Optimal Bell Functions for Biorthogonal Local Trigonometric Bases

KAI BITTNER*

Abstract

A general approach for biorthogononal local trigonometric bases in the two-overlapping setting was given by Chui and Shi. In this paper, we give error estimates for the approximation with such basis functions. In particular, it is shown that for a partition of the real axis into small intervals one obtains better approximation order if polynomials are reproduced locally. Furthermore, smooth trigonometric bases are constructed, which reproduce constants resp. linear functions by only one resp. a small number of basis functions for each interval.

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1 Introduction

Often it is useful to investigate the local properties of a signal. Therefore, many applications in signal and image processing use basis functions which are local in time and frequency. Because the most signals have both temporal and spectral correlation such basis functions have good approximation properties, i.e., one can obtain a reasonable approximation using only a few basis functions.

One example of special interest are wavelets, where the basis functions are translates and dilates of one particular function. Another way to construct an orthogonal basis is to consider functions of the type

$$\psi_j^k = w(\cdot - j)e^{2\pi ik(\cdot - j)}, \quad k, j \in \mathbb{Z},$$

where w is called the window or bell function. From a well-known theorem of Balian and Low it follows that if the family $\{\psi_{j,k}\}$ is a frame for $L^2(\mathbb{R})$ then either $t\,w(t) \notin L^2(\mathbb{R})$ or $\xi\,\hat{w}(\xi) \notin L^2(\mathbb{R})$. Thus, the functions $\psi_{j,k}$ cannot be well localized in both time and frequency. In particular, from $\xi\hat{w}(\xi) \notin L^2(\mathbb{R})$ it follows that $w'(t) \notin L^2(\mathbb{R})$, such that e.g. for bell functions with compact support we cannot obtain a frame or Riesz basis consisting of smooth functions.

One can overcome this problem using certain sets of cosine or sine functions which form an orthonormal basis of $L^2([0,1])$ instead of the exponential monomials $e^{2\pi ik\cdot}$. In this

*Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, D-18051 Rostock, Germany, (kai.bittner@mathematik.uni-rostock.de)

way, smooth orthogonal local trigonometric bases were introduced by Malvar [11]. Here the so called "two-overlapping setting" is considered, where the window functions have compact support such that $w(x-r)w(x-s) \equiv 0$ if $|s-r| > 1$. The "Malvar bases" were independently discovered by Coifman and Meyer [7] in a generalized nonuniform setting, where the uniform spacing is replaced by an arbitrary partition

$$\ldots < a_{-1} < a_0 < a_1 < \ldots$$

of $\mathbb{R}$ with $a_j \to \pm\infty$ for $j \to \pm\infty$. An expository representation of these results can be found in [2]. As another example let us mention the Wilson bases described by Daubechies, Jaffard and Journé [9] for window functions with arbitrary support. In [1], the connection of the both approaches is shown. Bivariate orthogonal local trigonometric bases are investigated in [14].

It is known that many applications only need the Riesz stability instead of the orthogonality of the basis functions. Furthermore, for non-orthogonal bases one obtains more freedom to choose the bell functions. Therefore, biorthogonal local trigonometric bases were introduced by Matviyenko [12] and by Jawerth and Sweldens [10], where one has orthogonality only between ψ_r^k and ψ_s^l for $r \neq s$. A more general approach is given by Chui and Shi [5, 6], where two arbitrary functions need not to be orthogonal. In [3] the results of Chui and Shi were applied to the bivariate case.

In this paper, we consider local trigonometric functions of the form

$$\psi_j^k(x) := w_j(x)\sqrt{\frac{2}{a_{j+1}-a_j}}\cos\left(\left(k+\tfrac{1}{2}\right)\frac{x-a_j}{a_{j+1}-a_j}\pi\right).$$

The following two questions arise, naturally.

1. What is the best choice for the splitting points a_j?
2. How one should choose the window functions w_j?

The first problem can be solved by an adaptive algorithm, where the splitting points are allowed to depend on the signal. Such algorithms where investigated for orthogonal bases by many authors, see e.g. Coifman and Wickerhauser [8, 13]. A disadvantage of this approach is that the determination of the best basis requires a higher amount of computation time. In particular for image processing in the bivariate situation it has to be investigated whether an adaptive algorithm is better than an explicit splitting in small uniform squares, like for JPEG or MPEG, with appropriate bell functions. This is a reason to discuss the second question. For that purpose one should investigate a large assortment of possible bell functions. Therefore, it is important to consider not only orthogonal bases, but more generally Riesz bases. In [10] and [12] window functions are introduced, for which the basis functions have good approximation properties for certain function classes. In particular, the approach of Jawerth and Sweldens [10] to consider smooth basis functions which reproduce constant functions is very powerful in many applications. Unfortunately, both papers [10, 12] are based on the assumption of orthogonality between ψ_r^k and ψ_s^l for $r \neq s$. This restriction leaves a lot of possible bell functions out of account.

The aim of this paper is to improve the approximation properties using the more flexible approach of Chui and Shi. To give a proper description of the approximation properties, we investigate the error for the approximation by local trigonometric bases. In particular, we are interested how the error behaves for a finer splitting of the real axis with a fixed number of basis functions for each interval. We show that for such approximations arbitrary smooth bell functions do not imply a small error. On the other hand, applications in image processing like e.g. JPEG work with a fixed number of basis functions for each interval. Therefore, in the second part of the paper we investigate local trigonometric bases which reproduce polynomials with a few basis functions.

The paper is organized as follows. In §2, we recall the approach investigated by Chui and Shi. As a main tool for our purposes, a folding operator is introduced. With the aid of this folding operator, we establish error estimates for the approximation with smooth basis functions in §3. In Theorem 3, it turns out that the approximation becomes better for smooth bell and test functions if we have sufficiently many basis functions per interval. Furthermore, we investigate the error for the uniform splitting in more detail. In §4, we show that local trigonometric bases which reproduce polynomials have better approximation properties. Therefore, we investigate in the following sections how one can construct such bases. In §5, we present a method to construct bell functions of arbitrary smoothness such that the constant is reproduced by only one bell function per interval. The reproduction of linear functions is investigated in §6. Note, that the new results in §5 and §6 could be only established using the very general approach of Chui and Shi. Finally, in §7 we show that for the two-overlapping setting there does not exist a local trigonometric basis such that a polynomial of higher degree can be reproduced by a finite number of basis functions per interval.

2 Biorthogonal local trigonometric bases

In the following, we will consider biorthogonal, local trigonometric bases in the so called two-overlapping setting, investigated by Chui and Shi [5, 6]. Here, we shortly recall the definition of these bases.

Let be given sequences $(a_j)_{j\in\mathbb{Z}}$, $(a_j^+)_{j\in\mathbb{Z}}$, $(a_j^-)_{j\in\mathbb{Z}}$ with

$$a_j < a_j^+ \le a_{j+1}^- < a_{j+1} \quad \text{and} \quad a_j - a_j^- = a_j^+ - a_j \quad \text{for } j \in \mathbb{Z}.$$

We define now bell functions $w_j : \mathbb{R} \to \mathbb{R}$ with support

$$[a_j, a_{j+1}] \subset \operatorname{supp}(w_j) \subset [a_j^-, a_{j+1}^+], \tag{1}$$

which yields a two-overlapping setting, i.e., $\operatorname{supp} w_j \cap \operatorname{supp} w_i$ has measure zero for $|j-i| > 1$. Further, we consider the functions

$$C_j^k(x) := \sqrt{\frac{2}{a_{j+1}-a_j}} \cos\left(\left(k+\tfrac{1}{2}\right) \frac{x-a_j}{a_{j+1}-a_j}\pi\right)$$

which for each $j \in \mathbb{Z}$ form an orthonormal basis of $L^2([a_j, a_{j+1}])$. Now we introduce the cosine wavelets

$$\psi_j^k(x) := w_j(x)\, C_j^k(x), \quad j \in \mathbb{Z},\ k \in \mathbb{N}_0.$$

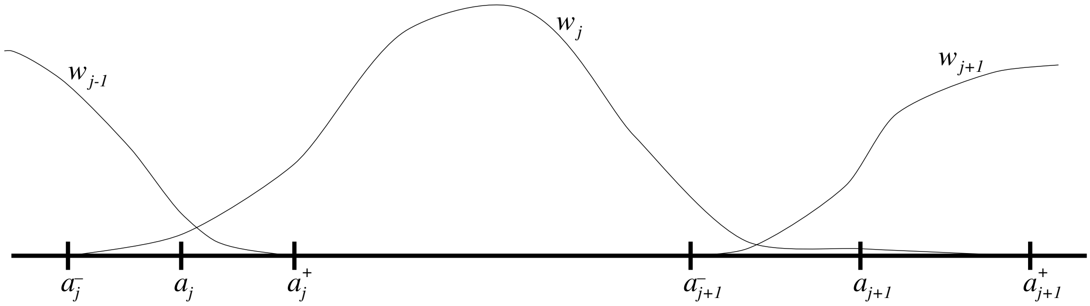


Figure 1: An example for bell functions in the two-overlapping setting.

Note, that the functions C_j^k can be replaced by

$$S_j^k(x) := \sqrt{\tfrac{2}{a_{j+1}-a_j}} \sin\left(\left(k+\tfrac{1}{2}\right)\tfrac{x-a_j}{a_{j+1}-a_j}\pi\right)$$

or

$$D_j^k(x) := \begin{cases} \sqrt{\tfrac{1}{a_{j+1}-a_j}}, & k=0;\ j \text{ even}, \\ \sqrt{\tfrac{2}{a_{j+1}-a_j}} \cos k\pi \tfrac{x-a_j}{a_{j+1}-a_j}, & k=1,2,\ldots;\ j \text{ even}, \\ \sqrt{\tfrac{2}{a_{j+1}-a_j}} \sin k\pi \tfrac{x-a_j}{a_{j+1}-a_j}, & k=1,2,\ldots;\ j \text{ odd} \end{cases}$$

as well as suited mixtures of them (cf. [2, 5, 6]). Because we obtain similar assertions for all these bases we will consider here only the cosine bases $\{C_j^k\}$.

