)} \end{aligned}$$

and

$$C_P(k,g) = \frac{\Gamma(M)}{(\pi m)^{k/2}\Gamma(M-k/2)},$$

respectively.

*Examples:*

1. The Kotz-type density generator $g_K$ with $M = 1, \beta = 0.5, \gamma = 1$ generates the Standard Gaussian law.

2. The Pearson-VII-type density generators $g_P$ with $M = (k+m)/2$ and $M = (k+1)/2$ and $m = 1$ generate the $k$-dimensional Student distribution and the $k$-dimensional Cauchy distribution, respectively.

**Remark 4:** Results concerning the behaviour of statistics which were well studied for the Gaussian sample distribution can be interpreted as robustness type results as far as, e.g., error probabilities do not change rapidly when the underlying sample distribution changes within the class of elliptically contoured distributions. On the other side, such results can be interpreted as sensitivity type results as far as one is interested in, e.g., how strong error probabilities for common statistical decisions change when the density generating function of the sample distribution changes in a well defined way within the class of admissible density generators. As the examples show, both heavy tails and light tails are allowed for the sample distribution to occure.

# 3 Tables of probability integrals for central and noncentral $g$-generalized chi-square distributions

Various kinds of algorithms for producing tables of the usual noncentral chi-square distribution functions are available from the literature. No related tables and algorithms are known to us, however, for the general case of a $g$-generalized noncentral chi-square distribution function when $g$ does not coincide with the density generating function $g_G$ of the Gaussian sample distribution. Based upon the geometric representation formula in Theorem 1, we are in a position to present now a new unified approach to establishing probability integrals for all noncentral $g$-generalized chi-square distributions with arbitrary density generating function $g$ satisfying assumption (1).

Our algorithm is mainly devided into two steps. The first one realizes the evaluation of the intersection-percentage function $\mathcal{F}(A(c), \nu)$, $\nu > 0$ for arbitrary $c > 0$. To this end, the representation formula for $\mathcal{F}$ from Theorem 1 has been implemented in a Borland Pascal program using the recursion relation for the function $f(\nu)$, $\nu > 0$, stated in Remark 1.
In the second step of our algorithm we use a standard numerical integration method for evaluating the integrals from the geometric representation formula for $CQ(k, \delta^2; g)(c^2)$ in Theorem 1. To be more specific, the integration is performed by Simpson's rule with 100000 steps.
Computations for realizing our algorithm have been done on a Hewlett Packard Pentium computer.
Our numerical results following from Theorem 1 for the "ordinary" case $g = g_G$ will be compared in Table 1 with the four-digit exact and the Incomplete Gamma function approximative results in Patnaik (1949), see columns Pat.-ex. and Pat.-appr., respectively, with the Cornish-Fisher type approximation results in Abdel-Aty (1954), column Abd.-Aty, and with the approximation results using infinite series of Poissonian weighted central chi-square distributions in Ashour and Abdel-Samad (1990), Ash. & Abd.-Sam. Note that the upper 5% significance points for $n = 1(1)7$ and $\delta = 0(0.2)5.0$ first published by Fisher (1928).

**Table 1** $CQ(k, \delta^2; g_G)(c^2)$-values

| $k$ | $\delta^2$ | $c^2$ | Pat.-ex. | Pat.-appr. | Abd.-Aty | Ash.&Abd.-Sam. | Theorem 1 |
|---|---|---|---|---|---|---|---|
| 4 | 4 | 1.765 | 0.0500 | 0.0399 | - | 0.04999937329 | 0.04999937471 |
| | 4 | 10.000 | 0.7118 | 0.7191 | 0.7123 | 0.7117928156 | 0.71179281648 |
| | 4 | 17.309 | 0.9500 | 0.9492 | - | 0.9499957033 | 0.94999570938 |
| | 4 | 24.000 | 0.9925 | 0.9913 | 0.9925 | 0.9924603701 | 0.99246037447 |
| | 10 | 10.000 | 0.3148 | 0.3178 | - | 0.3148206466 | 0.31482065003 |
| 7 | 1 | 4.000 | 0.1628 | 0.1621 | - | 0.1628330056 | 0.16283300701 |
| | 1 | 16.004 | 0.9500 | 0.9499 | - | 0.9500015423 | 0.95000154258 |
| | 16 | 10.257 | 0.0500 | 0.0430 | - | 0.04999417662 | 0.04999418181 |
| | 16 | 24.000 | 0.5898 | 0.5947 | 0.5894 | - | 0.58633683948 |
| | 16 | 38.970 | 0.9500 | 0.9482 | 0.9500 | 0.9499992082 | 0.94999921449 |
| 12 | 6 | 24.000 | 0.8174 | 0.8187 | - | 0.8173526185 | 0.81735262555 |
| | 18 | 24.000 | 0.2901 | 0.2936 | - | - | 0.29004949596 |
| 16 | 8 | 30.000 | 0.7880 | 0.7895 | - | 0.788001461 | 0.78800147228 |
| | 8 | 40.000 | 0.9632 | 0.9626 | 0.9632 | 0.9632254713 | 0.96322547499 |
| | 32 | 30.000 | 0.0609 | 0.0590 | - | - | 0.06284204877 |
| | 32 | 60.000 | 0.8316 | 0.8329 | - | 0.8315634526 | 0.83156347739 |
| 24 | 24 | 36.000 | 0.1567 | 0.1556 | - | 0.1567110344 | 0.15671106200 |
| | 24 | 48.000 | 0.5296 | 0.5333 | - | 0.5296283918 | 0.52962840920 |
| | 24 | 72.000 | 0.9667 | 0.9656 | - | 0.9666953909 | 0.96669542296 |

For easy comparison, the parameters $k$, $\delta^2$ and $c^2$ in Table 1 are choosen as in Patnaik (1949) and in the other papers of our comparison study. Let us remark, that at least 6 digits in the columns Ash. & Abd.-Sam. and Theorem 1 coincide and that the first four digits of these columns (after rounding) coincide with those of the column Pat.-ex. Exact results with more then four digits can also be obtained in the single case $g = g_G$ with the method of continued fractions as in the recent paper of Wang and Kennedy (1994).

**Table 2** Comparison with Wang and Kennedy (1994)

| $k$ | $\delta^2$ | $c^2$ | Wang & Kennedy | | Theorem 1 |
|---|---|---|---|---|---|
| 5 | 1 | 9.23636 | 0.8272818751175547 | 0.8272818751175549 | 0.827281875117556416 |
| 11 | 21 | 24.72497 | 0.2539481822183125 | 0.2539481822183127 | 0.253948182218312699 |
| 31 | 6 | 44.98534 | 0.81251987850649685 | 0.81251987850649695 | 0.812519878506495555 |
| 51 | 1 | 38.56038 | 0.08519497361859116 | 0.08519497361859120 | 0.085194973618591443 |
| 100 | 16 | 82.35814 | 0.01184348822747822 | 0.01184348822747826 | 0.011843488227478244 |
| 300 | 16 | 331.78852 | 0.73559567103067085 | 0.73559567103067095 | 0.735595671030671229 |
| 500 | 21 | 459.92612 | 0.02797023600800058 | 0.02797023600800062 | 0.027970236008000547 |

Note that the results in Table 2 following from Theorem 1 coincide at least in 13 digits with the midpoint-approximations of Wang and Kennedy. In two cases the results of Theorem 1 belong to the intervals derived by Wang and Kennedy.
Let us turn now to the non Gaussian case, i.e. to the case $g \neq g_G$. The first step of our algorithm is not influenced in any way by the special choise of the density generating function $g$ because in its first step our algorithm exploits only the pure geometric properties of the underlying spherical sampling distribution. Table 3 contains some typical values of the intersection-percentage function $\mathcal{F}(A(c),\nu)$ which have been used for calculating the values of the column Theorem 1 in Table 1.

