From Wavelets to Multiwavelets

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Abstract. This paper gives an overview of recent achievements of the multiwavelet theory. The construction of multiwavelets is based on a multiresolution analysis with higher multiplicity generated by a scaling vector. The basic properties of scaling vectors such as L^2-stability, approximation order and regularity are studied. Most of the proofs are sketched.

§1. Introduction

Wavelet theory is based on the idea of *multiresolution analysis* (MRA). Usually it is assumed that an MRA is generated by one scaling function, and dilates and translates of only one wavelet $\psi \in L^2(\mathbb{R})$ form a stable basis of $L^2(\mathbb{R})$. This paper considers a recent generalization allowing several wavelet functions $\psi_1, \ldots, \psi_r$. The vector $\mathbf{\Psi} = (\psi_1, \ldots, \psi_r)^{\mathrm{T}}$ is then called *multiwavelet.*

Multiwavelets have more freedom in their construction and thus can combine more useful properties than the scalar wavelets. Symmetric scaling functions constructed by Geronimo, Hardin, and Massopust [16] (Figure 1) have short support, generate an orthogonal MRA, and provide approximation order 2. These properties are very desirable in many applications but cannot be achieved by one scaling function. Thus, multiwavelets can be useful for various practical problems [11,53].

Our purpose is to give a survey of the basic ideas of multiwavelet theory and to show how naturally multiwavelets generalize the scalar ones. We start with a simple example of piecewise linear multiwavelets and the definition of MRA. Then we discuss necessary properties of a function vector $\mathbf{\Phi} = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ in order to generate an MRA. In particular, $\mathbf{\Phi}$ has to be *refinable* and *L^2-stable*, i.e., $\mathbf{\Phi}$ can be seen as a stable solution vector of a *matrix refinement equation.* Such vector is then called *scaling vector* or *multi-scaling function.* The *mask symbol* of the matrix refinement equation is closely associated with the scaling vector. Similarly to the scalar case, the mask symbol

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and two linear operators, *transition operator* and the *subdivision operator*, are the main tools in multiwavelet theory.

In Section 2, we discuss matrix refinement equations with compactly supported, L^2-stable solutions.

Section 3 is devoted to the approximation properties of scaling vectors. We show how approximation comes from the Strang-Fix conditions and that it implies certain factorization of the mask symbol. This factorization generalizes zeros at $\omega = \pi$ of the scalar symbols.

In Section 4 we overview basic methods of estimation of the scaling vectors' smoothness.

Aiming to explain main ideas behind the theory, we will sketch the proofs of most of the assertions.

Unfortunately the space does not allow us to consider more applied aspects of the multiwavelets such as decomposition and reconstruction algorithms, construction of biorthogonal bases, lifting, preprocessing. Let us only mention that the application of multiwavelets is a fast developing field and the literature on this subject is rapidly growing.

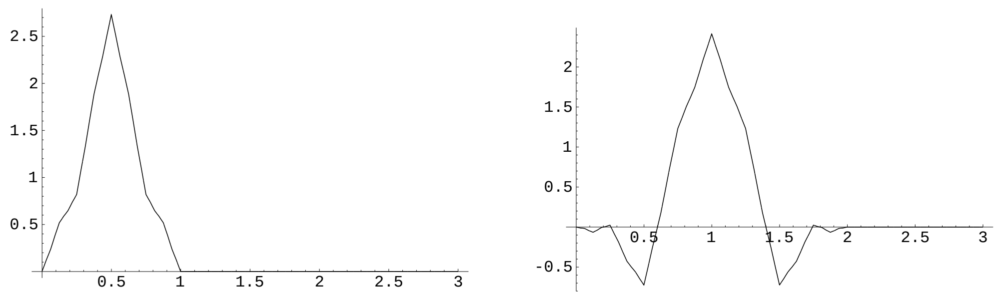

Fig. 1. GHM symmetric orthogonal multi–scaling function with approximation order 2.

1.1. Example of linear multiwavelets

Let us start with a simple example taken from [1]. Consider two piecewise linear functions

$$\phi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \phi_2(t) = \begin{cases} 2\sqrt{3}(t-\frac{1}{2}) & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(Figure 2). Their integer translates $\phi_1(\cdot - l), \phi_2(\cdot - l)$, $l \in \mathbb{Z}$ form an orthonormal basis of the closed subspace $V_0 \subset L^2(\mathbb{R})$ containing functions piecewise linear on integer intervals. Furthermore, let V_j be the closed subspace of $L^2(\mathbb{R})$ spanned by $2^{j/2}\,\phi_1(2^j \cdot - l)$, $2^{j/2}\,\phi_2(2^j \cdot - l)$ ($l \in \mathbb{Z}$), and containing all functions which are piecewise linear on intervals $[2^{-j}l,\, 2^{-j}(l+1)]$ ($l \in \mathbb{Z}$). It is easy to see that

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2\cdot) \\ \phi_2(2\cdot) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \phi_1(2\cdot - 1) \\ \phi_2(2\cdot - 1) \end{bmatrix}.$$

This relation represents the fact that $\phi_1, \phi_2 \in V_0 \subset V_1$. Analogously,

$$V_j \subset V_{j+1} \quad (j \in \mathbb{Z}). \tag{1}$$

Orthogonal projections $f_j(t)$ of any function $f \in L^2(\mathbb{R})$ on subspaces V_j are successive piecewise linear approximations converging to $f(t)$ as j goes to ∞. Thus,

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbf{R}), \qquad \bigcap_{j=-\infty}^{\infty} V_j = \{0\}\,. \tag{2}$$

Such nested structure of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ is usually referred as a multiresolution analysis (MRA). In our case it is generated by two functions ϕ_1, ϕ_2 and thus is of *multiplicity* 2. Observe, that the spaces V_j considered here, can not be spanned by integer translates of only one function ϕ.

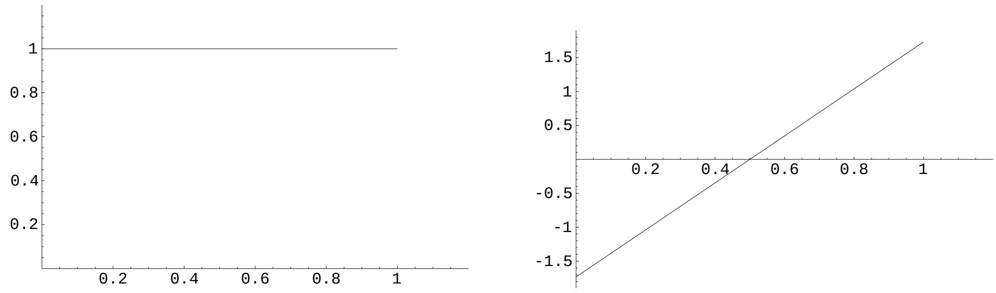

Fig. 2. Piecewise linear orthogonal scaling functions ϕ_1, ϕ_2 with approximation order 2.

Consider two more piecewise linear functions

$$\psi_1(t) = \begin{cases} 6t-1 & 0 \le t < 1/2 \\ 6t-5 & 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \psi_2(t) = \begin{cases} 2\sqrt{3}(2t-\frac{1}{2}) & 0 \le t < 1/2 \\ -2\sqrt{3}(2t-\frac{3}{2}) & 1/2 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

(Figure 3). Let W_j $(j \in \mathbb{Z})$ be the closed subspaces of $L^2(\mathbb{R})$ spanned by $2^{j/2}\,\psi_1(2^j\cdot - l)$ and $2^{j/2}\,\psi_2(2^j\cdot - l)$ $(l \in \mathbb{Z})$.