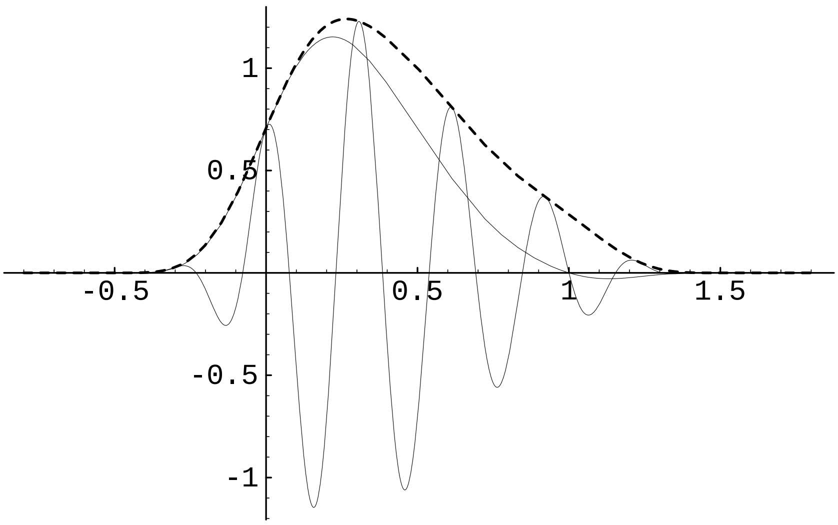


Figure 2: Bell function (dashed line) with corresponding basis functions ψ_j^0 and ψ_j^6 (solid lines).

To investigate the basis properties of ψ_j^k we introduce the matrices

$$\boldsymbol{M}_j(x) := \boldsymbol{M}_j^w(x) := \begin{pmatrix} w_j(x) & w_j(2a_j - x) \\ -w_{j-1}(x) & w_{j-1}(2a_j - x) \end{pmatrix}$$

(cf. [5, 6]), as well as the total folding Operator $\mathcal{T}_w$ (cf. [3, 10]) defined by

$$\begin{aligned} \mathcal{T}_w f(x) &:= w_j(x)\, f(x), & \text{for } a_j^+ < x < a_{j+1}^-, \\ \begin{pmatrix} \mathcal{T}_w f(x) \\ \mathcal{T}_w f(2a_j - x) \end{pmatrix} &:= \boldsymbol{M}_j(x) \begin{pmatrix} f(x) \\ f(2a_j - x) \end{pmatrix}, & \text{for } a_j < x < a_j^+. \end{aligned} \tag{2}$$

Note that this definition determines $\mathcal{T}_w f$ uniquely a.e. on $\mathbb{R}$. We want to emphasize that the introduction of $\mathcal{T}_w$ is motivated by the following observation. Since C_j^k is even with respect to a_j and odd with respect to a_{j+1} we obtain that

$$\int_{\mathbb{R}} \psi_j^k(x)\, f(x)\, dx = \int_{a_j}^{a_{j+1}} C_j^k(x)\, \mathcal{T}_w f(x)\, dx \tag{3}$$

if the integrals are well–defined. Furthermore, if f and w_j are smooth, then $T_w f$ is smooth in $[a_j, a_{j+1}]$ and there exists a smooth extension which is even with respect to a_j and odd with respect to a_{j+1}.

The functions ψ_j^k form a Riesz basis of $L^2(\mathbb{R})$ iff the folding operator $\mathcal{T}_w$ and its inverse operator are bounded, (cf. [3]). In [3, 5, 6] one finds also the following statements on the Riesz bounds. The best lower Riesz bound is given by

$$A_0 := \|T_w^{-1}\|_{L^2 \to L^2}^{-2} = \inf_{j \in \mathbb{Z}} \min\{A_j^1, A_j^2\}$$

with

$$\begin{aligned} A_j^1 &:= \operatorname*{ess\,inf}_{x \in (a_j^+, a_{j+1}^-)} |w_j(x)|^2, \\ A_j^2 &:= \operatorname*{ess\,inf}_{x \in (a_j^-, a_j^+)} \|\boldsymbol{M}_j^{-1}(x)\|_2^{-2} = \operatorname*{ess\,inf}_{x \in (a_j^-, a_j^+)} \frac{\Delta_j(x)}{2} - \sqrt{\frac{\Delta_j^2(x)}{4} - |\det \boldsymbol{M}_j(x)|^2}, \end{aligned}$$

where $\|\boldsymbol{A}\|_2 := \sqrt{\rho(\boldsymbol{A}^H \boldsymbol{A})}$ is the spectral norm of the matrix $\boldsymbol{A}$ and

$$\Delta_j(x) := |w_j(x)|^2 + |w_j(2a_j - x)|^2 + |w_{j-1}(x)|^2 + |w_{j-1}(2a_j - x)|^2.$$

Analogously, the best upper Riesz bound is given by

$$B_0 := \|T_w\|_{L^2 \to L^2}^2 = \sup_{j \in \mathbb{Z}} \max\{B_j^1, B_j^2\}$$

with

$$\begin{aligned} B_j^1 &:= \operatorname*{ess\,sup}_{x \in (a_j^+, a_{j+1}^-)} |w_j(x)|^2, \\ B_j^2 &:= \operatorname*{ess\,sup}_{x \in (a_j^-, a_j^+)} \|\boldsymbol{M}_j(x)\|_2^2 = \operatorname*{ess\,sup}_{x \in (a_j^-, a_j^+)} \frac{\Delta_j(x)}{2} + \sqrt{\frac{\Delta_j^2(x)}{4} - |\det \boldsymbol{M}_j(x)|^2}. \end{aligned}$$

In particular, the operator $\mathcal{T}_w$ is bounded iff all bell functions w_j are contained in $L^\infty(\mathbb{R})$.

Furthermore, dual basis functions are given by $\tilde{\psi}_j^k := \tilde{w}_j C_j^k$ with the dual bell

$$\tilde{w}_j(x) = \begin{cases} \frac{1}{w_j(x)}, & \text{if } a_j^+ \leq x < a_{j+1}^-, \\ \frac{w_{j-1}(2a_j - x)}{\det \boldsymbol{M}_j(x)}, & \text{if } a_j^- \leq x < a_j^+, \\ \frac{w_{j+1}(2a_{j+1} - x)}{\det \boldsymbol{M}_{j+1}(x)}, & \text{if } a_{j+1}^- \leq x < a_{j+1}^+, \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the question how the smoothness of the bell functions is connected with the smoothness of the dual bells. For this, we establish the following assertion.

Theorem 1 *Let the functions $\{\psi_{j,k}\}$ form a Riesz basis. The dual bell functions $\tilde{w}_j$, $j \in \mathbb{Z}$ are contained in $C^m(\mathbb{R})$ iff $w_j \in C^m(\mathbb{R})$ for all $j \in \mathbb{Z}$.*

Proof. Assume that $w_j \in C^m(\mathbb{R})$. From the Riesz basis property and the continuity of w_j it follows that there exists values $\varepsilon_j > 0$ such that $w_j(x) \neq 0$ for $x \in [a_j^+ - \varepsilon_j, a_{j+1}^- + \varepsilon_{j+1}]$ and $\det \boldsymbol{M}_j(x) \neq 0$ for $x \in [a_{j-1}^+ - \varepsilon_{j-1}, a_{j+1}^- + \varepsilon_{j+1}]$. Hence, $\tilde{w}_j$ is piecewise smooth. Indeed we have

$$\begin{aligned} f_1(x) &:= \frac{w_{j-1}(2a_j - x)}{\det \boldsymbol{M}_j(x)} \in C^m([a_{j-1}^+ - \varepsilon_{j-1}, a_{j+1}^- + \varepsilon_{j+1}]), \\ f_2(x) &:= \frac{1}{w_j(x)} \in C^m([a_j^+ - \varepsilon_j, a_{j+1}^- + \varepsilon_{j+1}]), \\ f_3(x) &:= \frac{w_{j+1}(2a_{j+1} - x)}{\det \boldsymbol{M}_{j+1}(x)} \in C^m([a_j^+ - \varepsilon_j, a_{j+2}^- + \varepsilon_{j+2}]). \end{aligned}$$

Obviously, the functions f_1, f_2, f_3 and 0 coincide piecewise, i.e.,

$$\begin{array}{ll} f_1(x) = 0 & \text{for } x \in [a_{j-1}, a_j^-], \\ f_2(x) = f_1(x) & \text{for } x \in [a_j^+, a_{j+1}], \\ f_3(x) = f_2(x) & \text{for } x \in [a_j, a_{j+1}^-], \\ f_3(x) = 0 & \text{for } x \in [a_{j+1}^+, a_{j+2}]. \end{array}$$

Therefore, it holds that $f_1^{(k)}(a_j^-) = 0$, $f_1^{(k)}(a_j^+) = f_2^{(k)}(a_j^+)$, $f_2^{(k)}(a_{j+1}^-) = f_3^{(k)}(a_{j+1}^-)$ and $f_3^{(k)}(a_{j+1}^+) = 0$ for $k = 0, \ldots, m$. Hence, we obtain $\tilde{w}_j \in C^m(\mathbb{R})$. Note that this assertion holds even if $a_j^+ = a_{j+1}^-$, because none of the intervals above shrinks to one point in this case.

Analogously, we can show that $w_j \in C^m(\mathbb{R})$ follows from $\tilde{w}_j \in C^m(\mathbb{R})$, because the functions w_j are the dual bells with respect to $\tilde{w}_j$. ■

3 Error estimates

3.1 The non-uniform grid

Let us now investigate the approximation properties of local trigonometric bases. Here we will consider the error in the norm of $L^2(\mathbb{R})$ for the approximation of a smooth function by a finite number of smooth basis functions for each interval. Because the bell functions have compact support one can obtain analogous local estimates in the same way.

In the sequel, we consider the Sobolev spaces $W_p^m(\mathbb{R})$ with the norm

$$\|f\|_{W_p^m(\mathbb{R})} := \left(\sum_{\nu=0}^{m} \|f^{(\nu)}\|_{L^p(\mathbb{R})}^p \right)^{1/p} .$$

For our further investigations we need the following assertion.

Lemma 2 *If $f \in W_2^m(\mathbb{R})$ and $w_j \in W_\infty^m(\mathbb{R})$ then*

$$\sum_{j\in\mathbb{Z}} \|D^m(\mathcal{T}_w f)\|_{L^2([a_j,a_{j+1}])}^2 \le \sum_{\nu=0}^{m} \binom{m}{\nu}^2 \|\mathcal{T}_{w^{(\nu)}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}^2 \|f^{(m-\nu)}\|_{L^2(\mathbb{R})}^2 < \infty.$$

Proof. Because the matrices $D^\nu M_j^w(x)$ and $M_j^{w^{(\nu)}}(x)$ only differ in the sign of the second column, we have the equality

$$\|D^\nu M_j^w(x)\|_2 = \|M_j^{w^{(\nu)}}(x)\|_2$$

and it follows that

$$\|\mathcal{T}_w^{(\nu)}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|\mathcal{T}_{w^{(\nu)}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} ,$$

where $T_w^{(\nu)}$ is the operator we obtain by replacing in (2) w_j and M_j by $w^{(\nu)}$ and $D^\nu M_j$, respectively.