**Table 3** Intersection percentage function $\mathcal{F}(A(c),\nu)$

| $\nu$ | $k=7$, $\delta^2=16$ $c^2=10.257$ | $k=4$, $\delta^2=4$ $c^2=24$ |
|---|---|---|
| 0.00 | 0.000000000 | 1.000000000 |
| 0.80 | 0.000000000 | 1.000000000 |
| 1.00 | 0.004295470 | 1.000000000 |
| 1.40 | 0.029695285 | 1.000000000 |
| 1.80 | 0.049180979 | 1.000000000 |
| 2.00 | 0.054545474 | 1.000000000 |
| 2.40 | 0.058260564 | 1.000000000 |
| 2.80 | 0.055497901 | 1.000000000 |
| 3.00 | 0.052578892 | 0.985743021 |
| 3.40 | 0.044979398 | 0.868003984 |
| 3.80 | 0.036295444 | 0.727566204 |
| 4.00 | 0.031902881 | 0.657481179 |
| 4.40 | 0.023519620 | 0.523144707 |
| 4.80 | 0.016131808 | 0.399624626 |
| 5.00 | 0.012928522 | 0.342518821 |
| 5.40 | 0.007629064 | 0.238351701 |
| 5.80 | 0.003847733 | 0.148559576 |
| 6.00 | 0.002505124 | 0.109551019 |
| 6.40 | 0.000792158 | 0.045034599 |
| 6.80 | 0.000106013 | 0.003958628 |
| 7.00 | 0.000013900 | 0.000000000 |
| 7.20 | 0.000000000 | 0.000000000 |

Notice that exactly the same values of the intersection-percentage function $\mathcal{F}$ as in Table 3 are to be used for evaluating probability integrals of noncentral $g$-generalized chi-square distributions in all cases when the geometric representation formula from Theorem 1 applies, i.e. when the sample distribution has a density generating function $g$ satisfying assumption (1).

In the second step of our algorithm we integrate the weighted intersection-percentage function $\mathcal{F}$ in the sense of Theorem 1. The answer to the question whether the underlying sample distribution has light or heavy tails becomes interesting to a certain extent in this second step of our algorithm. While the first step reflects a certain invariance or robustness property of our method with respect to changes of the sample distribution within a certain subclass of elliptically contoured distributions, the second step takes into account a certain necessarily existing sensitivity of the method with respect to these changes. Table 4 deals with a comparison of results influenced by heavy and light tails of the underlying sampling distribution, respectively. Column $g_K$ summarizes probability integrals $CQ(k, \delta^2; g_K)(c^2)$ for the case that the sampling distribution has the considerably light tail density generating function of Kotz-type from Corollary 1 with parameters $M = 1$, $\gamma = 2$ and $\beta = 1$, i.e.

$$g_K(r) = e^{-r^2}, r > 0.$$

Column $g_P$ summarizes probability integrals $CQ(k, \delta^2; g_P)$ for the case that the underlying sampling distribution has the considerably heavy tails Pearson-VII-type with parameters $M = (k+1)/2$ and $m = 1$, i.e.

$$g_P(r) = \frac{1}{\left(1 + \frac{r}{m}\right)^M}, r > 0.$$

The last column in Table 4 coincides with that of Table 1.

**Table 4** Robustness-sensitivity study

| $k$ | $\delta^2$ | $c^2$ | $g_K$ | $g_P$ | $g_G$ |
|---|---|---|---|---|---|
| 4 | 4 | 1.765 | 0.034458 | 0.025625 | 0.04999937471 |
| | 4 | 10.000 | 0.993725 | 0.452918 | 0.71179281648 |
| | 4 | 17.309 | 1.000000 | 0.605000 | 0.94999570938 |
| | 4 | 24.000 | 1.000000 | 0.672473 | 0.99246037447 |
| | 10 | 10.000 | 1.000000 | 0.208772 | 0.31482065003 |
| 7 | 1 | 4.000 | 0.658953 | 0.178667 | 0.16283300701 |
| | 1 | 16.004 | 0.691367 | 0.515594 | 0.95000154258 |
| | 16 | 10.257 | 0.008145 | 0.024662 | 0.04999418181 |
| | 16 | 24.000 | 0.673476 | 0.379660 | 0.58633683948 |
| | 16 | 38.970 | 0.691367 | 0.587635 | 0.94999921449 |
| 12 | 6 | 24.000 | 1.000000 | 0.428322 | 0.81735262555 |
| | 18 | 24.000 | 0.907495 | 0.214391 | 0.29004949596 |
| 16 | 8 | 30.000 | 1.000000 | 0.405108 | 0.78800147228 |
| | 8 | 40.000 | 1.000000 | 0.488109 | 0.96322547499 |
| | 32 | 30.000 | 0.164262 | 0.044069 | 0.06284204877 |
| | 32 | 60.000 | 1.000000 | 0.455225 | 0.83156347739 |
| 24 | 24 | 36.000 | 0.998728 | 0.184009 | 0.15671106200 |
| | 24 | 48.000 | 1.000000 | 0.327561 | 0.52962840920 |
| | 24 | 72.000 | 1.000000 | 0.484215 | 0.96669542296 |

Note that the case $\delta^2 = 0$, i.e. $\mu^* = 0 \in \mathbb{R}^k$, can be dealt with our computer program, too. A related computional experience made the practical robustness of our computer program evident as $\delta$ approaches zero, see Table 5.

**Table 5** Small noncentrality parameters

| $\delta^2$ | $k = 15$ <br> $c^2 = 25$ | $k = 20$ <br> $c^2 = 31.41$ | $k = 4$ <br> $c^2 = 13.28$ | $k = 10$ <br> $c^2 = 23.21$ | $k = 25$ <br> $c^2 = 44.31$ |
|---|---|---|---|---|---|
| 1.0 | 0.925056 | 0.928948 | 0.971126 | 0.980030 | 0.984613 |
| 0.5 | 0.938208 | 0.939980 | 0.981702 | 0.985522 | 0.987501 |
| 0.1 | 0.947792 | 0.948073 | 0.988530 | 0.989184 | 0.989522 |
| 0.09 | 0.948021 | 0.948267 | 0.988682 | 0.989267 | 0.989570 |
| 0.08 | 0.948249 | 0.948461 | 0.988834 | 0.989351 | 0.989617 |
| 0.07 | 0.948477 | 0.948654 | 0.988984 | 0.989433 | 0.989664 |
| 0.06 | 0.948704 | 0.948847 | 0.989134 | 0.989516 | 0.989711 |
| 0.05 | 0.948931 | 0.949039 | 0.989283 | 0.989598 | 0.989758 |
| 0.04 | 0.949157 | 0.949231 | 0.989431 | 0.989680 | 0.989804 |
| 0.03 | 0.949383 | 0.949423 | 0.989578 | 0.989761 | 0.989851 |
| 0.02 | 0.949608 | 0.949614 | 0.989724 | 0.989842 | 0.989897 |
| 0.01 | 0.949832 | 0.949804 | 0.989870 | 0.989922 | 0.989943 |
| 0.001 | 0.950034 | 0.949976 | 0.990000 | 0.989995 | 0.989985 |
| 0.0001 | 0.950054 | 0.949993 | 0.990013 | 0.990002 | 0.989989 |
| 0.00001 | 0.950056 | 0.949995 | 0.990014 | 0.990003 | 0.989989 |
| 0.0 | 0.950054 | 0.949992 | 0.990014 | 0.990002 | 0.989989 |