The integer translates $\psi_1(\cdot - l), \psi_2(\cdot - n), l, n \in \mathbb{Z}$ are orthogonal to each other and to the integer translates of ϕ_1, ϕ_2 which makes W_0 orthogonal to V_0. ψ_1 and ψ_2 are piecewise linear on half integer intervals, thus $W_0 \subset V_1$. In particular,

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1(2\cdot) \\ \phi_2(2\cdot) \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1(2\cdot - 1) \\ \phi_2(2\cdot - 1) \end{bmatrix}.$$

Finally, generators of V_1 are linear combinations of translates of $\phi_1, \phi_2, \psi_1, \psi_2$,

$$\phi_1(2\cdot) = \frac{1}{2}\phi_1 - \frac{\sqrt{3}}{4}\phi_2 + \frac{1}{4}\psi_1, \quad \phi_1(2\cdot - 1) = \frac{1}{2}\phi_1 + \frac{\sqrt{3}}{4}\phi_2 - \frac{1}{4}\psi_1,$$

$$\phi_2(2\cdot) = \frac{1}{4}\phi_2 + \frac{\sqrt{3}}{4}\psi_1 + \frac{1}{2}\psi_2, \quad \phi_2(2\cdot - 1) = \frac{1}{4}\phi_2 + \frac{\sqrt{3}}{4}\psi_1 - \frac{1}{2}\psi_2.$$

Hence, V_1 is an orthogonal sum of V_0 and W_0. Analogously, it follows that $V_j \oplus W_j = W_{j+1}$ for all $j \in \mathbb{Z}$, and

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j = \operatorname{clos}_{L^2} \operatorname{span}\{2^{j/2}\psi_1(2^j \cdot -l),\, 2^{j/2}\psi_2(2^j \cdot -l) : l, j \in \mathbb{Z}\}.$$

All said above implies that the dilates and translates of ψ_1, ψ_2 form an orthogonal basis of $L^2(\mathbb{R})$.

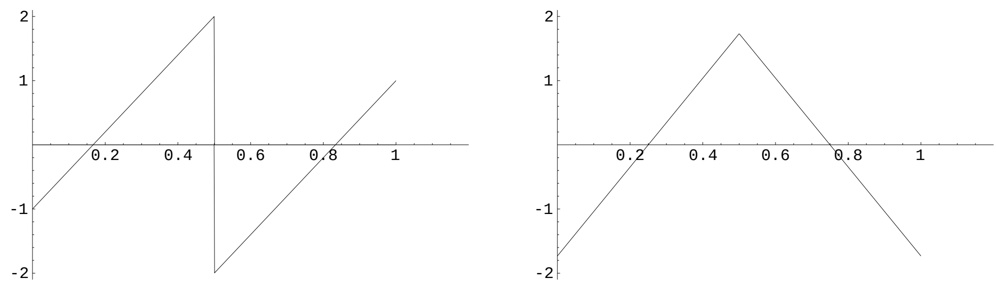

Fig. 3. Piecewise linear orthogonal wavelets ψ_1, ψ_2.

1.2 Multiresolution analysis with multiplicity r

Generally, a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is called *multiresolution analysis (MRA) of multiplicity* r if (1) and (2) are satisfied, and if there exists a vector $\mathbf{\Phi} = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ of L^2-functions such that $2^{j/2}\,\phi_\nu(2^j \cdot -l)$ ($\nu = 1, \ldots r$, $l \in \mathbb{Z}$) form an L^2-stable basis of V_j. The scaling spaces V_j are *finitely generated* $2^{-j}\,\mathbb{Z}$*-translation-invariant subspaces of* $L^2(\mathbb{R})$. Function vectors $\mathbf{\Phi}$ that generate an MRA with multiplicity $r \geq 1$, are called *scaling vectors* or *multi-scaling functions.*

Once an MRA $\{V_j\}_{j\in\mathbb{Z}}$ is given, we define the *wavelet spaces* W_j as complements of V_j in V_{j+1}. The wavelet spaces W_j are also finitely generated $2^{-j-1}\mathbb{Z}$-translation-invariant subspaces of $L^2(\mathbb{R})$. Moreover, the structure of MRA implies that W_j is the closure of the span of $2^{j/2}\psi_\nu(2^j \cdot -l)$ ($\nu = 1, \ldots, r$, $l \in \mathbb{Z}$) if only

$$W_0 = \operatorname{clos}_{L^2} \operatorname{span}\{\psi_\nu(\cdot - l) : \nu = 1, \ldots, r-1,\, l \in \mathbb{Z}\}$$

can be shown. If we can find a function vector $\mathbf{\Psi} = (\psi_1, \ldots, \psi_r)^{\mathrm{T}}$ such that $\{\psi_\nu(\cdot - l) : \nu = 1, \ldots, , r,\, l \in \mathbb{Z}\}$ forms an L^2-stable basis of W_0, then $\{2^{j/2}\,\psi_\nu(2^j \cdot -l) : \nu = 1, \ldots, r,\, l, j \in \mathbb{Z}\}$ forms an L^2-stable basis of $L^2(\mathbb{R})$.

In the Fourier domain, the problem of finding the multiwavelet $\mathbf{\Psi}$ can be reduced to an algebraic problem of matrix completion (see e.g. [36,51]). Moreover, if $\mathbf{\Phi}$ is compactly supported, then a corresponding compactly supported wavelet vector can be found.

Since $\psi_\nu \in W_0 \subset V_1$ is a linear combination of the dilated components of the scaling vector $\mathbf{\Phi}$, most properties of the multiwavelet $\mathbf{\Psi}$ are determined by the properties of $\mathbf{\Phi}$.

A scaling vector $\mathbf{\Phi}$ has to satisfy some special properties which are induced by the conditions of the MRA. Another set of properties such as compact support and smoothness of the components ϕ_ν of $\mathbf{\Phi}$ and polynomial reproduction in V_j is required by applications.

Condition (1) of the MRA implies that $\mathbf{\Phi}$ needs to satisfy a *matrix refinement equation* of the form

$$\mathbf{\Phi}(t) = \sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, \mathbf{\Phi}(2t-l). \tag{3}$$

Here $\boldsymbol{P}_l$ are $r \times r$ *mask* coefficient matrices. A function vector $\mathbf{\Phi}(t)$ satisfying (3) is called *refinable*. Application of the Fourier transform to (3) leads to

$$\widehat{\mathbf{\Phi}}(\omega) = \boldsymbol{P}\left(\frac{\omega}{2}\right) \widehat{\mathbf{\Phi}}\left(\frac{\omega}{2}\right), \tag{4}$$

where $\boldsymbol{P}$ denotes the *symbol* of the mask $\{\boldsymbol{P}_l\}_{l\in\mathbb{Z}}$,

$$\boldsymbol{P}(\omega) := \frac{1}{2}\sum_{l\in\mathbb{Z}} \boldsymbol{P}_l\, e^{-i\omega l}. \tag{5}$$

Here, the Fourier transform is taken componentwisely, i.e., $\hat{\mathbf{\Phi}} = (\hat\phi_1, \dots, \hat\phi_r)^{\mathrm{T}}$ with $\hat\phi_\nu(\omega) := \int_{-\infty}^{\infty} \phi_\nu(t)\, e^{-i\omega t}\, \mathrm{d}t$.