With the Leibniz formula we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \|D^m(\mathcal{T}_w f)\|_{L^2([a_j,a_{j+1}])}^2 &= \sum_{\nu=0}^{m} \|\tbinom{m}{\nu} \mathcal{T}_w^{(\nu)} f^{(m-\nu)}\|_{L^2(\mathbb{R})}^2 \\ &\le \sum_{\nu=0}^{m} \binom{m}{\nu}^2 \|\mathcal{T}_{w^{(\nu)}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}^2 \|f^{(m-\nu)}\|_{L^2(\mathbb{R})}^2 .\end{aligned}$$

From $w_j \in W_\infty^\nu(\mathbb{R})$ we conclude that $\mathcal{T}_{w^{(\nu)}}$ is bounded and the lemma is shown. ∎

If $\{\psi_j^k\}$ is a Riesz basis then any function $f \in L^2(\mathbb{R})$ has the representation

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} f_j^k \, \psi_j^k$$

with the coefficients

$$f_j^k = \langle f, \tilde{\psi}_j^k \rangle = \int_{a_j}^{a_{j+1}} \mathcal{T}_{\tilde{w}} f(x) C_j^k(x) \, dx \, .$$

Now we consider the approximation

$$S_{\mathbf{N}} f := \sum_{j\in\mathbb{Z}} \sum_{k<N_j} f_j^k \, \psi_j^k \tag{4}$$

with $\mathbf{N} = (N_j)_{j\in\mathbb{Z}}$ and establish the following error estimate.

Theorem 3 *Let $f \in W_2^m(\mathbb{R})$ and $\tilde{w}_j \in W_\infty^m(\mathbb{R})$. Then the approximation error for the partial sum in* (4) *is given by*

$$\|f - S_{\mathbf{N}} f\|_{L^2(\mathbb{R})} \leq \frac{\sqrt{B_0}}{\pi^m} \left(\sup_{j\in\mathbb{Z}} \left(\frac{h_j}{N_j + \frac{1}{2}} \right) \right)^m \max_{\nu \leq m} \left(\binom{m}{\nu} \|\mathcal{T}_{\tilde{w}^{(\nu)}}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \right) \|f\|_{W_2^m(\mathbb{R})} . \quad (5)$$

Proof. If $f \in W_2^m(\mathbb{R})$ and $\tilde{w}_j \in W_\infty^m(\mathbb{R})$ it follows that $\mathcal{T}_{\tilde{w}} f \in W_2^m([a_j, a_{j+1}])$, $j \in \mathbb{Z}$. Because $\mathcal{T}_{\tilde{w}} f$ has a smooth even (resp. odd) extension in the left (resp. right) endpoint of each interval (a_j, a_{j+1}) it holds further that

$$\begin{aligned} D^{2\nu+1}\, \mathcal{T}_{\tilde{w}} f(a_j + 0) &= 0 \qquad \text{for} \quad \nu \leq \tfrac{m-2}{2}, \\ D^{2\nu}\, \mathcal{T}_{\tilde{w}} f(a_{j+1} - 0) &= 0 \qquad \text{for} \quad \nu \leq \tfrac{m-1}{2}. \end{aligned}$$

With these assertions we obtain by partial integration

$$f_j^k = \frac{h_j^m}{\pi^m \left(k + \frac{1}{2}\right)^m} \int_{a_j}^{a_{j+1}} D^m\, \mathcal{T}_{\tilde{w}} f(x) \sqrt{\tfrac{2}{h_j}}\, c_m \left(\left(k + \tfrac{1}{2}\right) \tfrac{\pi x}{h_j} \right) dx$$

with $c_m(x) := \frac{d^m}{dx^m} \cos x$. Because

$$\left\{ \sqrt{\tfrac{2}{h_j}} c_m \left(\left(k + \tfrac{1}{2}\right) \frac{\pi\, \cdot}{h_j} \right) :\ k \in \mathbb{N}_0 \right\}$$

is an orthonormal basis of $L^2([a_j, a_{j+1}])$ we conclude from Parseval's equation

$$\sum_{k=0}^{\infty} \left| \int_{a_j}^{a_{j+1}} D^m\, \mathcal{T}_{\tilde{w}} f(x) \sqrt{\tfrac{2}{h_j}}\, c_m \left(\left(k + \tfrac{1}{2}\right) \tfrac{\pi x}{h_j} \right) dx \right|^2 = \|D^m\, \mathcal{T}_{\tilde{w}} f\|_{L^2([a_j, a_{j+1}])} < \infty . \qquad (6)$$

With the Riesz stability and the behavior of the coefficients f_j^k we can estimate the approximation error by

$$\begin{aligned} \|f - S_{\mathbf{N}} f\|_{L^2(\mathbb{R})}^2 \ &\leq\ B_0 \sum_{j\in\mathbb{Z}} \sum_{k \geq N_j} \left| f_j^k \right|^2 \\ &= \frac{B_0}{\pi^{2m}} \sum_{j\in\mathbb{Z}} h_j^{2m} \sum_{k\geq N_j} \frac{1}{\left(k + \frac{1}{2}\right)^{2m}} \left| \int_{a_j}^{a_{j+1}} D^m\, \mathcal{T}_{\tilde{w}} f(x) \sqrt{\tfrac{2}{h_j}}\, c_m \left(\left(k + \tfrac{1}{2}\right) \tfrac{\pi x}{h_j} \right) dx \right|^2 \\ &\leq \frac{B_0}{\pi^{2m}} \sum_{j\in\mathbb{Z}} \left(\frac{h_j}{N_j + \frac{1}{2}} \right)^{2m} \sum_{k\geq N_j} \left| \int_{a_j}^{a_{j+1}} D^m\, \mathcal{T}_{\tilde{w}} f(x) \sqrt{\tfrac{2}{h_j}}\, c_m \left(\left(k + \tfrac{1}{2}\right) \tfrac{\pi x}{h_j} \right) dx \right|^2 . \end{aligned}$$

Using (6) and Lemma 2 we obtain

$$\|f - S_{\mathbf{N}} f\|_{L^2(\mathbb{R})} \leq \frac{\sqrt{B_0}}{\pi^m} \left(\sup_{j\in\mathbb{Z}} \left(\frac{h_j}{N_j + \frac{1}{2}} \right) \right)^m \left(\sum_{\nu=0}^{m} \binom{m}{\nu}^2 \|\mathcal{T}_{\tilde{w}^{(\nu)}}\|_{L^2 \to L^2}^2 \|f^{(m-\nu)}\|_{L^2}^2 \right)^{\frac{1}{2}} .$$

By Hölder's inequality the assertion (5) follows. ■

The theorem states that for a fixed grid (a_j) and fixed bells w_j the approximation order is $\mathcal{O}(N^{-m})$, where $N = \max_j N_j$. However, many applications allow only a fixed maximal N_j. In this case, it would be useful to have a better approximation for a lower gridsize h_j. But if the grid changes the bell functions have to change, too. We study this question in the next section. To simplify the notation, in the sequel we restrict ourselves to equidistant grids with shift invariant bell functions.

3.2 The equidistant setting

For a fixed $h \in \mathbb{R}$, we consider now the equidistant gridpoints $a_j = j\,h$ and the bell functions $w_j := w\left(\frac{\cdot}{h} - j\right)$. To obtain a two-overlapping setting we have to ensure, that $\operatorname{supp} w \subset [-\frac{1}{2}, \frac{3}{2}]$. Therefore, we set, without loss of generality, $a_j^+ = a_{j+1}^- = (j + \frac{1}{2})h$. Then, the total folding operator can be written as

$$\begin{pmatrix} \mathcal{T}_w f(x) \\ \mathcal{T}_w f(2jh - x) \end{pmatrix} = \boldsymbol{M}\left(\tfrac{\cdot}{h} - j\right) \begin{pmatrix} f(x) \\ f(2jh - x) \end{pmatrix}, \quad jh < x < \left(j + \tfrac{1}{2}\right) h$$

with

$$\boldsymbol{M}(x) := \boldsymbol{M}^w(x) := \begin{pmatrix} w(x) & w(-x) \\ -w(x+1) & w(1-x) \end{pmatrix}.$$

Thus, the upper and lower Riesz bounds are

$$A_0 = \operatorname*{ess\,inf}_{x \in [0,1/2]} \|\boldsymbol{M}^{-1}(x)\|_2^{-2} \quad \text{and} \quad B_0 = \operatorname*{ess\,sup}_{x \in [0,1/2]} \|\boldsymbol{M}(x)\|_2^2. \tag{7}$$

Furthermore, we obtain the dual bell functions $\tilde{w}_j := \tilde{w}\left(\frac{\cdot}{h} - j\right)$, where

$$\tilde{w}(x) = \begin{cases} \frac{w(1-x)}{\det \boldsymbol{M}(x)}, & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{w(1-x)}{\det \boldsymbol{M}(x-1)}, & \text{if } \frac{1}{2} < x < \frac{3}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Analogously to Theorem 3, we conclude the following error estimate.

Corollary 4 *For $N_j = N \in \mathbb{N}$, $f \in W_2^m(\mathbb{R})$ and $w \in W_\infty^m(\mathbb{R})$ we can estimate the approximation error of the partial sum $S_{\mathbf{N}} f$ by*

$$\|f - S_{\mathbf{N}} f\|_{L^2(\mathbb{R})} \le \frac{\sqrt{B_0}}{\pi^m} \left(N + \tfrac{1}{2}\right)^{-m} \max_{\nu \le m} \left(\binom{m}{\nu} h^{m-\nu} \operatorname*{ess\,sup}_{x \in [0,1/2]} \|M^{\tilde{w}^{(\nu)}}(x)\|_2 \right) \|f\|_{W_2^m(\mathbb{R})}. \tag{9}$$

Proof. With $\tilde{w}_j^{(\nu)} = h^{-\nu} \tilde{w}^{(\nu)}\left(\frac{\cdot}{h} - j\right)$ and $N_j = N$ the assertion (9) follows immediately from Theorem 3. ■

This corollary does not imply a good approximation order for a fixed N and $h \to 0$, because the right-hand side in (9) is a positive constant for sufficiently small h. In particular, for an arbitrary bell function we cannot expect that the approximation order depends on h, as stated in the following.