# 4 Large deviation approach to asymptotic approximations for large quantiles

In Sections 2 and 3, we have presented a geometric approach to exact evaluating the distribution functions $CQ(k,\delta^2;g)$. Combining this approach with standard numerical methods like the secant method we can obtain quite satisfactory numerical approximations for quantiles of these distributions. This is, however, not of primary interest in the present section. Rather we are interested in explicit approximation formulas or algorithms converging to the quantiles of interest. Furthermore we are interested in estimations for the occuring approximation errors.
The type of quantile approximations studied here is based upon exploiting an asymptotic expansion for large deviations. It turns out that large quantiles can be approximated in this way especially precisely. To estimate approximation errors we use exact numerical approximations based upon the results of Sections 2 and 3. However, we use this exact numerical quantile approximations for large quantiles rather then for small quantiles and suppress large tables for exact small quantiles.
In the theory of large deviations, amongst other things, one tries to describe quite precisely the asymptotic behaviour of tail probabilities of more or less complicated distributions when the argument of the distribution function approaches infinity. Assume that one could determine an explicitely known function

$$f = f_{k,\delta^2;g}|\mathbb{R}^+ \to [0,1]$$

such that

$$\frac{1-CQ(k,\delta^2;g)(c^2)}{f_{k,\delta^2;g}(c^2)} \to 1 \text{ as } c \to \infty. \tag{7}$$

If the function $f_{k,\delta;g}$ would have an appropriate structure then one could hope to get a suitable asymptotic approximation for the quantile $\chi_q^2$ by dealing with $\chi_q^2$ as the asymptotic solution of the asymptotic equation

$$f_{k,\delta^2;g}(\chi_q^2) = 1-q \text{ as } q \to 1.$$

For finding such a function $f_{k,\delta^2;g}$ we shall restrict our attention to the special class of sample distributions which are generated by the Kotz-type density generating function

$$g_K(r) = r^{M-1}e^{-\beta r^\gamma}, r > 0$$

with $\gamma > 0.5, \beta > 0$ and $2M+k > 2$. Furthermore we shall assume in the present and the next section that $\delta$ is positive. It has been shown in Richter and Schumacher (in preparation) that one can take

$$f_{k,\delta^2;g}(c^2) = \frac{\beta^{\frac{k}{2\gamma}+\frac{M-1}{\gamma}-\frac{k+1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\gamma^{\frac{k-1}{2}}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{k}{2\gamma}+\frac{M-1}{\gamma}\right)}c^{\frac{3k-1}{2}-\gamma(k+1)+2M-2}e^{-\beta(c-\delta)^{2\gamma}}. \tag{8}$$

Moreover, it has been proved there that the following asymptotic expansion formula for large deviations is a refinement of the asymptotic relation (7):

$$\begin{aligned}1 - CQ(k,\delta^2;g)(c^2) = f_{k,\delta^2;g}(c)\Big[&1 + D_{1-2\gamma}c^{1-2\gamma} + D_{-2\gamma}c^{-2\gamma} \\ &+ D_{-1-2\gamma}c^{-1-2\gamma} + D_{2-4\gamma}c^{2-4\gamma} + D_{1-4\gamma}c^{1-4\gamma} + D_{-4\gamma}c^{-4\gamma} + D_{-1-4\gamma}c^{-1-4\gamma} \\ &+ D_{-2-4\gamma}c^{-2-4\gamma} + O(c^{3-6\gamma})\Big], \text{ as } \quad c \to \infty,\end{aligned} \tag{9}$$

where

$$\begin{aligned}
D_{1-2\gamma} &= -\frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^3}\frac{k-3}{16}, \\
D_{-2\gamma} &= \frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^3}\frac{(k+1)(k-3)}{8(k-1)} + \frac{c_1^{\frac{k-1}{2}}c_2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)}{4(k-1)}, \\
D_{-1-2\gamma} &= -\frac{c_1^{\frac{k+3}{2}}}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^3}\frac{k+1}{16},
\end{aligned}$$

$$\begin{aligned}
D_{2-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k+3}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k-3)(k+1)}{512}, \\
D_{1-4\gamma} &= -\frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k-3)(k+3)}{128} + \frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k+1}{2}}(1-\frac{\delta}{c})^3}\frac{(k-3)(k+3)(k+5)}{64}, \\
D_{-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^5}\frac{3(k-5)(k-3)(k+1)(k+3)}{256(k-1)} \\
&\quad - \frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})^3}\frac{(k-3)(k+1)(k+3)(k+5)}{32(k-1)} \\
&\quad + \frac{c_1^{\frac{k-1}{2}}c_3}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)(k+5)}{8(k-1)} + \frac{c_1^{\frac{k-3}{2}}c_2^2}{b_1\delta^{\frac{k-1}{2}}(1-\frac{\delta}{c})}\frac{(k+1)(k+3)(k+5)}{32}, \\
D_{-1-4\gamma} &= -\frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^5}\frac{(k-5)(k+1)(k+3)}{128} + \frac{c_1^{\frac{k+1}{2}}c_2}{b_1\delta^{\frac{k-3}{2}}(1-\frac{\delta}{c})^3}\frac{(k+1)(k+3)(k+5)}{64}, \\
D_{-2-4\gamma} &= \frac{c_1^{\frac{k+5}{2}}}{b_1\delta^{\frac{k-5}{2}}(1-\frac{\delta}{c})^5}\frac{(k-3)(k+1)(k+3)}{512}
\end{aligned}$$

with

$$b_1 = \frac{1}{(k-1)(\gamma\beta)^{\frac{k+1}{2}}\delta^{\frac{k-1}{2}}} \tag{10}$$

and

$$\begin{aligned}
c_1 &= \frac{(1-\delta/c)^2}{\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}} \\
c_2 &= -\frac{1}{2}\frac{(1-\delta/c)^2\Big(\beta\gamma(\gamma-1)(1-\delta/c)^{2\gamma}+\frac{M-1}{c^{2\gamma}}\Big)}{\Big(\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\Big)^3} \\
c_3 &= -\frac{1}{12}\frac{(1-\delta/c)^2\Big(\beta\gamma(\gamma-1)(\gamma-2)(1-\delta/c)^{2\gamma}-2\frac{M-1}{c^{2\gamma}}\Big)}{\Big(\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\Big)^4} \\
&\quad +\frac{1}{4}\frac{(1-\delta/c)^2\Big(\beta\gamma(\gamma-1)(1-\delta/c)^{2\gamma}+\frac{M-1}{c^{2\gamma}}\Big)^2}{\Big(\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\Big)^5}.
\end{aligned}$$

This asymptotic expansion formula for large deviations is our starting point for first specifying the function $CQ_N(k,\delta^2;g)(.)$ occuring in the above mentioned $N$-th order approximative quantile equation (see Section 1) in such a way that

$$1-CQ_N(k,\delta^2;g)(c^2)=f_{k,\delta^2;g}(c)\left[1+\sum_{l=1}^{N}A_l(c)\right] \tag{11}$$

and second dealing with $\chi^2_{q,N}(k,\delta^2;g)$ as the asymptotic solution of the asymptotic equation

$$1-CQ_N(k,\delta^2;g)(\chi^2_{q,N}(k,\delta^2;g))=1-q \text{ as } q\to 1.$$

Here $A_l(c)$ denotes the $l$-th greatest term with respect to the power order of $c$, arising after the number one in the parentheses in formula (9). In the case $N=0$ we put $\sum\limits_{l=1}^{N}A_l(c)=0$. In accordance to the choise of the parameter $\gamma$ in the Kotz-type density generating function $g_K$ we obtain different $N$-th order approximative quantile equations. We give some examples to illustrate this fact.