For simplicity we assume that the sum in the refinement equation (3) is finite, or equivalently, the symbol $\boldsymbol{P}$ is a matrix of trigonometric polynomials $\boldsymbol{P}(\omega) = 2^{-1}\sum_{l=0}^{N} \boldsymbol{P}_l\, e^{-i\omega l}$. Being interested in the refinable function vectors, we will view the refinement equation as a functional equation. The finite mask already implies that solutions of (3) are compactly supported in the sense that each component ϕ_ν $(\nu = 1,\dots,r)$ is compactly supported.

The second condition on $\mathbf{\Phi}$ induced by the MRA is the L^2-stability. We say that a function vector $\mathbf{\Phi} \in (L^2(\mathbb{R}))^r$ is L^2-*stable*, if there are constants $0 < A \le B < \infty$, such that

$$A \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^\star \boldsymbol{c}_l \le \| \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^\star \mathbf{\Phi}(\cdot - l)\|_{L^2}^2 \le B \sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^\star \boldsymbol{c}_l \tag{6}$$

holds for any vector sequence $\{\boldsymbol{c}_l\}_{l\in\mathbb{Z}} \in l^2(\mathbb{Z})^r$. Here $l^2(\mathbb{Z})^r$ denotes the set of sequences of vectors $\boldsymbol{c}_l \in \mathbb{R}^r$ with $\sum_{l=-\infty}^{\infty} \boldsymbol{c}_l^\star \boldsymbol{c}_l < \infty$ and $\boldsymbol{c}_l^\star$ stands for $\overline{\boldsymbol{c}_l}^{\mathrm{T}}$.

Let us introduce the autocorrelation symbol

$$\mathbf{\Omega}(\omega) := \sum_{l=-\infty}^{\infty} \left(\langle \phi_\mu, \phi_\nu(\cdot - l)\rangle_{L^2}\right)_{\mu,\nu=1}^{r} e^{-i\omega l}.$$

Poisson summation formula gives

$$\mathbf{\Omega}(\omega) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{\Phi}}(\omega + 2\pi n)\, \hat{\mathbf{\Phi}}(\omega + 2\pi n)^\star \quad a.e.$$

Observe, that the autocorrelation symbol is positive semidefinite and, for compactly supported $\mathbf{\Phi}$, it is a matrix of trigonometric polynomials. L^2-stability of $\mathbf{\Phi}$ is equivalent to the assertion that $\mathbf{\Omega}(\omega)$ is strictly positive definite for all real ω (see [18]). In the Fourier domain, L^2-stability of $\mathbf{\Phi}$ is ensured if and only if the sequences $\{\hat{\phi}_\nu(\omega+2\pi l)\}_{l\in\mathbb{Z}}$ $(\nu = 1,\ldots,r)$ are linearly independent for each $\omega \in \mathbb{R}$ (see [29]).

Finally, let us consider the union and intersection properties (2) of the MRA. As shown in [17] the equality $\mathrm{clos}_{L^2} \cup_{j\in\mathbb{Z}} V_j = L^2(\mathbb{R})$ is satisfied if and only if $\cup_{\nu=1}^r \mathrm{supp}\,\hat{\phi}_\nu = \mathbb{R}$ (modulo a null set). This condition obviously holds for compactly supported $\mathbf{\Phi}$. The intersection condition $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ follows for nested sequences V_j if $\sum_{\nu=1}^r |\hat{\phi}_\nu(\omega)| > 0$ *a.e.* in some neighborhood of the origin (see [17], Theorem 3). Indeed, this condition is already satisfied if $\mathbf{\Phi}$ is refinable and L^2-stable (see Section 2).

Let us summarize: In order to obtain a scaling vector $\mathbf{\Phi}$ generating an MRA, it is enough to find a compactly supported, L^2-stable function vector which is a solution of a matrix refinement equation (3). In other words, we are looking for a mask $\{\boldsymbol{P}_l\}$ (a symbol $\boldsymbol{P}(\omega)$), such that a compactly supported L^2-solution vector of (3) is L^2-stable. This problem will be considered in Section 2.

Applications usually require several other features such as exact polynomial reproduction in V_j (vanishing moments of the wavelets) and smoothness of the elements of V_j. The corresponding properties of the scaling vector $\mathbf{\Phi}$ are considered in Sections 3 and 4.

1.3. Transition operator and subdivision operator

Besides the mask, two other important tools for studying the properties of a scaling vector are the *transition operator* T and the *subdivision operator* S. For a given mask symbol $\boldsymbol{P}(\omega) = 2^{-1}\sum_{l=0}^N \boldsymbol{P}_l\, e^{-i\omega l}$, the *transition operator* $T : (L^2_{2\pi})^{r\times r} \to (L^2_{2\pi})^{r\times r}$, acting on $r\times r$ matrices $\boldsymbol{H}(\omega)$ with 2π-periodic, quadratic integrable entries, is defined by

$$T\boldsymbol{H}(2\omega) := \boldsymbol{P}(\omega)\boldsymbol{H}(\omega)\boldsymbol{P}^\star(\omega) + \boldsymbol{P}(\omega+\pi)\boldsymbol{H}(\omega+\pi)\boldsymbol{P}^\star(\omega+\pi). \qquad (7)$$

Observe that the autocorrelation symbol $\mathbf{\Omega}(\omega)$ is an eigenmatrix of T to the eigenvalue 1. For $\boldsymbol{H} \in (L^2_{2\pi})^{r\times r}$ we find

$$(T^n\boldsymbol{H})(\omega) = \sum_{l=0}^{2^n-1} \mathbf{\Pi}_n(\omega+2\pi l)\,\boldsymbol{H}(\frac{\omega+2\pi l}{2^n})\,\mathbf{\Pi}_n(\omega+2\pi l)^\star, \qquad (8)$$

where $\mathbf{\Pi}_n(\omega) := \prod_{j=1}^n \boldsymbol{P}(2^{-j}\omega)$. If $\mathbb{H}_M$ is the space of $r\times r$ matrices of trigonometric polynomials of degree at most M, then for $M \geq N$, $\mathbb{H}_M$ is invariant under the action of T.

The transition operator T is a linear operator, and $T : \mathbb{H}_M \to \mathbb{H}_M$ can be represented by an $r^2(2M+1)\times r^2(2M+1)$ matrix $\boldsymbol{T}$

$$\boldsymbol{T} := (\frac{1}{2}\boldsymbol{A}_{2k-j})_{j,k=-M}^M$$

with

$$\boldsymbol{A}_j := \sum_{l=0}^{M} \boldsymbol{P}_{l-j} \otimes \boldsymbol{P}_l,$$

where $\boldsymbol{A} \otimes \boldsymbol{B}$ stands for the *Kronecker product* of matrices $\boldsymbol{A} = (a_{jk})_{j,k=1}^{r} \in \mathbb{C}^{r\times r}$ and $\boldsymbol{B} \in \mathbb{C}^{r\times r}$,

$$\boldsymbol{A} \otimes \boldsymbol{B} := \begin{bmatrix} a_{11}\boldsymbol{B} & \dots & a_{1r}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{r1}\boldsymbol{B} & \dots & a_{rr}\boldsymbol{B} \end{bmatrix}$$

(see [33,41]). As in the scalar case, the spectral properties of the transition operator govern stability and regularity of scaling vectors.