Lemma 5 *Let $N \in \mathbb{N}$ be fixed. There exists a bell function $\tilde{w} \in W^m_\infty(\mathbb{R})$ such that*

$$\sup_{\|f\|_{W_2^m}=1} \limsup_{h\to 0} \|f - S_N f\|_{L^2(\mathbb{R})} \geq c > 0.$$

Proof. We choose a function $f^\star \in W^m_2(\mathbb{R})$ with $\|f^\star\|_{W_2^m} = 1$ and $f^\star(x) = c_0$ for $-\frac{1}{2} < x < \frac{3}{2}$. For $h = \frac{1}{l}$, $l \in \mathbb{N}$ we estimate

$$\begin{aligned}
\|f^\star - S_N f^\star\|^2_{L^2(\mathbb{R})} &\geq B_0 \sum_{j=0}^{l-1} \sum_{k\geq N} |\langle f^\star, \psi_j^k\rangle|^2 = B_0 \sum_{j=0}^{l-1} \sum_{k\geq N} h c_0 \left| \int_{-1/2}^{3/2} \tilde{w}(x)\, C_j^k(x)\, dx \right|^2 \\
&= B_0 c_0 \sum_{k\geq N} \left| \int_{-1/2}^{3/2} \tilde{w}(x)\, C_j^k(x)\, dx \right|^2 = c^2.
\end{aligned}$$

Obviously, one can find a bell $\tilde{w}$, such that for every $N \in \mathbb{N}$ it exists a $k \geq N$ with

$$\int_{-1/2}^{3/2} \tilde{w}(x)\, C_j^k(x)\, dx \neq 0 \tag{10}$$

and thus $c > 0$. Because c is independent of l we have

$$\lim_{l\to\infty} \|f^\star - S_N f^\star\|_{L^2(\mathbb{R})} \geq c$$

and the assertion follows immediately. ■

Therefore, we investigate how the bell function w resp. the dual bell $\tilde{w}$ has to be chosen, such that our cosine wavelets have good approximation properties for a fixed small N, too. Obviously, this can be achieved only if (10) does not hold for any $k \geq N$ for a sufficiently large N.

4 The reproduction of polynomials

An important property of a basis is how polynomials are represented. Let Π_n be the class of (algebraic) polynomials of degree less or equal n. We say a basis reproduces the polynomials $p \in \Pi_n$, if on any finite domain these polynomials are represented with a finite number of basis functions. If a function f can be approximated well by a polynomial for each interval (a_j^-, a_{j+1}^+) we can expect a good approximation by the functions $\{\psi_{j,k} :\ j \in \mathbb{Z},\ k < N\}$. For example, suppose we have an image with a constant background. Surely, we do not want to spend many coefficients in the representation of the background. In what follows, we show that for a basis which reproduces polynomials one obtains a good approximation even if N is small.

Theorem 6 *Let $N_0 \in \mathbb{N}$ and a bell function w be given such that each polynomial $p \in \Pi_n$ has the representation*

$$p = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N_0-1} \langle p, \tilde{\psi}_j^k \rangle \, \psi_j^k. \tag{11}$$

Then for $N \geq N_0$, $m > n$, $f \in W_2^m(\mathbb{R})$, $\tilde{w} \in W_\infty^m(\mathbb{R})$ and for sufficiently small h it holds that

$$\|f - S_N f\|_{L^2(\mathbb{R})} \leq C \frac{h^{n+1}}{\left(N+\frac{1}{2}\right)^m} \|f^{(n+1)}\|_{W_2^{m-n-1}(\mathbb{R})}, \tag{12}$$

where C depends only on $\tilde{w}$, m and n.

Proof. First we will estimate the terms $\sum_{k\geq N} |f_j^k|^2$ separately. Without loss of generality, we restrict ourselves to $j = 0$. Because $m > n$ the function f is in $W_2^{n+1}(\mathbb{R})$ and has the Taylor expansion

$$f\left(x+\tfrac{h}{2}\right) = \underbrace{\sum_{\nu=0}^{n} \frac{f^{(\nu)}\left(\frac{h}{2}\right)}{k!} x^k}_{u_0(x+h/2)} + \underbrace{\int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}\left(t+\tfrac{h}{2}\right)\, dt}_{u_1(x+h/2)} .$$

Since u_0 is a polynomial of degree n we obtain for $k \geq N_0$ the coefficients $f_0^k = \langle u_1, \tilde{\psi}_{0,k}\rangle$. Analogously to Theorem 3, one shows

$$\sum_{k\geq N} |f_0^k|^2 \leq \left(\pi\left(N+\tfrac{1}{2}\right)\right)^{-2m} \sum_{\nu=0}^{m} \binom{m}{\nu}^2 \operatorname*{ess\,sup}_{x\in(0,1/2)} \left\| \boldsymbol{M}^{\tilde{w}^{(m-\nu)}}(x) \right\|_2^2 h^{2\nu} \left\| u_1^{(\nu)} \right\|_{L^2([-h/2,3h/2])}^2 . \tag{13}$$

The higher order derivatives of u_1 are given by

$$u_1^{(\nu)}\left(x+\tfrac{h}{2}\right) = \begin{cases} \displaystyle\int_0^x \frac{(x-t)^{n-\nu}}{(n-\nu)!} f^{(n+1)}\left(t+\tfrac{h}{2}\right)\, dt, & \text{for } \nu \leq n, \\ f^{(\nu)}\left(x+\tfrac{h}{2}\right), & \text{for } \nu > n. \end{cases}$$

Applying the Cauchy-Schwarz inequality, we obtain for $\nu \leq n$ that

$$\begin{aligned} \|u_1^{(\nu)}\|_{L^2([-h/2,3h/2])}^2 &= \int_{-h}^{h} \left| \int_0^x \frac{(x-t)^{n-\nu}}{(n-\nu)!} f^{(n+1)}\left(t+\tfrac{h}{2}\right)\, dt \right|^2 dx \\ &\leq \frac{1}{((n-\nu)!)^2} \int_{-h}^{h} \left| \int_0^x (x-t)^{2(n-\nu)}\, dt \right| \left| \int_0^x \left| f^{(n+1)}\left(t+\tfrac{h}{2}\right)\right| dt \right| dx. \end{aligned}$$

Using Hölder's inequality it follows that

$$\begin{aligned}
\|u_1^{(\nu)}\|^2_{L^2([-h/2,3h/2])} &\le \frac{1}{((n-\nu)!)^2} \int\limits_{-h}^{h} \left| \int\limits_0^x (x-t)^{2(n-\nu)}\,dt \right| dx \int\limits_{-h}^{h} \left| f^{(n+1)}\left(t+\tfrac{h}{2}\right)\right| dt \\
&= \frac{1}{((n-\nu)!)^2}\, 2\int\limits_0^h \frac{x^{2(n-\nu)+1}}{2(n-\nu)+1}\,dx\, \left\| f^{(n+1)} \right\|^2_{L^2([-h/2,3h/2])} \\
&= \frac{h^{2(n-\nu+1)}}{((n-\nu)!)^2(2n-2\nu+1)(n-\nu+1)} \left\| f^{(n+1)} \right\|^2_{L^2([-h/2,3h/2])}.
\end{aligned}$$

Inserting this into (13), we obtain

$$\begin{aligned}
&\|f - S_N f\|^2_{L^2(\mathbb{R})} \le B_0 \sum_{j\in\mathbb{Z}} \sum_{k\ge N} |f_j^k|^2 \\
&\le 2B_0 \left(\pi\left(N+\tfrac{1}{2}\right)\right)^{-2m} \left(\sum_{\nu=0}^{n} \binom{m}{\nu}^2 \operatorname*{ess\,sup}_{x\in(0,1/2)} \|\boldsymbol{M}_{\tilde{w}^{(m-\nu)}}(x)\|_2^2 \, \frac{h^{2(n+1)}}{((n-\nu+1)!)^2} \left\| f^{(n+1)} \right\|^2_{L^2(\mathbb{R})} \right. \\
&\qquad \left. + \sum_{\nu=n+1}^{m} \binom{m}{\nu}^2 \operatorname*{ess\,sup}_{x\in(0,1/2)} \|\boldsymbol{M}_{\tilde{w}^{(m-\nu)}}(x)\|_2^2 \; h^{2\nu} \left\| f^{(\nu)} \right\|^2_{L^2(\mathbb{R})} \right).
\end{aligned}$$

For sufficiently small h the assertion follows immediately. ∎

5 Reproduction of constants

We study now the question how a constant function is represented. In particular, we demand that the constant is already reproduced by the functions $\psi_{j,0}$ (cf. [10]), i.e.,

$$1 = \sum_{j\in\mathbb{Z}} \psi_{j,0}(x), \quad x \in \mathbb{R}. \tag{14}$$

By (3), this is equivalent to

$$\delta_{k,0} = \langle \tilde{\psi}_j^k, p_0 \rangle = \int\limits_j^{j+1} \mathcal{T}_{\tilde{w}} p_0(x)\, C_j^k(x)\,dx$$

with $p_0(x) \equiv 1$. Because the functions $C_j^k = C_{0,k}(\cdot - j)$, $k \in \mathbb{N}_0$, form an orthonormal basis of $L^2([j,j+1])$, the equality (14) is also equivalent to

$$T_{\tilde{w}} p_0(x) = C_{0,k}(x) = \sqrt{2}\cos\tfrac{\pi}{2}x, \quad x \in (0,1). \tag{15}$$

In the following, we want to determine smooth dual bell functions, which satisfy (15).

To describe such a bell, we consider the left part

$$l := \tilde{w}\Big|_{[-\frac{1}{2},\frac{1}{2}]}$$

and the right part

$$r := \tilde{w}(\cdot + 1)\Big|_{[-\frac{1}{2},\frac{1}{2}]}$$

of $\tilde{w}$ separately. From (15), we obtain

$$l(x) + l(-x) = \sqrt{2}\cos\tfrac{\pi}{2}x \quad \text{and} \quad r(x) - r(-x) = -\sqrt{2}\sin\tfrac{\pi}{2}x\,. \tag{16}$$

If we demand a symmetric bell function, i.e., $l(x) = r(-x)$, then we obtain only the function $\tilde{w}_0(x) = \sin(\frac{\pi}{2}x + \frac{\pi}{4})$ for $x \in (-\frac{1}{2}, \frac{3}{2})$ (cf. [10, 12]). This bell function generates an orthonormal basis. Unfortunately, $\tilde{w}_0$ is continuous but not continuously differentiable. To obtain a smoother bell we have to cut out the symmetry. It turns out, that with a non-symmetric bell we can construct cosine wavelets of arbitrary smoothness which reproduce a constant.