*Examples:*

1. For the Gaussian case $g_K=g_G$, i.e. for $\gamma=1$, $M=1$ and $\beta=0.5$, it holds

$$\begin{aligned}1-CQ_N(k,\delta^2;g_G)(c^2)=f_{k,\delta^2;g_G}(c)\Big[&1+D_{1-2\gamma}c^{1-2\gamma}+D_{-2\gamma}c^{-2\gamma}\\ &+D_{2-4\gamma}c^{2-4\gamma}+O(c^{3-6\gamma})\Big].\end{aligned} \tag{12}$$

2. For the nongaussian case of the Kotz-type density generating function $g_K$ with the parameters $\gamma=0.7$, $M=1$ and $\beta=0.5$ holds

$$\begin{aligned}1-CQ_N(k,\delta^2;g_K)(c^2)=f_{k,\delta^2;g_K}(c)\Big[&1+D_{1-2\gamma}c^{1-2\gamma}+D_{2-4\gamma}c^{2-4\gamma}\\ &+O(c^{3-6\gamma})\Big].\end{aligned} \tag{13}$$

3. For the nongaussian case of the Kotz-type density generating function $g_K$ with the parameters $\gamma = 1.5$, $M = 1$ and $\beta = 0.5$ holds

$$\begin{aligned} 1 - CQ_N(k, \delta^2; g_K)(c^2) = & f_{k,\delta^2;g_K}(c)\Big[1 + D_{1-2\gamma}c^{1-2\gamma} + D_{-2\gamma}c^{-2\gamma} \\ & + D_{-1-2\gamma}c^{-1-2\gamma} + D_{2-4\gamma}c^{2-4\gamma} + O(c^{3-6\gamma})\Big]. \end{aligned} \tag{14}$$

Recall that a general approach to quantile approximations using asymptotic expansions for large deviations has been discussed in Richter (in preparation). In accordance with the approach developed there we pass over now from the $N$-th order approximative quantile equation to the following Large Deviation Iteration Procedure.
Define $C_{n+1}$ from $C_n$ by

$$\alpha = a_0 C_n^{\frac{3k-1}{2}-\gamma(k+1)+2M-2} e^{-\beta(C_{n+1}-\delta)^{2\gamma}} \left[1 + A_1(C_n) + \ldots + A_N(C_n)\right] \tag{15}$$

for $n = 0, 1, 2, \ldots$, with the starting value

$$C_0 = \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta\right)^{\frac{1}{2\gamma}} \tag{16}$$

and the constants

$$a_0 = \frac{\beta^{\frac{k}{2\gamma}+\frac{M-1}{\gamma}-\frac{k+1}{2}}\Gamma\left(\frac{k}{2}\right)}{2\sqrt{\pi}\gamma^{\frac{k-1}{2}}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{k}{2\gamma}+\frac{M-1}{\gamma}\right)}, \qquad \theta = \frac{3k-1}{2} - \gamma(k+1) + 2M - 2. \tag{17}$$

The following Lemma 1 reflects our motivation for choosing the starting value $C_0$ as it has been just defined.

**Lemma 1**
*If $x \geq 1$ is a solution of the equation*

$$a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} = \alpha$$

*then for sufficiently small $\alpha > 0$ holds*

$$\delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta + \frac{\theta}{\beta}\ln\left(1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\right)^{\frac{1}{2\gamma}} \leq x \leq C_0^*$$

*with*

$$\begin{aligned} C_0^* = \delta + \Bigg(& -\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right) - \frac{\theta}{2\gamma\beta}\ln\beta \\ & + \frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln(-\ln\frac{\alpha}{a_0})}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} + \frac{|\theta|\theta(1)}{\beta(\ln\frac{\alpha}{a_0})^{\frac{1}{2\gamma}}}\Bigg)^{\frac{1}{2\gamma}}, \end{aligned} \tag{18}$$

*where $\theta(1)$ denotes a certain positive constant.*

For the proof of Lemma 1 see the Appendix.

**Remark 5:** Obviously, $C_0^* \sim C_0$ as $\alpha \to 0$.

Let us consider now the function

$$\phi(c) = \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0} + \frac{\theta}{\beta}\ln c + \frac{1}{\beta}\ln\left[1 + A_1(c) + \ldots + A_N(c)\right]\right)^{\frac{1}{2\gamma}}, \tag{19}$$

for $c > 1$. The iteration procedure can be equivalently defined then by

$$C_{n+1} = \phi(C_n),\ n = 0, 1, 2, \ldots$$

and taking the starting value $C_0$ as above. From the fixed point theory it follows that the sequence $(C_n)$ will convergence to $\chi^2_{q,N}(k, \delta^2; g)$ if we approximate sufficiently large quantiles $\chi^2_q(k, \delta^2; g)$, because then holds $|\phi'(c)| < 1$.

# 5 Tables of large quantiles

The iteration procedure $C_{n+1} = \phi(C_n)$ with $\phi$ from (19) was used to implement a Borland Pascal Program for the evaluation of the $N$-th order approximative quantiles $\chi^2_{q,N}(k, \delta^2; g)$ of the noncentral $g$-generalized chi-square distribution with Kotz-type density generating function $g = g_K$. The iteration algorithm stops as soon as for the relative error

$$\varepsilon_n = \frac{|C_{n+1} - C_n|}{C_{n+1}}$$

holds $\varepsilon_n \leq 10^{-8}$.

To check the accuracy of our numerical result additionally in terms of quantile orders we calculate the c.d.f. $CQ(k, \delta; g)(c^2)$ at the point $c^2 = C^2_{n+1}$ and the relative error for the quantile orders,

$$r_n(\alpha) = \frac{|1 - CQ(k, \delta^2; g)(C^2_{n+1}) - \alpha|}{\alpha}, \quad \alpha = 1 - q.$$

Table 6 gives a survey of our computational results in the Gaussian case $g_K = g_G$. For the computations we used the specific representation of the function $CQ_N(k, \delta^2; g_G)(c^2)$ in the $N$-th order approximative quantile equation, given in (12). We choose always the result with the smallest relative error $r_n(\alpha)$ among all results of the Large Deviation Iteration Procedure for different orders $N$. In all tables the footnotes 0, 1, 2 and $S$ at the results indicate wether the given value is the $N$-th order approximative quantile, $N \in \{0, 1, 2\}$, and the starting value $C_0$, respectively. For all numerical results given in this Table 6 holds $r_n(\alpha) \leq 0.35$. If no result is given in Table 6 this means that our result did not satisfy the inequality $r_n(\alpha) \leq 0.35$.

**Table 6** Approximated values for the $N$-th order approximative quantiles $\chi^2_{q,N}(k,1;g_G)$, $r_n(\alpha) \leq 0.35$

| $\alpha \setminus k$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 0.1 | $6.272647_1$ | - | - | - | - |
| 0.01 | $12.422780_S$ | $13.802919_S$ | $15.663581_0$ | $17.643733_0$ | $19.729094_0$ |
| 0.001 | $18.289315_S$ | $19.899377_S$ | $21.874278_0$ | $23.938525_0$ | $26.084156_0$ |
| 0.0001 | $23.970570_S$ | $25.727201_S$ | $27.787963_0$ | $29.921488_0$ | $32.122476_0$ |
| 0.00001 | $29.529460_S$ | $31.394053_S$ | $33.524108_0$ | $35.716181_0$ | $37.966403_0$ |

If we use a preliminary stage from the derivation of the asymptotic expansion (9) with

$$\tilde{f}_{k,\delta^2;g}(c^2) = \frac{\gamma\beta^{\frac{M-1}{\gamma}+\frac{k}{2\gamma}}\Gamma(k/2)}{2\sqrt{\pi}\delta^{\frac{k-1}{2}}\Gamma\left(\frac{M-1}{\gamma}+\frac{k}{2\gamma}\right)} \frac{(1-\delta/c)^{2M-2+k}c^{\frac{3k-1}{2}-\gamma(k+1)+2M-2}}{\left(\beta\gamma(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\right)^{\frac{k+1}{2}}}e^{-\beta(c-\delta)^{2\gamma}} \tag{20}$$

and

$$\tilde{b}_1 = \frac{(1-\delta/c)^k}{(k-1)\delta^{\frac{k-1}{2}}\left(\gamma\beta(1-\delta/c)^{2\gamma}-\frac{M-1}{c^{2\gamma}}\right)^{\frac{k+1}{2}}} \tag{21}$$

instead of $b_1$ we obtain a corresponding Large Deviation Iteration Procedure. With starting value $C_0$ from (16) we compute corrected values $\tilde{\chi}^2_{q,N}(k,\delta^2;g_G)$ for the $N$-th order approximative quantiles $\chi^2_{q,N}(k,\delta^2;g_G)$. For this values, given in Table 7, holds $r_n(\alpha) \leq 0.05$.