The *subdivision operator* $S = S_{\boldsymbol{P}}$ associated with the refinement mask $\{\boldsymbol{P}_l\}_{l\in\mathbb{Z}}$ of $\boldsymbol{\Phi}$ is a linear operator on $l(\mathbb{Z})^r$, defined by

$$(S_{\boldsymbol{P}}\boldsymbol{c})_\alpha = \sum_{\beta\in\mathbb{Z}} \boldsymbol{P}^\star_{\alpha-2\beta}\, \boldsymbol{c}_\beta, \tag{9}$$

where $\boldsymbol{c} \in l(\mathbb{Z})^r$. Here $l(\mathbb{Z})^r$ denotes the set of sequences with arbitrary vectors of $\mathbb{R}^r$ as entries. With the help of the double infinite matrix

$$\boldsymbol{L} := (\boldsymbol{P}_{\alpha-2\beta})_{\alpha,\beta=-\infty}^{\infty}$$

we find $S_{\boldsymbol{P}}\boldsymbol{c} = \boldsymbol{L}^\star\, \boldsymbol{c}$. Note that with the infinite vector $\boldsymbol{F} := (\dots, \boldsymbol{\Phi}(\cdot + 1)^{\mathrm{T}}, \boldsymbol{\Phi}(\cdot)^{\mathrm{T}}, \boldsymbol{\Phi}(\cdot - 1)^{\mathrm{T}}, \dots)^{\mathrm{T}}$ the refinement equation (3) can formally be written in the vector form $\boldsymbol{L}\,\boldsymbol{F}(2\cdot) = \boldsymbol{F}$.
Assuming that $\boldsymbol{c} \in l^2(\mathbb{Z})^r$, we can consider Fourier series $\hat{\boldsymbol{c}}(\omega) := \sum_l \boldsymbol{c}_l\, e^{-i\omega l}$; then (9) leads to

$$(\widehat{S_{\boldsymbol{P}}\boldsymbol{c}})(\omega) = 2\,\boldsymbol{P}(\omega)^\star\, \hat{\boldsymbol{c}}(2\omega).$$

In the scalar case, there is a simple connection between transition and subdivision operator, namely

$$(\widehat{S_{|P|^2}c})(\omega) = (T^\star \hat{c})(\omega) = 2\,|P(\omega)|^2\, \hat{c}(2\omega)$$

for $c \in l^2(\mathbb{Z})$. Here $T^\star$ denotes the adjoint operator of T. In vector case, we can find a similar close relation (see e.g. [47]): Let for the $r \times r$ matrix $\boldsymbol{H}$ with columns $\boldsymbol{h}_1, \dots, \boldsymbol{h}_r$ the vec-operator be defined as

$$\mathrm{vec}\,\boldsymbol{H} = (\boldsymbol{h}_1^{\mathrm{T}}, \dots, \boldsymbol{h}_r^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{C}^{r^2}.$$

Introduce the scalar product of two matrices $\boldsymbol{H}, \boldsymbol{G} \in L^2(\mathbb{R})^{r\times r}$ by

$$\langle \boldsymbol{H}, \boldsymbol{G}\rangle_2 := \langle \mathrm{vec}\,\boldsymbol{H}, \mathrm{vec}\,\boldsymbol{G}\rangle := \sum_{j=1}^{r^2} \langle (\mathrm{vec}\,\boldsymbol{H})_j, (\mathrm{vec}\,\boldsymbol{G})_j\rangle_{L^2}$$

$$= \sum_{j=1}^{r^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathrm{vec}\,\boldsymbol{H})_j(\omega)\, \overline{(\mathrm{vec}\,\boldsymbol{G})_j(\omega)}\, \mathrm{d}\omega$$

and recall the well-known property of Kronecker products $\mathrm{vec}\,(\boldsymbol{A}\,\boldsymbol{X}\,\boldsymbol{B}) = (\boldsymbol{B}^{\mathrm{T}} \otimes \boldsymbol{A})\,\mathrm{vec}\,\boldsymbol{X}$ for $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X} \in \mathbb{C}^{r\times r}$. Then we find

$$\begin{aligned}
\langle T\boldsymbol{H},\, \boldsymbol{G}\rangle_2 &= \Big\langle \mathrm{vec}\,\Big(\boldsymbol{P}(\tfrac{\cdot}{2})\,\boldsymbol{H}(\tfrac{\cdot}{2})\,\boldsymbol{P}(\tfrac{\cdot}{2})^\star \\
&\quad + \boldsymbol{P}(\tfrac{\cdot}{2}+\pi)\,\boldsymbol{H}(\tfrac{\cdot}{2}+\pi)\,\boldsymbol{P}(\tfrac{\cdot}{2}+\pi)^\star\Big)\,,\ \mathrm{vec}\,\boldsymbol{G}\Big\rangle \\
&= \Big\langle \Big(\overline{\boldsymbol{P}(\tfrac{\cdot}{2})} \otimes \boldsymbol{P}(\tfrac{\cdot}{2})\Big)\ \mathrm{vec}\,\boldsymbol{H}(\tfrac{\cdot}{2}) \\
&\quad + \Big(\overline{\boldsymbol{P}(\tfrac{\cdot}{2}+\pi)} \otimes \boldsymbol{P}(\tfrac{\cdot}{2}+\pi)\Big)\ \mathrm{vec}\,\boldsymbol{H}(\tfrac{\cdot}{2}+\pi),\ \mathrm{vec}\,\boldsymbol{G}\Big\rangle \\
&= \sum_{j=1}^{r^2} \frac{1}{2\pi}\int_{-2\pi}^{2\pi} \Big[\Big(\overline{\boldsymbol{P}(\tfrac{\omega}{2})} \otimes \boldsymbol{P}(\tfrac{\omega}{2})\Big)\ \mathrm{vec}\,\boldsymbol{H}(\tfrac{\omega}{2})\Big]_j\ \overline{(\mathrm{vec}\,\boldsymbol{G})_j(\omega)}\,\mathrm{d}\omega \\
&= \frac{2}{2\pi}\sum_{k=1}^{r^2}\int_{-\pi}^{\pi} [\mathrm{vec}\,\boldsymbol{H}(\omega)]_k\ \Big[\Big(\overline{\boldsymbol{P}(\omega)} \otimes \boldsymbol{P}(\omega)\Big)^\star\ \overline{\mathrm{vec}\,\boldsymbol{G}(2\omega)}\Big]_k\ \mathrm{d}\omega \\
&= 2\langle \mathrm{vec}\,\boldsymbol{H},\ \big(\overline{\boldsymbol{P}} \otimes \boldsymbol{P}\big)^{\mathrm{T}}\ \mathrm{vec}\,\boldsymbol{G}(2\cdot)\rangle.
\end{aligned}$$