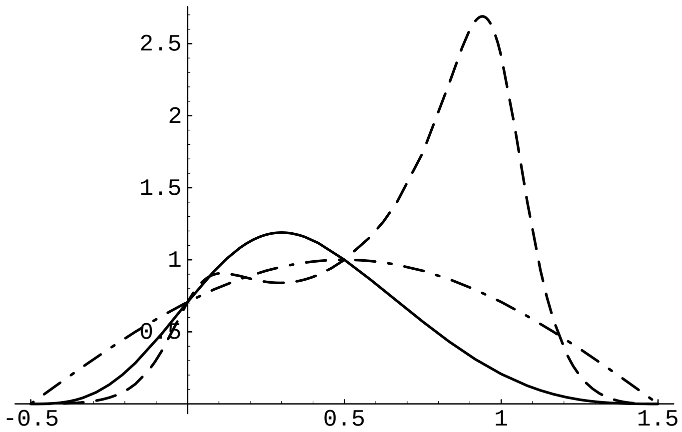


Figure 3: A bell function w (dashed line) and it's dual bell $\tilde{w}$ (solid line) for which the basis functions ψ_j^0 reproduce a constant. We have chosen $l_1(x) = \frac{1}{4}\sin 2\pi x$ and $r_1(x) = -\frac{1}{2}\cos\pi x$ such that $w, \tilde{w} \in C^2(\mathbb{R})$. The dashed-dotted line is the graph of $\tilde{w}_0(x) = \sin(\frac{\pi}{2}x + \frac{\pi}{4})$.

Theorem 7 *Assume the functions $l_1, r_1 \in C^m([-\frac{1}{2}, \frac{1}{2}])$ satisfy $l_1(x) = -l_1(-x)$ and $r_1(x) = r_1(-x)$ as well as*

$$l_1^{(k)}\left(-\tfrac{1}{2}\right) = \frac{d^k}{dx^k}\sin(\tfrac{\pi}{2}x + \tfrac{\pi}{4})\Big|_{x=-\frac{1}{2}}, \tag{17}$$

$$r_1^{(k)}\left(\tfrac{1}{2}\right) = \frac{d^k}{dx^k}\sin(\tfrac{\pi}{2}x + \tfrac{\pi}{4})\Big|_{x=\frac{3}{2}} \tag{18}$$

for $k = 0, \ldots, m$. Then the dual bell $\tilde{w}$ given by

$$\tilde{w}(x) = \begin{cases} \sin(\frac{\pi}{2}x + \frac{\pi}{4}) + l_1(x) & \textit{for } -\frac{1}{2} < x < \frac{1}{2}, \\ \sin(\frac{\pi}{2}x + \frac{\pi}{4}) + r_1(x-1) & \textit{for } \frac{1}{2} < x < \frac{3}{2}, \\ 0 & \textit{otherwise} \end{cases} \tag{19}$$

satisfies (14) and is contained in $C^m(\mathbb{R})$.

Proof. Because l_1 is odd and r_1 is even it holds that $l_1(x) + l_1(-x) = 0$ and $r_1(x) - r_1(-x) = 0$. Hence,

$$\mathcal{T}_{\tilde{w}} p_0(x) = \mathcal{T}_{\tilde{w}_0} p_0(x) = \sqrt{2} \cos \tfrac{\pi}{2} x,$$

i.e., (14) holds. Obviously, $\tilde{w}$ is piecewise smooth. Using that l_1 is odd and r_1 is even, one shows that $l_1^{(k)}\left(\frac{1}{2}\right) = r_1^{(k)}\left(-\frac{1}{2}\right)$ follows from (17) and (18) and therefore, it is shown that $\tilde{w} \in C^m(\mathbb{R})$. ■

6 Reproduction of linear functions

Now we want additionally, that our basis reproduces linear functions, i.e., for all $a, b \in \mathbb{R}$ there exist coefficients $\alpha_{j,k} \in \mathbb{R}$ such that

$$c_1 x + c_0 = \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \alpha_{j,k} \psi_j^k(x), \quad x \in \mathbb{R}.$$

This implies in particular, that our basis has to reproduce a constant. Therefore, let us assume that $\tilde{w}$ satisfies (15). Because $\mathcal{T}_w$ is linear, we need further that

$$\langle \tilde{\psi}_j^k, p_1(\cdot - j) \rangle = \int_0^1 \mathcal{T}_{\tilde{w}} p_1(x)\, C_0^k(x)\, dx = 0, \quad \text{for } k \geq N$$

with $p_1(x) = x$. Hence, we have to look for a bell $\tilde{w}$ which satisfies (15) and

$$\mathcal{T}_{\tilde{w}} p_1(x) = \sum_{k=0}^{N-1} a_k \, \cos\left(k + \tfrac{1}{2}\right) \pi x, \tag{20}$$

for certain $N \in \mathbb{N}$ and $a_k \in \mathbb{R}$. Then one obtains the coefficients $\alpha_{j,k}$ by $\alpha_{j,k} = \delta_{0,k}(c_0 + jc_1) + a_k c_1$.

The equality (20) leads to

$$x(l(x) - l(-x)) = \sum_{k=0}^{N-1} a_k \, \cos\left(k + \tfrac{1}{2}\right) \pi x$$

and

$$\begin{aligned} \sum_{k=0}^{N-1} (-1)^{k+1} a_k \, \sin\left(k + \tfrac{1}{2}\right) \pi x &= r(x) - r(-x) + x(r(x) + r(-x)) \\ &= -\sqrt{2} \sin \tfrac{\pi}{2} x + x(r(x) + r(-x)) \,. \end{aligned}$$

Together with (16), we conclude

$$2x\, l(x) = \sqrt{2}\, x \cos \tfrac{\pi}{2} x + \sum_{k=0}^{N-1} a_k \cos(k + \tfrac{1}{2}) \pi x, \tag{21}$$

$$2x\, r(x) = \sqrt{2}(1 - x) \sin \tfrac{\pi}{2} x - \sum_{k=0}^{N-1} (-1)^k a_k \, \sin(k + \tfrac{1}{2}) \pi x. \tag{22}$$

To obtain good approximation order we have to impose a certain smoothness on the bell. On the other hand, linear functions should be reproduced by a small number of basis functions. Therefore, we want to construct bell functions of highest smoothness for a fixed N.

Theorem 8 *For $N \geq 2$ and $m = N - 1$, there exists one and only one bell function $\tilde{w} \in W_\infty^m(\mathbb{R})$ with $\operatorname{supp} \tilde{w} \subset [-\frac{1}{2}, \frac{3}{2}]$ such that*

$$\begin{aligned} \langle p_0, \tilde{\psi}_j^k \rangle &= \delta_{k,0} \\ \langle p_1, \tilde{\psi}_j^k \rangle &= 0, \quad k \geq N\,. \end{aligned}$$

This function is given by

$$\tilde{w}(x) = \begin{cases} \frac{\sqrt{2}}{2} \cos \frac{\pi}{2} x + \sum\limits_{k=0}^{N-1} a_k \frac{\cos\left(k+\frac{1}{2}\right)\pi x}{2x} & \text{for } x \in [-\frac{1}{2}, \frac{1}{2}) \backslash \{0\}, \\ \frac{\sqrt{2}}{2} & \text{for } x = 0, \\ \frac{\sqrt{2}}{2} \frac{x-2}{x-1} \cos \frac{\pi}{2} x + \sum\limits_{k=0}^{N-1} a_k \frac{\cos\left(k+\frac{1}{2}\right)\pi x}{2(x-1)} & \text{for } x \in [\frac{1}{2}, \frac{3}{2}) \backslash \{1\}, \\ \frac{\pi}{4} \left(\sqrt{2} - \sum\limits_{k=0}^{N-1} (-1)^k (2k+1) a_k \right) & \text{for } x = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{23}$$

where the coefficients a_k are the solution of the linear system of equations

$$\sum_{k=0}^{N-1} a_k = 0 \tag{24}$$

$$\sum_{k=0}^{N-1} s(k-\mu)\,(2k+1)^\mu a_k = \frac{\pi\, s(-\mu) - 4\mu\, s(1-\mu)}{\sqrt{2}\pi}, \quad \mu = 0, \ldots, m-1 \tag{25}$$

with

$$s(k) = \sqrt{2} \cos\left(k + \tfrac{1}{2}\right) \tfrac{\pi}{2} = \begin{cases} 1 & \text{for } k \equiv 0 \mod 4, \\ -1 & \text{for } k \equiv 1 \mod 4, \\ -1 & \text{for } k \equiv 2 \mod 4, \\ 1 & \text{for } k \equiv 3 \mod 4. \end{cases}$$

Proof. **1.** First we show $\tilde{w} \in W_\infty^m$ for each solution of (24), (25). From (21) with $x = 0$ it follows that $\tilde{w}$ is bounded iff (24) holds. In this case,

$$l(x) = \begin{cases} \frac{\sqrt{2}}{2} & \text{for } x = 0, \\ \frac{\sqrt{2}}{2} \cos \frac{\pi}{2} x + \sum\limits_{k=0}^{N-1} a_k \frac{\cos\left(k+\frac{1}{2}\right)\pi x}{2x} & \text{otherwise} \end{cases} \tag{26}$$

and

$$r(x) = \begin{cases} \frac{\pi}{4} \left(\sqrt{2} - \sum_{k=0}^{N-1} (-1)^k (2k+1) a_k \right) & \text{for } x = 0, \\ \frac{\sqrt{2}}{2} \left(\frac{1}{x} - 1 \right) \sin \frac{\pi}{2} x - \sum\limits_{k=0}^{N-1} (-1)^k a_k \frac{\sin\left(k+\frac{1}{2}\right)\pi x}{2x} & \text{otherwise} \end{cases} \tag{27}$$

are both from $C^\infty(\mathbb{R})$. Therefore, $\tilde{w} \in C^{m-1}(\mathbb{R})$ iff

$$l^{(\mu)}\left(-\tfrac{1}{2}\right) = 0, \quad l^{(\mu)}\left(\tfrac{1}{2}\right) = r^{(\mu)}\left(-\tfrac{1}{2}\right), \quad r^{(\mu)}\left(\tfrac{1}{2}\right) = 0, \quad \mu = 0, \ldots, m-1\,. \tag{28}$$

In the following we denote $c_\mu(x) := \frac{d^\mu}{dx^\mu} cos(x)$. By induction one deduces from (26) and (27)

$$\begin{aligned} l^{(\mu)}(x) &= \frac{1}{2x} \sum_{k=0}^{N-1} a_k \left(\frac{(2k+1)\pi}{2}\right)^\mu c_\mu\left(\left(k+\tfrac{1}{2}\right)\pi x\right) + \left(\frac{\pi}{2}\right)^\mu \frac{\sqrt{2}}{2} c_\mu\left(\frac{\pi}{2}x\right) \\ &\quad - \frac{\mu}{x}\left(l^{(\mu-1)}(x) - \left(\frac{\pi}{2}\right)^{\mu-1} \frac{\sqrt{2}}{2} c_{\mu-1}\left(\frac{\pi}{2}x\right)\right) \end{aligned} \tag{29}$$

and

$$\begin{aligned} r^{(\mu)}(x) &= \frac{1}{2x} \sum_{k=0}^{N-1} (-1)^k a_k \left(\frac{(2k+1)\pi}{2}\right)^\mu c_{\mu+1}\left(\left(k+\tfrac{1}{2}\right)\pi x\right) \\ &\quad + \left(\frac{\pi}{2}\right)^\mu \frac{\sqrt{2}}{2}\left(1 - \frac{1}{x}\right) c_{\mu+1}\left(\frac{\pi}{2}x\right) \\ &\quad - \frac{\mu}{x}\left(r^{(\mu-1)}(x) - \left(\frac{\pi}{2}\right)^{\mu-1} \frac{\sqrt{2}}{2} c_\mu\left(\frac{\pi}{2}x\right)\right) \end{aligned} \tag{30}$$

for $\mu \in \mathbb{N}_0$, $x \neq 0$ and $l^{(-1)}(x) := 0$ resp. $r^{(-1)}(x) := 0$.