**Table 7** Corrected values for the $N$-th order approximative quantiles $\tilde{\chi}^2_{q,N}(k,1;g_G)$, $r_n(\alpha) \leq 0.05$

| $\alpha \setminus k$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 0.35 | $3.286958_2$ | - | - | - |
| 0.3 | $3.675375_2$ | - | - | - |
| 0.2 | $4.843447_2$ | - | - | - |
| 0.1 | $6.796464_2$ | - | - | - |
| 0.05 | $8.672945_2$ | $10.326004_0$ | - | - |
| 0.03 | $10.025478_2$ | $11.710646_0$ | $13.306141_2$ | - |
| 0.025 | $10.503386_2$ | $12.200220_0$ | $13.811369_2$ | - |
| 0.01 | $12.874054_2$ | $14.627922_0$ | $16.309541_2$ | - |
| 0.005 | $14.638875_2$ | $16.432776_0$ | $18.160458_2$ | - |
| 0.0025 | $16.383801_2$ | $18.214753_0$ | $19.983714_2$ | - |
| 0.001 | $18.664860_2$ | $20.540546_0$ | $22.358242_2$ | $24.187268_2$ |
| 0.0005 | $20.373838_2$ | $22.280522_0$ | $24.131499_2$ | $25.986788_2$ |
| 0.0001 | $24.296394_2$ | $26.267231_0$ | $28.186259_2$ | $30.098434_2$ |
| 0.00001 | $29.815791_1$ | $31.868323_0$ | $33.868263_2$ | $35.852319_2$ |

Note, that $A_i(c) = 0$, $i = 1,2$ for $k = 3$. The 0-indexed values of the column for $k = 3$ coincide therefore with the respective 2-indexed values.
Our geometric representation formula for the distribution function $CQ(k,\delta^2;g_K)(x)$, $x \in \mathbb{R}$, given in Corollary 1, Section 2, enables us to calculate more exact approximations

$\chi^2_{q,*} = \chi^2_{q,*}(k, \delta^2; g_K)$ for the quantiles $\chi^2_q = \chi^2_q(k, \delta^2; g_K)$ by using the secant method for approximating the solution $\chi^2_q$ of the equation

$$1 - CQ(k, \delta^2; g_K)(\chi^2_q) = 1 - q.$$

The algorithm of the secant method stops as soon as for the increment with respect to the latest value

$$dC_n := \frac{(C_n - C_{n+1})CQ(k, \delta^2; g_K)(C_{n+1})}{CQ(k, \delta^2; g_K)(C_{n+1}) - CQ(k, \delta^2; g_K)(C_n)}$$

holds $|dC_n| < 10^{-13}$ or $1 - CQ(k, \delta^2; g_K)(C_{n+1}) = 1 - q$. The values of the distribution function $CQ(k, \delta^2; g_K)(\chi^2_{q,*}(k, \delta^2; g_K))$ coincide with $q$ at least for 12 digits. In this way, we can compute the more interesting relative approximation error

$$\varepsilon^*_n(\alpha) = \frac{|c_n^2 - \chi^2_{q,*}(k, \delta^2; g_K)|}{\chi^2_{q,*}(k, \delta^2; g_K)}.$$

If we use this relative error instead of $r_n(\alpha)$ for controling our corrected quantile approximations $\tilde{\chi}^2_q(k, \delta^2; g_G)$ we obtain Table 8. In this table holds $\varepsilon^*_n(\alpha) \leq 0.02$ for all values.

**Table 8** Corrected values for the $N$-th order approximative quantiles $\tilde{\chi}^2_{q,N}(k, 1; g_G)$, $\varepsilon^*_n \leq 0.02$

| $\alpha \setminus k$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 0.35 | $3.286958_2$ | - | - | - | - |
| 0.3 | $3.675375_2$ | - | - | - | - |
| 0.2 | $4.843447_2$ | - | - | - | - |
| 0.1 | $6.796464_2$ | - | - | - | - |
| 0.05 | $8.672945_2$ | $10.326004_0$ | $11.874000_2$ | - | - |
| 0.03 | $10.025478_2$ | $11.710646_0$ | $13.306141_2$ | - | - |
| 0.025 | $10.503386_2$ | $12.200220_0$ | $13.811369_2$ | $15.515107_2$ | - |
| 0.01 | $12.874054_2$ | $14.627922_0$ | $16.309541_2$ | $18.046690_2$ | - |
| 0.005 | $14.638875_2$ | $16.432776_0$ | $18.160458_2$ | $19.925361_2$ | - |
| 0.0025 | $16.383801_2$ | $18.214753_0$ | $19.983714_2$ | $21.776609_2$ | $23.715079_2$ |
| 0.001 | $18.664860_2$ | $20.540546_0$ | $22.358242_2$ | $24.187268_2$ | $26.131113_2$ |
| 0.0005 | $20.373838_2$ | $22.280522_0$ | $24.131499_2$ | $25.986788_2$ | $27.938659_2$ |
| 0.0001 | $24.296394_2$ | $26.267231_0$ | $28.186259_2$ | $30.098434_2$ | $32.076165_2$ |
| 0.00001 | $29.815791_1$ | $31.868323_0$ | $33.868263_2$ | $35.852319_2$ | $37.874289_2$ |

For a comparison of the $N$-th order approximative quantiles $\chi^2_{q,N}(k, 1; g_G)$ with the corrected values $\tilde{\chi}^2_{q,N}(k, 1; g_G)$ we give in Table 9 both values with its relative approximation errors $\varepsilon^*_n(\alpha)$ and the result of the application of the secant method, $\chi^2_{q,*}(k, 1; g_G)$, too. Notice that in the Gaussian case the corrected values $\tilde{\chi}^2_{q,N}(k, 1; g_G)$ have a smaller relative approximation error $\varepsilon^*_n(\alpha)$ than the $N$-th order approximative quantiles $\chi^2_{q,N}(k, 1; g_G)$.