On the other hand, for $\boldsymbol{a}, \boldsymbol{b} \in l^2(\mathbb{Z})^r$,

$$\begin{aligned}
\langle \boldsymbol{H},\, (\widehat{S\boldsymbol{a}})\,(\widehat{S\boldsymbol{b}})^\star\rangle_2 &= 4\,\langle \mathrm{vec}\,\boldsymbol{H},\ \mathrm{vec}\,(\boldsymbol{P}^\star\,\hat{\boldsymbol{a}}(2\cdot)\,\hat{\boldsymbol{b}}(2\cdot)^\star\,\boldsymbol{P})\rangle \\
&= 4\,\langle \mathrm{vec}\,\boldsymbol{H}, \big(\overline{\boldsymbol{P}} \otimes \boldsymbol{P}\big)^{\mathrm{T}}\ \mathrm{vec}\,(\hat{\boldsymbol{a}}\hat{\boldsymbol{b}}^\star)(2\cdot)\rangle.
\end{aligned}$$

Together, these observations imply the relation

$$2\,T^\star(\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}}^\star) = (\widehat{S\boldsymbol{a}})\,(\widehat{S\boldsymbol{b}})^\star.$$

Both operators play a crucial role in characterization of scaling vectors and lead to deep insight in the structure of the solutions of matrix refinement equations. As in the scalar case, the subdivision operator implies an efficient algorithm for the iterative computation of $\boldsymbol{\Phi}$ in time domain. The corresponding subdivision algorithm is closely related with the cascade algorithm (see e.g. [38], Theorem 2.1, [39]). Actually, there is no reason to restrict the subdivision operator to the L^2-case, general solution vectors with components in L^p can also be handled (see e.g. [12,30,31]). However, in $L^2(\mathbb{R})$, the transition operator often provides simpler results. Since both, T and S are linear operators, their spectral properties are computable by considering their representing matrices.

§2. Existence, Uniqueness, and Stability of Scaling Vectors

In this section we summarize some basic results on existence and uniqueness of solutions $\boldsymbol{\Phi}$ of (3) in terms of the symbol $\boldsymbol{P}(\omega)$. We also are going to characterize the L^2-stability of $\boldsymbol{\Phi}$.

Let us assume that the mask symbol of $\mathbf{\Phi}$ is given in the form

$$\boldsymbol{P}(\omega) := \frac{1}{2}\sum_{k=0}^{N} \boldsymbol{P}_k e^{-i\omega k}. \tag{10}$$

The following theorem states a necessary and sufficient condition of existence of a solution vector of (3).

Theorem 1. *The matrix refinement equation* (3) *has a compactly supported distributional solution vector* $\mathbf{\Phi}$ *if and only if* $\boldsymbol{P}(0)$ *has an eigenvalue of the form* 2^n, $n \in \mathbb{Z}, n \geq 0$.

Proof: The necessary part of Theorem 1 was first obtained in [21]. Since $\mathbf{\Phi}$ is compactly supported, $\hat{\mathbf{\Phi}}$ is analytic. Hence, $\hat{\mathbf{\Phi}} \neq \mathbf{0}$ implies that there is an integer $\alpha \geq 0$ with $\mathrm{D}^\alpha \hat{\mathbf{\Phi}}(0) \neq \mathbf{0}$ and $\mathrm{D}^n \hat{\mathbf{\Phi}}(0) = \mathbf{0}$ for $n = 0, \ldots, \alpha - 1$. Thus,

$$\begin{aligned}\mathrm{D}^\alpha \hat{\mathbf{\Phi}}(0) &= \mathrm{D}^\alpha \left[\boldsymbol{P}(\frac{\omega}{2})\, \hat{\mathbf{\Phi}}(\frac{\omega}{2})\right] |_{\omega=0} = \frac{1}{2^\alpha}\sum_{n=0}^{\alpha}\binom{\alpha}{n} \mathrm{D}^{n-\alpha}\boldsymbol{P}(0)\, \mathrm{D}^n \hat{\mathbf{\Phi}}(0) \\ &= \frac{1}{2^\alpha}\boldsymbol{P}(0)\, \mathrm{D}^\alpha \hat{\mathbf{\Phi}}(0).\end{aligned}$$

The sufficient part was proved in [34]. Consider

$$\hat{\mathbf{\Phi}}_n(\omega) = \prod_{j=1}^{n} \boldsymbol{P}(\frac{\omega}{2^j})\, \hat{\mathbf{\Phi}}_0(\frac{\omega}{2^n})$$

with suitable compactly supported $\mathbf{\Phi}_0$. It can be shown that $\hat{\mathbf{\Phi}}_n$ converges for $n \to \infty$ to an entire function $\hat{\mathbf{\Phi}}$ with polynomial growth. Moreover, for $n > 0$, the solution $\mathbf{\Phi}$ of the refinement equation (3) with symbol $\boldsymbol{P}(\omega)$ is n-th distributional derivative of the solution of the dilation equation with symbol $\frac{1}{2^n}\boldsymbol{P}(\omega)$. □

Theorem 1 is analogous to the condition of existence in the scalar case $r = 1$. However, unlike the scalar case when the uniqueness (up to multiplication by a constant) of a distributional solution is also guaranteed, in the vector case ($r > 1$) the number of linearly independent solutions of the matrix refinable equation (3) is determined by the multiplicity of eigenvalue 2^n of $\boldsymbol{P}(0)$.

Without loss of generality we restrict ourselves to the case $n = 0$ and assume that 1 is the only eigenvalue of $\boldsymbol{P}(0)$ of the form 2^n, $n \in \mathbb{Z}$, $n \geq 0$. (As we will see later, L^2-stability implies that the spectral radius of $\boldsymbol{P}(0)$ is equal to 1). Then uniqueness can be ensured as follows (see [21,34]).

Theorem 2. *Let* 1 *be the only eigenvalue of* $\boldsymbol{P}(0)$ *of the form* 2^n, $n \in \mathbb{Z}$, $n \geq 0$. *The matrix refinement equation (3) has a unique (up to a constant) solution* $\mathbf{\Phi}$ *with* $\widehat{\mathbf{\Phi}}(0) = \boldsymbol{r}$ *if and only if 1 is a simple eigenvalue of* $\boldsymbol{P}(0)$ *and* $\boldsymbol{r} = \boldsymbol{P}(0)\boldsymbol{r}$. *In particular,*

$$\hat{\mathbf{\Phi}}(\omega) = \lim_{L\to\infty} \prod_{j=1}^{L} \boldsymbol{P}\left(\frac{\omega}{2^j}\right) \boldsymbol{r}. \tag{11}$$

The convergence of the infinite product (11) is also studied in [9,23,56].

We are especially interested in L^2-stable solutions. Let us introduce the following definition. A matrix (or a linear operator) $\boldsymbol{A}$ is said to satisfy *Condition E* if it has a simple eigenvalue 1 and the moduli of all its other eigenvalues are less then 1. First we observe some necessary conditions (see e.g. [12,25,34]).