Now we consider the derivatives of $l(x)$ and $r(x)$ for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. For this purpose we use several equalities for $s(k)$ in the following. Obviously, the correctness of these equalities for all integer numbers follows from the correctness for 0, 1, 2 and 3.

From (29) and with $c_\mu(-(k+\frac{1}{2})\pi/2) = s(k-\mu)$ it follows that

$$\begin{aligned} &-\left(\tfrac{2}{\pi}\right)^\mu \sqrt{2}\;\; l^{(\mu)}\left(-\tfrac{1}{2}\right) \\ &= \sum_{k=0}^{N-1} s(k-\mu)\,(2k+1)^\mu a_k - \frac{\pi\, s(-\mu) - 4\mu\, s(1-\mu)}{\sqrt{2}\pi} - 2\mu \left(\tfrac{2}{\pi}\right)^\mu \sqrt{2}\;\; l^{(\mu-1)}\left(-\tfrac{1}{2}\right)\,, \end{aligned}$$

such that (25) is equivalent to $l^{(\mu)}(-\frac{1}{2}) = 2\mu\, l^{(\mu-1)}(-\frac{1}{2})$, $\mu = 0, \ldots, m-1$, and therefore also equivalent to $l^{(\mu)}(-\frac{1}{2}) = 0$, $\mu = 0, \ldots, m-1$. Further, with $c_\mu((k+\frac{1}{2})\pi/2) = s(k+\mu)$ we obtain from (29) and (30)

$$\begin{aligned} &\left(\tfrac{2}{\pi}\right)^\mu \sqrt{2}\;\; \left(l^{(\mu)}\left(\tfrac{1}{2}\right) - r^{(\mu)}\left(-\tfrac{1}{2}\right)\right) \\ &= \sum_{k=0}^{N-1} (2k+1)^\mu a_k \Big(s(k+\mu) + (-1)^k s(k-\mu-1)\Big) + \frac{1}{\sqrt{2}}\Big(s(\mu) - 3s(-\mu-1)\Big) \\ &\quad + \frac{2m}{\sqrt{2}\pi}\Big(s(\mu-1) + s(-\mu)\Big) + \left(\tfrac{2}{\pi}\right)^\mu \sqrt{2}\; 2\mu\Big(l^{(\mu-1)}\left(\tfrac{1}{2}\right) + r^{(\mu-1)}\left(-\tfrac{1}{2}\right)\Big). \end{aligned}$$

In the same way, one shows with $s(k+\mu) = (-1)^k s(k-\mu-1)$ that for $\mu > 0$

$$\left(\tfrac{2}{\pi}\right)^{\mu-1} \sqrt{2}\;\; \left(l^{(\mu-1)}\left(\tfrac{1}{2}\right) + r^{(\mu-1)}\left(-\tfrac{1}{2}\right)\right) = \frac{4}{\sqrt{2}} s(\mu-1)$$

and with $s(k+\mu) = (-1)^\mu s(k-\mu)$ it follows that

$$
\begin{aligned}
&\left(\tfrac{2}{\pi}\right)^\mu \sqrt{2}\ \left(l^{(\mu)}(\tfrac{1}{2}) - r^{(\mu)}(-\tfrac{1}{2}) \right) \\
&= \ 2(-1)^\mu \left(\sum_{k=0}^{N-1} (2k+1)^\mu a_k\, s(k-\mu) - \frac{1}{\sqrt{2}} s(-\mu) + \frac{2m}{\sqrt{2}\pi} s(1-\mu) \right)
\end{aligned}
$$

such that (25) is equivalent to $l^{(\mu)}(1/2) = r^{(\mu)}(-1/2)$, $\mu = 0, \ldots, m-1$. Analogously, we show the equivalence of (25) to $r^{(\mu)}(1/2) = 0$, $\mu = 0, \ldots, m-1$. Hence, (28) is equivalent to (25). Therefore, we obtain that $\tilde{w} \in C^{m-1}(\mathbb{R})$ iff (24) and (28) are fulfilled. Because $l, r \in C^\infty(\mathbb{R})$ the derivatives $l^{(m)}$ and $r^{(m)}$ are bounded. Hence, $\tilde{w}^{(m)} \in L^\infty(\mathbb{R})$ and thus $\tilde{w} \in W_\infty^m(\mathbb{R})$.

2. We still have to show that the system of equations in Theorem 8 has a unique solution, i.e., the determinant of the matrix $\boldsymbol{C} = (c_{ij})_{i,j=0}^{N-1}$ with

$$
c_{ij} = \begin{cases} 1 & \text{for } i = N-1 \\ s(j-i)\,(2j+1)^i & \text{for } i = 0, \ldots, N-2 \end{cases}
$$

does not vanish. To achieve this goal we show that the system of equations

$$
\sum_{i=0}^{N-1} \gamma_i\, c_{ij} = 0, \quad j = 1, \ldots, N-1,
$$

has only the trivial solution. If we substitute $\gamma_i = \tilde{\gamma}_i\, s(i-1)$ and multiply each equation with $s(j)$ we obtain the equivalent system of equations

$$
\sum_{i=0}^{N-2} \tilde{\gamma}_i\, s(j)\, s(i-1)\, s(j-i)\, (2j+1)^i = -\gamma_{N-1}\, s(j), \quad j = 1, \ldots, N-1.
$$

With $s(j)\, s(i-1)\, s(j-i) = (-1)^{ij}$, we have

$$
\sum_{i=0}^{N-2} \tilde{\gamma}_i \left((-1)^j (2j+1) \right)^i = -\gamma_{N-1}\, s(j), \quad j = 0, \ldots, N-1. \tag{31}
$$

Note that the coefficient matrix of this system is not a Vandermonde matrix, because the $(N-1)$-th row has the entries $s(j)$. However, the remaining entries of this matrix are elements of a Vandermonde matrix. Using this property we can formulate the following interpolation problem which is equivalent to (31). For the knot set

$$
\left\{ x_j := 4j+1 :\ j = -\left\lfloor \tfrac{N}{2} \right\rfloor, \ldots, \left\lfloor \tfrac{N-1}{2} \right\rfloor \right\} = \left\{ (-1)^j (2j+1) :\ j = 0, \ldots, N-1 \right\}
$$

the polynomial

$$
q(x) := \sum_{i=0}^{N-2} \tilde{\gamma}_i\, x^i
$$

should satisfy the interpolation conditions

$$
q(x_j) = (-1)^{j+1}\, \gamma_{N-1}, \quad j = -\left\lfloor \tfrac{N}{2} \right\rfloor, \ldots, \left\lfloor \tfrac{N-1}{2} \right\rfloor .
$$

Let us assume that $\gamma_{N-1} \neq 0$. Then we have $q(x_j)q(x_{j+1}) < 0$ and therefore q has a zero in (x_j, x_{j+1}), $j = -\left\lfloor \frac{N}{2} \right\rfloor, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor - 1$. This means we obtain $N-1$ zeros for a polynomial with degree $N-2$ and thus we conclude $\gamma_{N-1} = 0$. Then it follows that $q(x) \equiv 0$, i.e.,

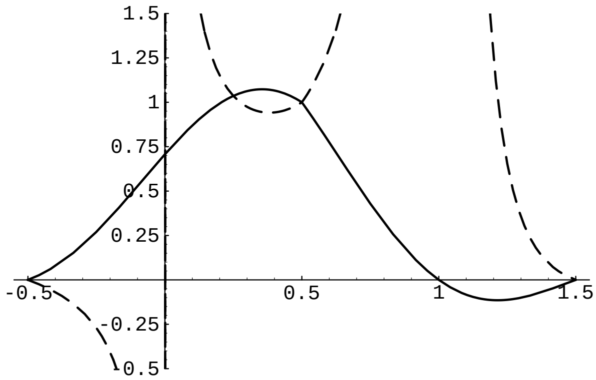
$$\gamma_i = \tilde{\gamma}_i \, s(i-1) = 0, \quad i = 0, \ldots, N-2$$

and therefore $\det \boldsymbol{C} \neq 0$.

Substituting (26) and (27) in

$$\tilde{w}(x) = \begin{cases} l(x) & \text{for } -\frac{1}{2} \leq x < \frac{1}{2} \\ r(x-1) & \text{for } \frac{1}{2} \leq x < \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

we obtain finally (23). ■



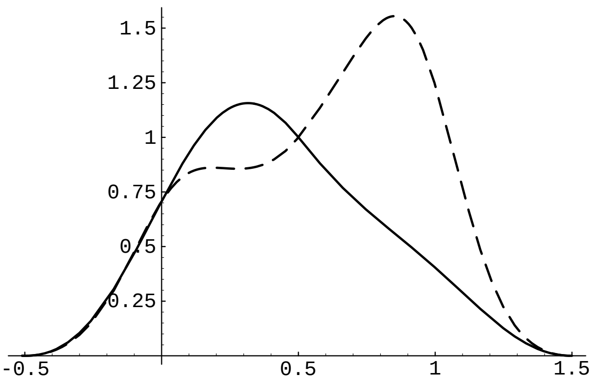


Figure 4: The bell function w (dashed line) and the dual bell $\tilde{w}$ (solid line) for $N = 2$ and $N = 3$. Note, that for $N = 2$ the bell w has poles in 0 and 1.

The bell function w can be determined from the dual bell $\tilde{w}$ by (8). We have still to investigate the question whether the bell function w resp. $\tilde{w}$ generates a Riesz basis. It is well known that the functions $\{\psi_j^k\}$ form a Riesz basis with the bounds A and B iff the dual system $\{\tilde{\psi}_j^k\}$ forms a Riesz basis with the bounds B^{-1} and A^{-1}. Therefore, we consider here only the dual functions. For odd N we obtain a Riesz basis as stated in the following theorem.

Theorem 9 *Let $N \in \mathbb{N}$ be odd. Then a Riesz basis is formed by the functions $\{\tilde{\psi}_j^k\}$ which are generated by the bell $\tilde{w}$ from* Theorem 8.