**Table 9** Comparison of approximations for the quantiles $\chi^2_{1-\alpha}(k,1;g_G)$

| $k$ | $\alpha$ | $\chi^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha))$ | $\tilde{\chi}^2_{1-\alpha,N}(k,1;g_G)(\varepsilon^*_n(\alpha))$ | $\chi^2_{1-\alpha,*}(k,1;g_G)$ |
|---|---|---|---|---|
| 2 | 0.1 | $6.27264722_1(0.1986528)$ | $6.79646412_2(0.0038897)$ | 6.77013045 |
| | 0.01 | $12.32348549_1(0.0408692)$ | $12.87405417_2(0.0019813)$ | 12.84859769 |
| | 0.001 | $18.18213409_1(0.0249444)$ | $18.66486018_2(0.0009428)$ | 18.64727974 |
| | 0.0001 | $23.86645384_1(0.0171785)$ | $24.29639463_2(0.0005265)$ | 24.28360971 |
| | 0.00001 | $29.43002096_1(0.0127647)$ | $29.81579086_1(0.0001761)$ | 29.81054242 |
| 3 | 0.1 | $7.09432971_0(0.4747291)$ | $8.41418910_0(0.0215007)$ | 8.23708570 |
| | 0.01 | $13.80291938_0(0.0520500)$ | $14.62792229_0(0.0046091)$ | 14.56081028 |
| | 0.001 | $19.89937706_0(0.0294528)$ | $20.54054658_0(0.0018188)$ | 20.50325620 |
| | 0.0001 | $25.72720143_0(0.0196455)$ | $26.26723128_0(0.0009327)$ | 26.24275474 |
| | 0.00001 | $31.39405313_0(0.0143369)$ | $31.86832308_0(0.0005535)$ | 31.85069342 |
| 4 | 0.1 | $8.77957771_0(0.3272683)$ | $9.88670591_2(0.0244904)$ | 9.65036473 |
| | 0.01 | $15.66358088_0(0.0341762)$ | $16.30954149_2(0.0056541)$ | 16.21784458 |
| | 0.001 | $21.87427839_0(0.0194037)$ | $22.35824209_2(0.0022918)$ | 22.30711968 |
| | 0.0001 | $27.78796302_0(0.0129564)$ | $28.18625958_2(0.0011913)$ | 28.15272218 |
| | 0.00001 | $33.52410758_0(0.0094561)$ | $33.86826331_2(0.0007127)$ | 33.84414107 |
| 5 | 0.1 | $10.66620456_0(0.1188386)$ | $11.79048626_2(0.2157233)$ | 11.02543135 |
| | 0.01 | $17.64373258_0(0.0005866)$ | $18.04668978_2(0.0121399)$ | 17.83023266 |
| | 0.001 | $23.93852494_0(0.0053071)$ | $24.18726823_2(0.0050287)$ | 24.06624588 |
| | 0.0001 | $29.92148807_0(0.0032500)$ | $30.09843452_2(0.0026445)$ | 30.01904880 |
| | 0.00001 | $35.71618091_0(0.0022094)$ | $35.85231948_2(0.0015939)$ | 35.79526706 |
| 6 | 0.01 | $19.72909378_0(0.0166664)$ | $20.44110682_2(0.0533571)$ | 19.40567026 |
| | 0.001 | $26.08415552_0(0.0115361)$ | $26.13111319_2(0.0133571)$ | 25.78667865 |
| | 0.0001 | $32.12247646_0(0.0086629)$ | $32.07616452_2(0.0072087)$ | 31.84659272 |
| | 0.00001 | $37.96640295_0(0.0068504)$ | $37.87428860_2(0.0044076)$ | 37.70808815 |

Fisher (1928) published the upper 5% significance points of the noncentral chi-square distribution function for $n = 1(1)7$ and $\delta = 0(0.2)5.0$. Several methods for approximating the quantiles of the usual noncentral chi-square distribution function have been developed by Patnaik (1949), Abdel-Aty (1954) and Sankaran (1963). In these papers the upper and the lower 5% points for several parameters are given. In Table 10 we use the choise of parameters studied by other authors to compare our upper 5% points with their results. In the column Fisher we quoted the results of Fisher (1928), which called 'exact values' by other authors. Modified central chi-square-approximated percentage points from Patnaik (1949) are given in the column Patnaik. The next column contains the Cornish-Fisher type approximations in Abdel-Aty (1954). In the columns Sankaran ('closer' and 'normal') the results of a modification of Abdel-Aty's method and a normal approximation by Sankaran were listed. The last columns contain our corrected values $\tilde{\chi}^2_{q,N}(k,\delta^2;g_G)$ of the $N$-th order approximative quantiles based upon the Large Deviation Iteration Procedure from Section 4, indexed by the actual value of $N$, and the results of the application of the secant method $\chi^2_{q,*}(k,\delta^2;g_G)$, respectivly.

**Table 10** Comparison study for the upper 5% points of $CQ(k,\delta^2;g_G)$

| $k$ | $\delta^2$ | Fisher | Patnaik | Abd.-Aty | Sankaran 'closer' | Sankaran 'normal' | $\tilde{\chi}^2_{0.95,N}$ | $\chi^2_{0.95,*}$ |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 8.64242404 | 8.63 | 8.38 | 8.38 | 8.87 | $8.67_2$ | 8.64220388 |
| | 4 | 14.64057169 | 14.72 | 14.62 | 14.62 | 14.68 | $14.61_2$ | 14.64021080 |
| | 16 | 33.05445049 | 33.35 | 33.08 | 33.08 | 33.057 | $32.95_2$ | 33.05421469 |
| | 25 | 45.30770721 | 45.66 | 45.33 | 45.33 | 45.309 | $45.17_2$ | 45.30823043 |
| 4 | 1 | 11.70734656 | 11.72 | 11.67 | 11.67 | 11.96 | $11.87_2$ | 11.70722775 |
| | 4 | 17.30892816 | 17.38 | 17.24 | 17.27 | 17.39 | $17.57_0$ | 17.30932288 |

For the nongaussian case Table 11 and Table 12 contain the values for the $N$-th order approximative quantiles $\chi^2_{q,N}(k,1;g_K)$, the corrected values of the $N$-th order approximative quantiles $\tilde{\chi}^2_{q,N}(k,1;g_K)$ and the results of the secant method $\chi^2_{q,*}(k,1;g_K)$ for two types of noncentral $g$-generalized chi-square distributions with Kotz-type density generating function $g=g_K$. For the computations we used the specific representations of the function $CQ_N(k,\delta^2;g_K)(c^2)$ in the $N$-th order approximative quantile equations, given in (14) for $\gamma=0.7$ and in (15) for $\gamma=1.5$, respectively.

**Table 11** Comparison of the approximations for the quantiles $\chi^2_{1-\alpha}(k,\delta^2;g_K)$ with parameters $M=1$, $\gamma=0.7$ and $\beta=0.5$

| $k$ | $\alpha$ | $\chi^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$ | $\tilde{\chi}^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$ | $\chi^2_{1-\alpha,*}(k,\delta^2;g_K)$ |
|---|---|---|---|---|
| 2 | 0.1 | $14.86493108_2(0.0138557)$ | $15.03970710_2(0.0044661)$ | 15.07378935 |
| | 0.01 | $34.19873621_2(0.0114173)$ | $34.34522854_2(0.0071826)$ | 34.59370194 |
| | 0.001 | $56.86025772_2(0.0074745)$ | $56.86974403_2(0.0073089)$ | 57.28846018 |
| | 0.0001 | $82.13730812_2(0.0052510)$ | $82.25693756_2(0.0038022)$ | 82.57088609 |
| | 0.00001 | $109.65677689_2(0.0039086)$ | $109.76886436_2(0.0028904)$ | 110.08706075 |
| | 0.000001 | $139.16973174_2(0.0030313)$ | $139.27604534_2(0.0022697)$ | 139.59287967 |
| | 0.0000001 | $170.49336517_2(0.0023906)$ | $170.59505395_2(0.0017955)$ | 170.90191646 |
| | 0.00000001 | $203.48575659_2(0.0016130)$ | $203.58361901_2(0.0011328)$ | 203.81450020 |
| 3 | 0.1 | $20.48896227_0(0.0728529)$ | $19.97599483_0(0.0960652)$ | 22.09893365 |
| | 0.01 | $44.61580028_0(0.0220916)$ | $44.26005612_0(0.0298889)$ | 45.62369965 |
| | 0.001 | $70.87841451_0(0.0108747)$ | $70.57753289_0(0.0150736)$ | 71.65767114 |
| | 0.0001 | $99.24700884_0(0.0065600)$ | $98.97691909_0(0.0092424)$ | 99.90236631 |
| | 0.00001 | $129.54627391_0(0.0043943)$ | $129.29678540_0(0.0063227)$ | 130.11804641 |
| | 0.000001 | $161.61916804_0(0.0031652)$ | $161.38481453_0(0.0046106)$ | 162.13234805 |
| | 0.0000001 | $195.33528895_0(0.0023427)$ | $195.11272269_0(0.0034795)$ | 195.79397925 |
| | 0.00000001 | $230.58629918_0(0.0013308)$ | $230.37328617_0(0.0022533)$ | 230.89356607 |
| 4 | 0.1 | $30.99758965_0(0.0488901)$ | $29.77961958_0(0.0076767)$ | 29.55275239 |
| | 0.01 | $55.01652743_1(0.0318210)$ | $58.10434137_0(0.0225182)$ | 56.82474969 |
| | 0.001 | $84.76452186_1(0.0140760)$ | $87.70180328_0(0.0200885)$ | 85.97470181 |
| | 0.0001 | $116.09481535_1(0.0076020)$ | $118.98183151_0(0.0170767)$ | 116.98413171 |
| | 0.00001 | $149.04874406_1(0.0045731)$ | $148.24652675_1(0.0099308)$ | 149.73350183 |
| | 0.000001 | $183.56415836_1(0.0029344)$ | $182.82286252_1(0.0069609)$ | 184.10439473 |
| | 0.0000001 | $219.56307306_1(0.0019022)$ | $218.86790295_1(0.0050624)$ | 219.98153002 |
| | 0.00000001 | $256.97007013_1(0.0006834)$ | $256.31146862_1(0.0032446)$ | 257.14581454 |
| 5 | 0.1 | $44.44110650_0(0.1888110)$ | - | 37.38281904 |
| | 0.01 | $75.79376247_0(0.1110184)$ | $74.43577764_0(0.0911125)$ | 68.22007379 |
| | 0.001 | $108.20870185_0(0.0786034)$ | $107.03985864_0(0.0669526)$ | 100.32297939 |
| | 0.0001 | $129.46034399_1(0.0335465)$ | $141.01234963_0(0.0526921)$ | 133.95403325 |
| | 0.00001 | $165.32686432_1(0.0223372)$ | $176.42262992_0(0.0432778)$ | 169.10418118 |
| | 0.000001 | $203.03064098_S(0.0130377)$ | $213.24332827_0(0.0366077)$ | 205.71266184 |
| | 0.0000001 | $240.79659339_1(0.0118989)$ | $238.94491736_1(0.0194972)$ | 243.69630082 |
| | 0.00000001 | $280.40996409_1(0.0085487)$ | $278.70840807_1(0.0145649)$ | 282.82777037 |