Theorem 3. *Let $\boldsymbol{\Phi}$ be a compactly supported, L^2-stable solution vector of (3). Then for the corresponding symbol $\boldsymbol{P}(\omega)$ we have:*
(a) *$\boldsymbol{P}(0)$ satisfies Condition E.*
(b) *There exists a nonzero vector $\boldsymbol{y} \in \mathbb{R}^r$ such that $\boldsymbol{y}^\star \boldsymbol{P}(0) = \boldsymbol{y}^\star$ and $\boldsymbol{y}^\star \boldsymbol{P}(\pi) = \mathbf{0}^{\mathrm{T}}$.*

Proof: Recall that the autocorrelation symbol $\boldsymbol{\Omega}$ is an eigenmatrix of the transition operator T to the eigenvalue 1 and, by L^2-stability, $\boldsymbol{\Omega}(\omega)$ is invertible for all $\omega \in \mathbb{R}$. For a given eigenvalue λ of $\boldsymbol{P}(0)$ with left eigenvector $\boldsymbol{y}^\star$ we find

$$
\begin{aligned}
\boldsymbol{y}^\star \boldsymbol{\Omega}(0)\,\boldsymbol{y} &= \boldsymbol{y}^\star (T\boldsymbol{\Omega})(0)\,\boldsymbol{y} \\
&= \boldsymbol{y}^\star\, \boldsymbol{P}(0)\,\boldsymbol{\Omega}(0)\,\boldsymbol{P}(0)^\star\,\boldsymbol{y} + \boldsymbol{y}^\star\,\boldsymbol{P}(\pi)\,\boldsymbol{\Omega}(\pi)\,\boldsymbol{P}(\pi)^\star\,\boldsymbol{y} \\
&\geq |\lambda|^2\, \boldsymbol{y}^\star \boldsymbol{\Omega}(0)\,\boldsymbol{y}
\end{aligned}
$$

which means that $|\lambda| \leq 1$. For $\lambda = 1$, $\boldsymbol{y}^\star\,\boldsymbol{P}(\pi)\,\boldsymbol{\Omega}(\pi)\,\boldsymbol{P}(\pi)^\star\,\boldsymbol{y} = 0$, and since $\boldsymbol{\Omega}(\pi)$ is invertible, $\boldsymbol{y}^\star\,\boldsymbol{P}(\pi) = 0$. The hypothesis, that there are other eigenvalues of $\boldsymbol{P}(0)$ on the unit circle can be shown to contradict the L^2-stability. $\square$

Necessary conditions for existence of L^2-stable (or more general L^p-stable) solution vectors of (3) can also be given in terms of the subdivision operator (see e.g. [31], Theorem 2.1).

Obviously, a necessary condition for L^2-stability of $\boldsymbol{\Phi}$ is that the entries of $\boldsymbol{\Phi}$ are in $L^2(\mathbb{R})$. The following theorem generalizes the results of Villemoes [54].

Theorem 4. *Let $\boldsymbol{P}(\omega)$ be a matrix of trigonometric polynomials satisfying the assertions (a), (b) of Theorem 3 and $\boldsymbol{P}(0)\boldsymbol{r} = \boldsymbol{r}$. Let $\boldsymbol{U}$ be an invertible matrix with first column $\boldsymbol{r}$ such that $\boldsymbol{U}^{-1}\,\boldsymbol{P}(0)\,\boldsymbol{U}$ is the Jordan matrix of $\boldsymbol{P}(0)$ with leading entry 1. Further, let $\boldsymbol{\Phi}$ be the corresponding solution vector of (3) with $\hat{\boldsymbol{\Phi}}(0) = \boldsymbol{r}$. Then $\boldsymbol{\Phi} \in L^2(\mathbb{R})^r$ if and only if there exists an $r \times r$ matrix $\boldsymbol{H} \in \mathbb{H}_N$ satisfying $T\boldsymbol{H} = \boldsymbol{H}$, and the leading entry of $\boldsymbol{U}^{-1}\,\boldsymbol{H}(0)\,(\boldsymbol{U}^{-1})^\star$ is positive.*

Proof: We refer to [34], Proposition 3.14. If $\boldsymbol{y}^\star$ is a left eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1 with $\boldsymbol{y}^\star\,\boldsymbol{r} = 1$, then $\boldsymbol{U}^{-1}$ has $\boldsymbol{y}^\star$ as its first row. Theorem 3 implies that $\boldsymbol{y}^\star\,\boldsymbol{P}(\pi) = 0$. Repeated application of (4) easily leads to $\boldsymbol{y}^\star\,\hat{\boldsymbol{\Phi}}(2\pi l) = \delta_{0,l}$.
1. If $\boldsymbol{\Phi} \in L^2(\mathbb{R})^r$ then its autocorrelation symbol $\boldsymbol{\Omega}$ is a matrix of trigonometric polynomials in $\mathbb{H}_N$ satisfying $T\boldsymbol{\Omega} = \boldsymbol{\Omega}$ and

$$
\boldsymbol{y}^\star\,\boldsymbol{\Omega}(0)\boldsymbol{y} = \boldsymbol{y}^\star \sum_{l=-\infty}^{\infty} \hat{\boldsymbol{\Phi}}(2\pi l)\,\hat{\boldsymbol{\Phi}}(2\pi l)^\star\,\boldsymbol{y} = 1.
$$

2. Conversely, assume that there is a matrix $\boldsymbol{H} \in \mathbb{H}_N$ with $T\boldsymbol{H} = \boldsymbol{H}$, and the first entry of $\boldsymbol{U}^{-1}\,\boldsymbol{H}(0)\,(\boldsymbol{U}^{-1})^\star$ is $c > 0$. Set $\boldsymbol{\Pi}_n(\omega) := \prod_{j=1}^n \boldsymbol{P}(2^{-j}\omega)$, $\boldsymbol{\Pi}(\omega) := \lim_{n\to\infty} \boldsymbol{\Pi}_n(\omega)$ and observe that, by Theorem 3 (a), $\boldsymbol{\Pi}(\omega)\,\boldsymbol{U} = (\hat{\boldsymbol{\Phi}}(\omega), \boldsymbol{0}, \ldots, \boldsymbol{0})$. Hence,

$$\boldsymbol{\Pi}(\omega)\,\boldsymbol{H}(0)\,\boldsymbol{\Pi}(\omega)^\star = \boldsymbol{\Pi}(\omega)\boldsymbol{U}\,\boldsymbol{U}^{-1}\,\boldsymbol{H}(0)\,(\boldsymbol{U}^{-1})^\star\,\boldsymbol{U}^\star\,\boldsymbol{\Pi}(\omega) = c\,\hat{\boldsymbol{\Phi}}(\omega)\hat{\boldsymbol{\Phi}}(\omega)^\star.$$

Using (8) and $T\boldsymbol{H} = \boldsymbol{H}$ we get

$$\begin{aligned} c\int_{-\infty}^{\infty} \hat{\boldsymbol{\Phi}}(\omega)\,\hat{\boldsymbol{\Phi}}(\omega)^\star\,\mathrm{d}\omega &= \lim_{n\to\infty}\int_{-2^n\pi}^{2^n\pi} \boldsymbol{\Pi}_n(\omega)\,\boldsymbol{H}(\frac{\omega}{2^n})\,\boldsymbol{\Pi}_n(\omega)^\star\,\mathrm{d}\omega \\ &= \lim_{n\to\infty}\int_{-\pi}^{\pi}(T^n\boldsymbol{H})(\omega)\mathrm{d}\omega = \int_{-\pi}^{\pi}\boldsymbol{H}(\omega)\mathrm{d}\omega, \end{aligned}$$

and the assertion can be derived. □

As in the scalar case, there are three different methods for characterization of necessary and sufficient conditions of L^2-stability of $\boldsymbol{\Phi}$. The first is based on spectral properties of the transition operator T associated with $\boldsymbol{P}$ (see e.g. [48]). Using the representing matrix of T (see e.g. [41]), the resulting stability condition can be seen as a generalization of Lawton's criteria [35] for scaling functions. Another way is to try to find explicit conditions on the mask symbol $\boldsymbol{P}$. This idea generalizes the criteria of Cohen [8] and that of Jia and Wang [32]. However, these conditions are rather complicated. One has not only to struggle with non commuting matrix products, but also needs to ensure the algebraic linear independence of the components ϕ_ν of $\boldsymbol{\Phi}$ and their translates in terms of $\boldsymbol{P}(\omega)$ [24,43,55]. Both of these problems do not occur in the scalar case.