To prove this theorem we need the following assertion.

Lemma 10 *Let $N \in \mathbb{N}$ be odd. If the coefficients a_k satisfy* (25) *then it holds that*

$$a_{2k} - a_{2k-1} > 0, \quad k = 1, \ldots, \tfrac{N-1}{2}.$$

Proof. From the proof of Theorem 8, namely from the regularity of the matrix $\boldsymbol{C}$, it follows that for $m = 1, \ldots, \frac{N-1}{2}$ there exist uniquely determined coefficients $\gamma_{m,\mu}$ such that

$$\gamma_{m,-1} + \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1)s(k-\mu)(2k+1)^{\mu} = \delta_{k,2m} - \delta_{k,2m-1} \tag{32}$$

for $k = 0, \ldots, N-1$ and $n \in \mathbb{N}$. We multiply each equation with $s(k)$ and the equations read as follows

$$s(k)\gamma_{m,-1} + \sum_{\mu=0}^{N-2} \gamma_{m,\mu} \left((-1)^k(2k+1)\right)^{\mu} = (-1)^m(\delta_{k,2m} - \delta_{k,2m-1}).$$

Analogously to the proof of Theorem 8 we replace now $(-1)^k(2k+1)$ by $4n+1$ with a suited $n \in \mathbb{Z}$ and obtain

$$(-1)^n\, \gamma_{m,-1} + \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\,(4n+1)^{\mu} = (-1)^n(\delta_{n,m} - \delta_{n,-m})$$

for $n = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}$. Now we define

$$q_m(x) := \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\,(4x+1)^{\mu}.$$

Then it holds that $(-1)^n\,\gamma_{m,-1} + q_m(n) = (-1)^n(\delta_{n,m} - \delta_{n,-m})$ Thus, we obtain for the even part of q_m that $-(-1)^n\, 2\gamma_{m,-1} = q_m(n) + q_m(-n)$. Analogously to the proof of Theorem 8 we conclude now that $\gamma_{-1} = 0$. Furthermore, it follows that q_m is an odd function.

With (25) and (32) we obtain

$$\begin{aligned} a_{2m} - a_{2m-1} &= \sum_{k=0}^{N-1} \left(\sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1)s(k-\mu)(2k+1)^{\mu} \right) a_k \\ &= \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1) \left(\frac{s(-\mu)}{\sqrt{2}} - \frac{4\mu\, s(1-\mu)}{\sqrt{2}\pi} \right). \end{aligned}$$

From (32) with $k=0$ it follows that $\sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1)\, s(-\mu) = 0$ such that

$$\begin{aligned} a_{2m} - a_{2m-1} &= -\frac{1}{\sqrt{2}\pi} \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\,(-1)^{\mu}\,4\mu \\ &= -\frac{1}{\sqrt{2}\pi} q_m'\left(-\tfrac{1}{2}\right). \end{aligned}$$

Thus, to prove the lemma we have to show that $q_m'(\frac{1}{2}) < 0$. To achieve this goal we use that the polynomial $q_m \in \Pi_{m-1}$ is uniquely determined by the interpolation conditions

$$q_m(n) = (-1)^n(\delta_{n,m} - \delta_{n,-m}), \quad n = -\tfrac{N-1}{2}, \ldots, \tfrac{N-1}{2}.$$

Let polynomials $\ell_{j,m}$ be given by $\ell_{0,m}(x) = -\frac{x}{n}$ and

$$\ell_{j,m}(x) = \begin{cases} -\ell_{j-1,m}(x)\, \frac{x^2-j^2}{m^2-j^2} & \text{for } j < m, \\ \ell_{j-1,m}(x) & \text{for } j = m, \\ \ell_{j-1,m}(x)\, \frac{x^2-j^2}{m^2-j^2} & \text{for } j > m. \end{cases}$$

Obviously, it holds that $q_m = \ell_{(N-1)/2,m}$. By induction on j one shows for an arbitrary $m \in \mathbb{N}$ that $\ell'_{j,m}\left(-\frac{1}{2}\right) < 0$ for $j \in \mathbb{N}_0$. Hence,

$$a_{2m} - a_{2m-1} = -\ell'_{(N-1)/2,m}\left(-\tfrac{1}{2}\right) > 0$$

which proves the lemma. ∎

We use now this result to verify Theorem 9.

Proof. **1.** In the first part of the proof we deduce a condition for the coefficients a_k, which ensures Riesz stability. Because the bell functions are bounded we know that the upper Riesz bound B_0 is finite. Thus, we have still to show that the lower Riesz bound A_0 does not vanish. For

$$A_0 = \operatorname*{ess\,inf}_{x\in(0,1/2)} \frac{\Delta(x)}{2} - \sqrt{\frac{\Delta^2(x)}{4} - (\det \boldsymbol{M}(x))^2} \;>\; 0, \quad 0 < x < \tfrac{1}{2}$$

to be positive, it is sufficient that $\Delta(x) > 0$ and $\det \boldsymbol{M}(x) \neq 0$. Because $\Delta^2(x) \geq 4(\det \boldsymbol{M}(x))^2$ we have only to show that $\det \boldsymbol{M}(x) \neq 0$ for $0 < x < \frac{1}{2}$. For $x \neq 0$ the even function $\det \boldsymbol{M}(x)$ is given by

$$\begin{aligned} \det \boldsymbol{M}(x) &= l(x)r(-x) + l(-x)r(x) \\ &= \frac{\cos\frac{\pi}{2}x \sin\frac{\pi}{2}x}{x} - \sum_{k=0}^{N-1}(-1)^k a_k \frac{\cos\frac{\pi}{2}x \sin\left(k+\frac{1}{2}\right)\pi x}{\sqrt{2}x} \\ &\quad + \sum_{k=0}^{N-1} a_k \frac{\sin\frac{\pi}{2}x \cos\left(k+\frac{1}{2}\right)\pi x}{\sqrt{2}x}. \end{aligned}$$

By addition formulas we deduce

$$\det \boldsymbol{M}(x) = \frac{\sin \pi x}{2x} + \sum_{k=1}^{(N-1)/2} (a_{2k-1} - a_{2k}) \frac{\sin 2k\pi x}{\sqrt{2}x}.$$

Thus, $\det \boldsymbol{M}(x) > 0$ is equivalent to

$$\frac{\sin \pi x}{2x} > \sum_{k=1}^{(N-1)/2} (a_{2k} - a_{2k-1}) \frac{\sin 2k\pi x}{\sqrt{2}x}$$

and hence for $0 < x < \frac{1}{2}$ equivalent to

$$1 > \sum_{k=1}^{(N-1)/2} \sqrt{2}(a_{2k} - a_{2k-1}) \frac{\sin 2k\pi x}{\sin \pi x}.$$

One easily shows that

$$\sup_{0<x<1/2} \frac{\sin 2k\pi x}{\sin \pi x} = \lim_{x\to 0} \frac{\sin 2k\pi x}{\sin \pi x} = 2k.$$

Therefore, $\det \boldsymbol{M}(x) > 0$ iff

$$\frac{\sqrt{2}}{4} > \sum_{k=1}^{(N-1)/2} k\, |a_{2k} - a_{2k-1}|.$$

By Lemma 10 this is equivalent to

$$\frac{\sqrt{2}}{4} > \sum_{k=1}^{(N-1)/2} k\, (a_{2k} - a_{2k-1}).$$

2. In the second part of the proof we derive from this inequality regarding the coefficients a_k a condition which does not explicitly depend on these coefficients. To achieve this aim we have to consider the system of equations (25). From the proof of Lemma 10 it follows that

$$\sum_{k=1}^{(N-1)/2} k\, (a_{2k} - a_{2k-1}) = -\frac{1}{\sqrt{2}\pi} \sum_{k=1}^{(N-1)/2} k\, q_k' \left(-\tfrac{1}{2}\right) = -\frac{1}{\sqrt{2}\pi} q' \left(-\tfrac{1}{2}\right)$$

with

$$q := \sum_{k=1}^{(N-1)/2} k\, q_k.$$

Thus, the polynomial $q \in \Pi_{N-1}$ is uniquely determined by the interpolation conditions

$$q(n) = \sum_{k=1}^{(N-1)/2} q_m(n) = (-1)^n\, n, \quad n = -\tfrac{N-1}{2}, \ldots, \tfrac{N-1}{2}. \tag{33}$$

3. To prove the theorem we have still to show, that

$$-q'\left(-\tfrac{1}{2}\right) < \frac{\pi}{2}.$$

To achieve this goal we show first that the polynomial q has the form

$$q(x) = \sum_{l=0}^{(N-3)/2} c_l r_l(x),$$

with $r_0(x) := x$ and $r_l(x) := r_{l-1}(x)\,(x-l)(x+l)$ as well as

$$c_l := \frac{(-1)^{l+1} 4^l}{(2l+1)!}.$$

Because q and r_l are odd functions, we have only to show that the polynomial above satisfies (33) for $n > 0$. For $n \leq \frac{N-1}{2}$ we obtain

$$\begin{aligned}\sum_{l=0}^{(N-3)/2} c_l r_l(n) &= \sum_{l=0}^{n-1} \frac{(-1)^{l+1}4^l}{(2l+1)!}\frac{(n+l)!}{(n-l-1)!}\\ &= \sum_{l=0}^{n-1}(-1)^{l+1}4^l\binom{n+l}{2l+1}.\end{aligned}$$

By induction we show now together that

$$\sum_{l=0}^{n-1}(-1)^{l+1}4^l\binom{n+l}{2l+1} = (-1)^n\, n = q(n)$$

and

$$\sum_{l=0}^{n}(-1)^{l+1}4^l\binom{n+l}{2l} = (-1)^{n+1}\,(2n+1).$$

Obviously, the assertions are true for $n=0$ and $n=1$. Then we conclude

$$\begin{aligned}\sum_{l=0}^{n}(-1)^{l+1}4^l\binom{n+l}{2l} &= \sum_{l=1}^{n}(-1)^{l+1}4^l\binom{n+l-1}{2l-1} + \sum_{l=0}^{n-1}(-1)^{l+1}4^l\binom{n+l-1}{2l}\\ &= -4\,(-1)^n\, n + (-1)^n\,(2n-1) = (-1)^{n+1}\,(2n+1)\end{aligned}$$

and

$$\begin{aligned}\sum_{l=0}^{n}(-1)^{l+1}4^l\binom{n+l+1}{2l+1} &= \sum_{l=0}^{n}(-1)^{l+1}4^l\binom{n+l}{2l} + \sum_{l=0}^{n-1}(-1)^{l+1}4^l\binom{n+l}{2l+1}\\ &= (-1)^n\, n - (-1)^n\,(2n+1) = (-1)^{n+1}\,(n+1) = q(n+1).\end{aligned}$$

Finally, we consider the value of $q'(x)$ for $x = -\frac{1}{2}$. Again, by induction one shows that

$$r_l'\left(-\tfrac{1}{2}\right) = (-1)^l\left(\frac{(2l)!}{4^l l!}\right)^2.$$

With the Taylor expansion of arcsin we obtain

$$\begin{aligned}-q'\left(-\tfrac{1}{2}\right) &= \sum_{l=0}^{(N-3)/2}\frac{(2l)!}{4^l(l!)^2\,(2l+1)}\\ &< \sum_{l=0}^{\infty}\frac{(2l)!}{4^l(l!)^2\,(2l+1)} = \arcsin(1) = \frac{\pi}{2}\end{aligned}$$

and the assertion is proved. ■

Unfortunately, for even N we have the following negative result.