In Table 11 we can observe that the relative approximation errors $\varepsilon^*_n(\alpha)$ of the $N$-th order approximative quantiles $\chi^2_{q,N}(k,1;g_K)$ for $\alpha \le 0.00001$ and $n \ge 3$ are smaller than the relative approximation errors for the corrected values $\tilde{\chi}^2_{q,N}(k,1;g_K)$. Recall that we choose always the result with the smallest relative error $r_n(\alpha)$ among all results of the Large Deviation Iteration Procedure for different orders $N$. The footnotes 0, 1, 2 and $S$ at the results indicate wether the given value is the $N$-th order approximative quantile, $N \in 0,1,2$, or the starting value $C_0$, respectively.

**Table 12** Comparison of the approximations for the quantiles $\chi^2_{1-\alpha}(k,\delta^2;g_K)$ with parameters $M=1$, $\gamma=1.5$ and $\beta=0.5$

| $k$ | $\alpha$ | $\chi^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$ | $\tilde{\chi}^2_{1-\alpha,N}(k,\delta^2;g_K)(\varepsilon^*_n(\alpha))$ | $\chi^2_{1-\alpha,*}(k,\delta^2;g_K)$ |
|---|---|---|---|---|
| 2 | 0.01 | $6.87185579_S(0.0439408)$ | $7.00401848_2(0.0255534)$ | 7.18768859 |
| | 0.001 | $9.08949622_S(0.0253340)$ | $9.28401156_2(0.0044761)$ | 9.32575474 |
| | 0.0001 | $10.95907778_S(0.0168096)$ | $11.13001266_2(0.0014742)$ | 11.14644503 |
| | 0.00001 | $12.61004396_S(0.0121966)$ | $12.75760974_2(0.0006370)$ | 12.76574211 |
| | 0.000001 | $14.10932345_S(0.0093829)$ | $14.23838034_2(0.0003218)$ | 14.24296372 |
| 3 | 0.01 | $6.97295539_S(0.0624126)$ | $7.70944597_1(0.0366163)$ | 7.43712605 |
| | 0.001 | $9.17244144_S(0.0371421)$ | $9.47454759_2(0.0054291)$ | 9.52626679 |
| | 0.0001 | $11.03103680_S(0.0252909)$ | $11.29734857_2(0.0017594)$ | 11.31726070 |
| | 0.00001 | $12.67467647_S(0.0187121)$ | $12.90685583_2(0.0007365)$ | 12.91636851 |
| | 0.000001 | $14.16865014_S(0.0146180)$ | $14.37371688_2(0.0003563)$ | 14.37883986 |
| 4 | 0.01 | $7.12728617_S(0.0731333)$ | $7.92854649_1(0.0310665)$ | 7.68965615 |
| | 0.001 | $9.29926645_S(0.0451198)$ | $9.65864734_2(0.0082173)$ | 9.73867307 |
| | 0.0001 | $11.14131720_S(0.0314827)$ | $11.47203477_2(0.0027333)$ | 11.50347734 |
| | 0.00001 | $12.77386654_S(0.0236982)$ | $13.06857978_2(0.0011734)$ | 13.08393280 |
| | 0.000001 | $14.25978045_S(0.0187522)$ | $14.52388030_2(0.0005789)$ | 14.53229278 |
| 5 | 0.001 | $9.45864078_S(0.0499704)$ | $10.05698513_1(0.0101275)$ | 9.95615395 |
| | 0.0001 | $11.28032661_S(0.0357057)$ | $11.64054052_2(0.0049130)$ | 11.69801286 |
| | 0.00001 | $12.89912969_S(0.0273258)$ | $13.23219154_2(0.0022109)$ | 13.26151169 |
| | 0.000001 | $14.37500650_S(0.0218898)$ | $14.67979190_2(0.0011514)$ | 14.69671418 |
| 6 | 0.001 | $9.64307315_S(0.0522953)$ | $10.17998657_1(0.0004716)$ | 10.17518772 |
| | 0.0001 | $11.44176859_S(0.0382587)$ | $11.92194900_1(0.0021030)$ | 11.89692968 |
| | 0.00001 | $13.04492550_S(0.0297637)$ | $13.47204137_1(0.0020036)$ | 13.44510213 |
| | 0.000001 | $14.50931410_S(0.0241365)$ | $14.89327338_1(0.0016877)$ | 14.86818063 |
| 7 | 0.0001 | $11.62118942_S(0.0394037)$ | $12.01881527_1(0.0065363)$ | 12.09789079 |
| | 0.00001 | $13.20734770_S(0.0311663)$ | $13.59840267_1(0.0024802)$ | 13.63221395 |
| | 0.000001 | $14.65917783_S(0.0255830)$ | $15.02884453_1(0.0010107)$ | 15.04404979 |

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# Appendix

*Proof of Lemma 1 : Case 1:* $\theta \leq 0$. We consider

$$a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} = \alpha \tag{22}$$

with

$$\theta \leq 0, \;\; x > 0.$$

In (22) holds $\alpha \to +0$ if and only if $x \to \infty$.
Let $x \geq 1$. Then it follows from (23) that

$$a_0 e^{-\beta(x-\delta)^{2\gamma}} \geq \alpha.$$

This is equivalent to

$$x \leq \delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}.$$

Inserting the last inequality into (22) at the place of $x^\theta$ it follows

$$\alpha \geq a_0 e^{-\beta(x-\delta)^{2\gamma}} \left(\delta + \left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}\right)^\theta$$

or

$$
\begin{aligned}
\ln\frac{\alpha}{a_0} &\geq -\beta(x-\delta)^{2\gamma}+\theta\ln\left[\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}\left(1+\frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\right],\\
\beta(x-\delta)^{2\gamma} &\geq -\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma}\ln\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)+\theta\ln\left(1+\frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),\\
\beta(x-\delta)^{2\gamma} &\geq -\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma}\ln\left(-\ln\frac{\alpha}{a_0}\right)+\frac{\theta}{2\gamma}\ln\frac{1}{\beta}+\theta\ln\left(1+\frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
x\geq\delta+\Bigg(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right)-\frac{\theta}{2\gamma\beta}\ln\beta \\
+\frac{\theta}{\beta}\ln\left(1+\delta\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{-\frac{1}{2\gamma}}\right)\Bigg)^{\frac{1}{2\gamma}}.
\end{aligned}
\tag{23}
$$