Finally, there is a close relation between stability and convergence of subdivision schemes (see e.g. [10,12,38,39]).

We only want to present the stability conditions in terms of the transition operator, proved in [48].

Theorem 5. *The refinable function vector $\boldsymbol{\Phi}$ is L^2-stable if and only if its symbol $\boldsymbol{P}(0)$ satisfies the Condition E and the corresponding transition operator T restricted to $\mathbb{H}_N$ satisfies Condition E where the eigenmatrix corresponding to the eigenvalue 1 is positive definite for all $\omega \in \mathbb{R}$.*

Proof: 1. The sufficiency of the assertions is obvious: Since $\boldsymbol{P}(0)$ satisfies Condition E, Theorem 2 implies that the matrix refinement equation (3) has a unique, compactly supported solution vector $\boldsymbol{\Phi}$. Let $\boldsymbol{H} \in \mathbb{H}_N$ be a positive definite eigenmatrix of T with eigenvalue 1, then by Theorem 4, $\boldsymbol{\Phi} \in L^2(\mathbb{R})^r$. Thus, the autocorrelation symbol $\boldsymbol{\Omega}$ of $\boldsymbol{\Phi}$ exists, and it equals $\boldsymbol{H}$ up to multiplication by a constant.

2. The necessity is more complicated. The assertion on $\boldsymbol{P}(0)$ follows from Theorem 3 (a). Theorem 4 implies that T possesses an eigenvalue 1. If there were an eigenvalue λ of T with $|\lambda| > 1$, $T\tilde{\boldsymbol{H}} = \lambda\tilde{\boldsymbol{H}}$, then since

$\mathbf{\Pi}(\omega)\,\tilde{\boldsymbol{H}}(0)\,\mathbf{\Pi}(\omega)^\star = \tilde{c}\,\hat{\mathbf{\Phi}}(\omega)\,\hat{\mathbf{\Phi}}(\omega)^\star$ with some finite constant $\tilde{c}$, the divergence of

$$\| \int_{-\infty}^{\infty} \mathbf{\Pi}(\omega)\,\tilde{\boldsymbol{H}}(0)\,\mathbf{\Pi}(\omega)^\star\,\mathrm{d}\omega\| = \| \lim_{n\to\infty} \int_{-\pi}^{\pi} T^n \tilde{\boldsymbol{H}}(\omega)\,\mathrm{d}\omega\|$$

would contradict the assertion $\mathbf{\Phi} \in L^2(\mathbb{R})^r$ (see the proof of Theorem 4). Similar arguments apply for showing that 1 is the only eigenvalue of T on the unit circle. Finally, the assertion follows by recalling that the autocorrelation $\mathbf{\Omega}$ of $\mathbf{\Phi}$ is positive definite and satisfies $T\mathbf{\Omega} = \mathbf{\Omega}$. □

§3. Approximation Order and Factorization of the Symbol

In this section we overview the polynomial reproduction properties of a scaling vector $\mathbf{\Phi}$. We start with definitions and notation. Let $S_0(\mathbf{\Phi})$ denote the linear space of all functions of the form $\sum_{\alpha\in\mathbb{Z}} \boldsymbol{b}_\alpha^\star\, \mathbf{\Phi}(\cdot-\alpha)$, where $\boldsymbol{b} \in l(\mathbb{Z})^r$ is an arbitrary vector sequence on $\mathbb{Z}$.

Assume Π_k to be the space of algebraic polynomials of degree at most k. A vector sequence $\boldsymbol{y} \in l(\mathbb{Z})^r$ is called *polynomial vector sequence of degree* k, if there exists a vector of algebraic polynomials $\boldsymbol{Y} \in (\Pi_k)^r$ with $\boldsymbol{y}_l = \boldsymbol{Y}(l)$ $(l \in \mathbb{Z})$.

We say that a function vector $\mathbf{\Phi}$ has *accuracy* k if $\Pi_{k-1} \subseteq S_0(\mathbf{\Phi})$.

A function vector $\mathbf{\Phi} \in (L^2(\mathbb{R}))^r$ is said to provide *approximation order* k if, for any sufficiently smooth function $g \in L^2(\mathbb{R})$,

$$\mathrm{dist}(g,\, S_h(\mathbf{\Phi})) = O(h^k).$$

Here $S(\mathbf{\Phi}) = L^2(\mathbb{R}) \cap S_0(\mathbf{\Phi})$ and

$$S_h(\mathbf{\Phi}) := \{f(\cdot/h) :\, f \in S(\mathbf{\Phi})\}.$$

As shown in [27], $\mathbf{\Phi}$ provides approximation order k if and only if it has accuracy k.

There are two equivalent methods for characterization of accuracy of a scaling vector $\mathbf{\Phi}$; the first method uses the subdivision operator S, the second gives conditions on the symbol $\boldsymbol{P}(\omega)$ in the Fourier domain.

Theorem 6. *Let $\mathbf{\Phi} = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ be a compactly supported, L^2-stable scaling vector with a finite refinement mask $\{\boldsymbol{P}_l\}_{l\in\mathbb{Z}}$. Then following statements are equivalent:*

(i) *$\mathbf{\Phi}$ provides approximation order k.*

(ii) *There exists vector sequences $\boldsymbol{y}_n = \{(\boldsymbol{y}_n)_\alpha\}_{\alpha\in\mathbb{Z}} \in l(\mathbb{Z})^r$ $(n = 0, \ldots k-1)$ such that*

$$\sum_{\alpha\in\mathbb{Z}} (\boldsymbol{y}_n)_\alpha^\star\, \mathbf{\Phi}(t-\alpha) = t^n \quad (t \in \mathbb{R}, n = 0, \ldots, k-1). \tag{12}$$

These series converge absolutely and uniformly on any compact interval of $\mathbb{R}$. In particular, $\boldsymbol{y}_n$ are polynomial vector sequences of degree n with

$$(\boldsymbol{y}_n)_\alpha = \sum_{l=0}^{n} \binom{n}{l}\, \alpha^{n-l}\, (\boldsymbol{y}_l)_0. \tag{13}$$

(iii) *There is a unique superfunction* $f \in S(\boldsymbol{\Phi})$ *which is a finite linear combination of shifts* $\boldsymbol{\Phi}(\cdot - l)$ $(l = 0, \ldots, k-1)$*, and satisfies*

$$\mathrm{D}^{\mu} f(2\pi l) = \delta_{\mu,0}\, \delta_{l,0} \quad (l \in \mathbb{Z}, \mu = 0, \ldots, k-1). \tag{14}$$

(iv) *There exists a nontrivial vector of trigonometric polynomials* $\boldsymbol{A}(\omega) := \sum_{l=0}^{k-1} \boldsymbol{a}_l\, e^{-i\omega l}$ *with coefficient vectors* $\boldsymbol{a}_l \in \mathbb{R}^r$*, such that the symbol* $\boldsymbol{P}$ *of* $\boldsymbol{\Phi}$ *satisfies*

$$\begin{aligned} \mathrm{D}^n [\boldsymbol{A}(2\omega)^\star\, \boldsymbol{P}(\omega)]|_{\omega=0} &= \mathrm{D}^n\, \boldsymbol{A}(0)^\star, \\ \mathrm{D}^n [\boldsymbol{A}(2\omega)^\star\, \boldsymbol{P}(\omega)]|_{\omega=\pi} &= \mathbf{0}^{\mathrm{T}}. \end{aligned} \tag{15}$$

for $n = 0, \ldots, k-1$.