Theorem 11 *Let $N \in \mathbb{N}$ be even and the functions $\tilde{\psi}_j^k$ be generated by the bell $\tilde{w}$ from* Theorem 8. *Then the family $\{\tilde{\psi}_j^k\}$ is not a Riesz basis for $L^2(\mathbb{R})$.*

Proof. Analogously, to the proof of Lemma 10 we conclude that for $m = 0, \ldots, \frac{N}{2} - 1$ there exist uniquely determined coefficients $\gamma_{m,\mu}$ such that

$$\gamma_{m,-1} + \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1)s(k-\mu)(2k+1)^{\mu} = \delta_{k,2m} - \delta_{k,2m+1} \tag{34}$$

for $k = 0, \ldots, N-1$ and $n \in \mathbb{N}$. We multiply each equation with $s(k)$ and obtain analogously to the proof of Lemma 10

$$(-1)^n \gamma_{m,-1} + \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\,(4n+1)^{\mu} = (-1)^n(\delta_{n,m} + \delta_{n,-m-1})$$

for $n = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$. With

$$q_m(x) := \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\,(4x+1)^{\mu}$$

it holds that $(-1)^n \gamma_{m,-1} + q_m(n) = (-1)^n(\delta_{n,m} + \delta_{n,-m-1})$ and thus, we have $q_m(n) - q_m(-n-1) = -(-1)^n\, 2\gamma_{m,-1}$. Analogously to the proof of Theorem 8 we conclude now that $\gamma_{-1} = 0$. Furthermore, it follows that $q_m(\cdot - \frac{1}{2})$ is an even function. With (25) and (34) we obtain

$$\begin{aligned} a_{2m} - a_{2m+1} &= \sum_{\mu=0}^{N-2} \gamma_{m,\mu}\, s(\mu-1) \left(\frac{s(-\mu)}{\sqrt{2}} - \frac{4\mu\, s(1-\mu)}{\sqrt{2}\pi} \right) \\ &= \frac{1}{\sqrt{2}}\delta_{m,0} - \frac{1}{\sqrt{2}\pi} q_m' \left(-\tfrac{1}{2}\right). \end{aligned}$$

Because $q_m(\cdot - \frac{1}{2})$ is even, it follows that $q_m'\left(-\frac{1}{2}\right) = 0$, and we deduce that for the solution of (24), (25) it holds $a_0 = \frac{1}{\sqrt{2}} + a_1$ and $a_{2k} = a_{2k+1}$, $k = 1, \ldots, \frac{N}{2} - 1$.

Using this assertion, we show that $\tilde{w}$ vanishes for $x = 1$. By addition formulas we conclude

$$\begin{aligned} r(0) &= -\frac{\sqrt{2}}{2} \sin \tfrac{\pi}{2} x + \frac{1}{2x} \Bigg(\frac{\sqrt{2}}{2} \sin \tfrac{\pi}{2} x \\ &\quad + \sum_{k=0}^{N/2-1} a_{2k+1} \Big(\sin\left(2k+1+\tfrac{1}{2}\right)\pi x - \sin\left(2k+1-\tfrac{1}{2}\right)\pi x \Big) \Bigg) \\ &= -\frac{\sqrt{2}}{2} \sin \tfrac{\pi}{2} x + \frac{\sin \frac{\pi}{2} x}{2x} \left(\frac{\sqrt{2}}{2} + \sum_{k=0}^{N/2-1} 2a_{2k+1} \cos(2k+1)\pi x \right). \end{aligned}$$

Because the coefficients a_k satisfy (24) we obtain finally for $x = 0$

$$w(1) = r(0) = \tfrac{\pi}{4} \sum_{k=0}^{N-1} a_k = 0.$$

Thus, the matrix $M^{\tilde{w}}(x)$ is singular for $x = 0$. Because $\tilde{w}$ is continuous, the assertion follows immediately from (7). ■

The table below shows the best possible Riesz bounds for small N which one can determine by (7) sufficiently precise. Apparently, for small odd N we obtain better Riesz constants. This and the smaller number of basis functions which are needed to reproduce a linear function are reasons to choose N not too large.

N	$a_k, \quad k = 0, \dots, N-1$	m	A_0	B_0
2	$\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}$	1	0	1
3	$\frac{1}{2\sqrt{2}}, -\frac{2+\pi}{4\sqrt{2\pi}}, \frac{2-\pi}{4\sqrt{2\pi}}$	2	0.326...	1.347...
4	$\frac{5\pi-2}{8\sqrt{2\pi}}, -\frac{2+3\pi}{8\sqrt{2\pi}}, \frac{2-\pi}{8\sqrt{2\pi}}, \frac{2-\pi}{8\sqrt{2\pi}}$	3	0	1.477...
5	$\frac{5\pi-2}{8\sqrt{2\pi}}, -\frac{13+6\pi}{24\sqrt{2\pi}}, \frac{13-6\pi}{24\sqrt{2\pi}}, \frac{5-3\pi}{48\sqrt{2\pi}}, \frac{7-3\pi}{48\sqrt{2\pi}}$	4	0.163...	1.541...
6	$\frac{-19+33\pi}{48\sqrt{2\pi}}, -\frac{19+15\pi}{48\sqrt{2\pi}}, \frac{31-15\pi}{96\sqrt{2\pi}}, \frac{31-15\pi}{96\sqrt{2\pi}}, \frac{7-3\pi}{96\sqrt{2\pi}}, \frac{7-3\pi}{96\sqrt{2\pi}}$	5	0	1.590...
7	$\frac{-19+33\pi}{48\sqrt{2\pi}}, -\frac{151+60\pi}{256\sqrt{2\pi}}, \frac{397-180\pi}{768\sqrt{2\pi}}, \frac{161-90\pi}{960\sqrt{2\pi}}, \frac{73-30\pi}{320\sqrt{2\pi}}, \frac{131-60\pi}{3840\sqrt{2\pi}}, \frac{149-60\pi}{3840\sqrt{2\pi}}$	6	0.108...	1.637...
8	$\frac{-757+1116\pi}{1536\sqrt{2\pi}}, -\frac{757+420\pi}{1536\sqrt{2\pi}}, \frac{2629-1260\pi}{7680\sqrt{2\pi}}, \frac{2629-1260\pi}{7680\sqrt{2\pi}}, \frac{1007-420\pi}{7680\sqrt{2\pi}}, \frac{1007-420\pi}{7680\sqrt{2\pi}}, \frac{149-60\pi}{7680\sqrt{2\pi}}, \frac{149-60\pi}{7680\sqrt{2\pi}}$	7	0	1.671...
9	$\frac{-757+1116\pi}{1536\sqrt{2\pi}}, -\frac{9731+3360\pi}{15360\sqrt{2\pi}}, \frac{2473-1120\pi}{5120\sqrt{2\pi}}, \frac{3097-1680\pi}{15360\sqrt{2\pi}}, \frac{835-336\pi}{3072\sqrt{2\pi}}, \frac{1523-672\pi}{21504\sqrt{2\pi}}, \frac{8569-3360\pi}{107520\sqrt{2\pi}}, \frac{2011-840\pi}{215040\sqrt{2\pi}}, \frac{2161-840\pi}{215040\sqrt{2\pi}}$	8	0.081...	1.696...

Table 1: Coefficients a_k and corresponding Riesz bounds for small N.

7 Reproduction of polynomials with higher degree

Now the question is, whether we can reproduce polynomials from Π_n with $n > 1$. For the two-overlapping setting it turns out that we cannot construct a bell function such that polynomials of degree two are reproduced.

Theorem 12 *There exists no basis $\{\psi_j^k\}$ such that for a certain $N \in \mathbb{N}$ each polynomial $p \in \Pi_n$ with $n \geq 2$ has the representation*

$$p = \sum_{j \in \mathbb{Z}} \sum_{k=0}^{N-1} \langle p, \tilde{\psi}_j^k \rangle \; \psi_j^k.$$

Proof. It is sufficient to find two polynomials from Π_2 such that for any basis ψ_j^k only one of this polynomials has a representation of the above form. Indeed, the functions $p_0(x) = 1$ and $p_2(x) = x^2$ are such polynomials as we will show in the following.

Let us assume there exists a bell $\tilde{w}$ such that

$$p_0 = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} a_k \, \psi_j^k \quad \text{and} \quad p_2 = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} c_{j,k} \, \psi_j^k.$$

If we set $c_k = c_{0,k}$ it follows for $x \in [-\frac{1}{2}, \frac{1}{2}]$ that

$$\begin{aligned} \mathcal{T}_{\tilde{w}} p_0 = l(x) + l(-x) &= \sum_{k=0}^{N-1} a_k \cos\left(k + \tfrac{1}{2}\right) \pi x \\ \mathcal{T}_{\tilde{w}} p_2 = x^2(l(x) + l(-x)) &= \sum_{k=0}^{N-1} c_k \cos\left(k + \tfrac{1}{2}\right) \pi x. \end{aligned}$$

From this equalities we deduce

$$x^2 \sum_{k=0}^{N-1} a_k \cos\left(k + \tfrac{1}{2}\right) \pi x = \sum_{k=0}^{N-1} c_k \cos\left(k + \tfrac{1}{2}\right) \pi x \tag{35}$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$. By analytic continuation we conclude, that the equation holds for all $x \in \mathbb{R}$. Since the right-hand side of (35) is 4π-periodic for arbitrary coefficients c_k and the left-hand side is only if $a_k = 0$, $k = 0, \ldots, N-1$, it follows that all coefficients a_k vanish. Hence, we do not have a basis. This proves the theorem. ∎

The last result is a reason to consider not only the two-overlapping setting. If for three or more bells the intersection of the supports is not only a single point then one should have more freedom for the choice of a basis which reproduces linear functions. Using this freedom we can possibly find a basis which reproduces polynomials of higher degree. In this way it could also be possible to construct smooth bases which reconstruct linear functions with fewer coefficients than in the two-overlapping setting.

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