Inserting the last inequality again into (22) for $x^\theta$ supplies

$$
\begin{aligned}
\alpha\leq a_0e^{-\beta(x-\delta)^{2\gamma}}\Bigg[\delta+\Bigg(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right)-\frac{\theta}{2\gamma\beta}\ln\beta \\
+\frac{\theta}{\beta}\ln\left(1+\delta\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{-\frac{1}{2\gamma}}\right)\Bigg)^{\frac{1}{2\gamma}}\Bigg]^\theta.
\end{aligned}
$$

It follows with $\theta(1)=O(1)$, $\alpha\to 0$ that

$$
\alpha\leq a_0e^{-\beta(x-\delta)^{2\gamma}}\left[\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right)\right)^{\frac{1}{2\gamma}}\left(1+\frac{\theta(1)}{\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\right]^\theta,
$$

respectivly

$$
\begin{aligned}
\ln\frac{\alpha}{a_0}&\leq-\beta(x-\delta)^{2\gamma}\\
&+\frac{\theta}{2\gamma}\ln\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right)\right)+|\theta|\ln\left(1+\frac{\theta(1)}{\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),\\
\ln\frac{\alpha}{a_0}&\leq-\beta(x-\delta)^{2\gamma}\\
&+\frac{\theta}{2\gamma}\ln\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\left[1+\frac{\theta}{2\gamma\beta}\frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)}\right]\right)+|\theta|\left(\frac{\theta(1)}{\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),
\end{aligned}
$$

or

$$x \le \delta + \left[ -\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\gamma\beta} \ln \left( -\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right) \right.$$
$$\left. + \frac{|\theta|}{2\beta\gamma} \ln \left( 1 + \frac{|\theta|}{2\gamma\beta} \frac{\ln\left(-\ln \frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} \right) + \frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}},$$

$$x \le \delta + \left[ -\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma} \ln \left( -\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\beta\gamma} \ln \beta \right.$$
$$\left. + \frac{|\theta|^2}{(2\beta\gamma)^2} \frac{\ln\left(-\ln \frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} + \frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}}. \tag{24}$$

From the inequality (24) follows the upper bound

$$C_0^* = \delta + \left[ -\frac{1}{\beta} \ln \frac{\alpha}{a_0} + \frac{\theta}{2\beta\gamma} \ln \left( -\ln \frac{\alpha}{a_0} \right) - \frac{\theta}{2\beta\gamma} \ln \beta \right.$$
$$\left. + \frac{|\theta|^2}{(2\beta\gamma)^2} \frac{\ln\left(-\ln \frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)} + \frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right]^{\frac{1}{2\gamma}} \tag{25}$$

for the case $\theta \le 0$.
*Case 2:* $\theta > 0$. Let $x \ge 1$ then follows from (22) for $\theta > 0$

$$a_0 e^{-\beta(x-\delta)^{2\gamma}} \le \alpha.$$

This is equivalent to

$$x \ge \delta + \left( -\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}}. \tag{26}$$

If we insert the last inequality (26) for $x^\theta$ into (22), we obtain

$$\begin{aligned}
\alpha &\ge a_0 \left( \delta + \left( -\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \right)^{\theta} e^{-\beta(x-\delta)^{2\gamma}}, \\
\ln \frac{\alpha}{a_o} &\ge -\beta(x-\delta)^{2\gamma} + \theta \ln \left( \delta + \left( -\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \right), \\
\ln \frac{\alpha}{a_o} &\ge -\beta(x-\delta)^{2\gamma} + \theta \ln \left[ \left( -\frac{1}{\beta} \ln \frac{\alpha}{a_0} \right)^{\frac{1}{2\gamma}} \left( 1 + \frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}} \right) \right],
\end{aligned}$$

$$
\begin{aligned}
\ln\frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma}+\frac{\theta}{2\gamma}\ln\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)+\theta\ln\left(1+\frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right)\\
\ln\frac{\alpha}{a_o} &\geq -\beta(x-\delta)^{2\gamma}+\frac{\theta}{2\gamma}\ln\left(-\ln\frac{\alpha}{a_0}\right)-\frac{\theta}{2\gamma}\ln\beta+\theta\ln\left(1+\frac{\delta}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
x \geq \delta+\Bigg(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\gamma\beta}\ln\left(-\ln\frac{\alpha}{a_0}\right)-\frac{\theta}{2\gamma\beta}\ln\beta\\
+\frac{\theta}{\beta}\ln\left(1+\delta\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{-\frac{1}{2\gamma}}\right)\Bigg)^{\frac{1}{2\gamma}}.
\end{aligned}
\tag{27}
$$

Now we want prove $x \leq C_0^*$. We show $x < (1+\varepsilon)C_0^*$, $\forall\varepsilon > 0$, leading the now stated assumption $x \geq (1+\varepsilon)C_0^*$ for $\varepsilon > 0$ to a contradiction. Because in (22) $\alpha \to +0$ if and only if $x \to \infty$, it holds for a sufficiently small $\alpha$ or a sufficiently large $x$, respectivly,

$$
\alpha = a_0 x^\theta e^{-\beta(x-\delta)^{2\gamma}} \leq a_0(1+\varepsilon)^\theta C_0^{*\theta} e^{-\beta((1+\varepsilon)C_0^*-\delta)^{2\gamma}}
$$

or

$$
\begin{aligned}
\frac{\alpha}{a_0} \leq (1+\varepsilon)^\theta C_0^{*\theta}\exp\Bigg\{-\beta\Bigg((1+\varepsilon)\delta+(1+\varepsilon)\Bigg[-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\beta\gamma}\ln\left(-\ln\frac{\alpha}{a_0}\right)\\
-\frac{\theta}{2\beta\gamma}\ln\beta+\frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)}+\frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\Bigg]^{\frac{1}{2\gamma}}-\delta\Bigg)^{2\gamma}\Bigg\}
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
(1+\varepsilon)^{-\theta}\frac{\alpha}{a_0} \leq & C_0^{*\theta}\exp\Bigg\{-\beta(1+\varepsilon)^{2\gamma}\Bigg[-\frac{1}{\beta}\ln\frac{\alpha}{a_0}+\frac{\theta}{2\beta\gamma}\ln\left(-\ln\frac{\alpha}{a_0}\right)-\frac{\theta}{2\beta\gamma}\ln\beta\\
&+\frac{|\theta|^2}{(2\beta\gamma)^2}\frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)}+\frac{|\theta|\theta(1)}{\beta\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{1}{2\gamma}}}\Bigg]\Bigg\}\\
=& C_0^{*\theta}\left(\frac{\alpha}{a_0}\right)^{(1+\varepsilon)^{2\gamma}}\beta^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}}\frac{\exp\left\{-\frac{|\theta|^2}{\beta(2\gamma)^2}\frac{\ln\left(-\ln\frac{\alpha}{a_0}\right)}{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)}-\frac{|\theta|\theta(1)}{\beta(-\ln\frac{\alpha}{a_0})^{\frac{1}{2\gamma}}}\right\}}{\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}}}\\
\asymp& \frac{\left(-\frac{1}{\beta}\ln\frac{\alpha}{a_0}\right)^{\frac{\theta}{2\gamma}}}{\left(-\ln\frac{\alpha}{a_0}\right)^{\frac{\theta}{2\gamma}(1+\varepsilon)^{2\gamma}}}\alpha^{(1+\varepsilon)^{2\gamma}}
\end{aligned}
$$

This, however, doesn't hold for $\alpha \to 0$ and so we have for sufficiently small $\alpha$ the upper bound $C_0^*$ for $x$. $\square$