(v) *There exists a polynomial vector sequence* $\boldsymbol{b}$ *satisfying*

$$S_{\boldsymbol{P}}\, \boldsymbol{b} = \left(\frac{1}{2}\right)^{k-1} \boldsymbol{b}$$

and $\boldsymbol{b}^{\mathrm{T}}\, \hat{\boldsymbol{\Phi}}(0)$ *has degree* $k-1$.

Proof: We are going to give a sketch of the proof:

(i) $\Rightarrow$ (ii): The existence of sequences $\boldsymbol{y}_n \in l(\mathbb{Z})^r$ satisfying (12) follows directly from the definition of accuracy. Only their polynomial structure (13) has to be shown. For $n = 0$ we find (by replacing of t by $t+1$)

$$1 = \sum_{\alpha \in \mathbb{Z}} (\boldsymbol{y}_0)_\alpha^\star\, \boldsymbol{\Phi}(t - 1 - \alpha) = \sum_{\alpha \in \mathbb{Z}} (\boldsymbol{y}_0)_{\alpha+1}^\star\, \Phi(t - \alpha).$$

Hence $(\boldsymbol{y}_0)_\alpha = (\boldsymbol{y}_0)_{\alpha+1}$ for all $\alpha \in \mathbb{Z}$. For $n > 0$ the assertion easily follows by induction (see [42], Lemma 2.1). Here we need to use the assumed L^2-stability of $\boldsymbol{\Phi}$ which also applies to polynomial sequences.

(ii)$\Rightarrow$(iii): Take the trigonometric polynomial vector

$$\boldsymbol{A}(\omega) = \sum_{l=0}^{k-1} \boldsymbol{a}_l\, e^{-i\omega l} \tag{16}$$

with $\boldsymbol{a}_l \in \mathbb{R}^r$ determined by

$$\mathrm{D}^n \boldsymbol{A}(0) = i^n\, (\boldsymbol{y}_n)_0 \quad (n = 0, \ldots, k-1). \tag{17}$$

Consider the function $f \in S(\boldsymbol{\Phi})$ with $\hat{f}(\omega) := \boldsymbol{A}(\omega)^\star\, \hat{\boldsymbol{\Phi}}(\omega)$. Then f is the desired superfunction satisfying (14).

For $n = 0$, (12) and Poisson summation formula imply

$$1 = \sum_{\alpha \in \mathbb{Z}} \boldsymbol{A}(0)^\star\, \boldsymbol{\Phi}(t - \alpha) = \boldsymbol{A}(0)^\star \sum_{j \in \mathbb{Z}} e^{2\pi i j x}\, \hat{\boldsymbol{\Phi}}(2\pi j).$$

Hence, $\hat{f}(0) = \boldsymbol{A}(0)^\star\, \hat{\boldsymbol{\Phi}}(0) = 1$ and $\hat{f}(2\pi j) = \boldsymbol{A}(0)^\star\, \hat{\boldsymbol{\Phi}}(2\pi j) = 0$ for $j \in \mathbb{Z} \backslash \{0\}$. For $n > 0$, an induction proof using (12) and differentiated versions of Poisson summation formula can be applied (see [42], Theorem 2.2). The uniqueness of f is shown in [6], Theorem 4.2.

(iii) $\Rightarrow$ (i): This statement is true, since f satisfies the Strang-Fix conditions of order k.

(iii) $\Leftrightarrow$ (iv): We show that the trigonometric polynomial vector $\boldsymbol{A}(\omega)$ determined by (16)–(17) also satisfies the conditions (15). First, note that $\hat{f}(2\omega) = \boldsymbol{A}(2\omega)^\star\, \hat{\boldsymbol{\Phi}}(2\omega) = \boldsymbol{A}(2\omega)^\star\, \boldsymbol{P}(\omega)\, \hat{\boldsymbol{\Phi}}(\omega)$. Using (14), we find for $l \in \mathbb{Z}$

$$0 = \hat{f}(4\pi l + 2\pi) = \boldsymbol{A}(0)^\star\, \boldsymbol{P}(\pi)\, \hat{\boldsymbol{\Phi}}(2\pi l + \pi).$$

The linear independence of the sequences $\{\hat{\phi}_\nu(2\pi l + \pi)\}_{l\in\mathbb{Z}}$ $(\nu = 1, \ldots, r)$ implies $\boldsymbol{A}(0)^\star\, \boldsymbol{P}(\pi) = \boldsymbol{0}^{\mathrm{T}}$. To get the second equality in (15) for $n > 0$, we proceed by induction, using $\mathrm{D}^n \hat{f}(4\pi l + 2\pi) = 0$.

The first equality in (15) for $n = 0$ follows by comparison of $\hat{f}(4\pi l) = \boldsymbol{A}(0)^\star\, \boldsymbol{P}(0)\, \hat{\boldsymbol{\Phi}}(2\pi l) = \delta_{0,l}$ and $\hat{f}(2\pi l) = \boldsymbol{A}(0)^\star\, \hat{\boldsymbol{\Phi}}(2\pi l) = \delta_{0,l}$. For $n > 0$, the assertion is a consequence of $\mathrm{D}^n \hat{f}(4\pi l) = \mathrm{D}^n\, \hat{f}(2\pi l) = 0$ (see [40], Theorem 2). Conversely, take $\boldsymbol{A}(\omega)$ of the form (16) satisfying (15) for $n = 0, \ldots, k-1$ and with $\boldsymbol{A}(0)^\star\, \hat{\boldsymbol{\Phi}}(0) = 1$. This last equality is only a normalization, since $\boldsymbol{A}(0)^\star$ and $\hat{\boldsymbol{\Phi}}(0)$ are left and right eigenvector of $\boldsymbol{P}(0)$ to the simple eigenvalue 1, respectively. (Indeed, $\boldsymbol{A}$ is already uniquely determined by these conditions.) Then it can be shown that f, determined by $\hat{f}(\omega) = \boldsymbol{A}(\omega)^\star\, \hat{\boldsymbol{\Phi}}(\omega)$ is the superfunction.

(ii) $\Rightarrow$ (v): Let $\boldsymbol{b} := \boldsymbol{y}_{k-1}$ with $\boldsymbol{y}_{k-1}$ as in Theorem 6 (ii). On the one hand, with $\gamma = 2l + n$,

$$\begin{aligned} t^{k-1} &= \sum_{l\in\mathbb{Z}} \boldsymbol{b}_l^\star \boldsymbol{\Phi}(t-l) = \sum_{l\in\mathbb{Z}} \boldsymbol{b}_l^\star \sum_{n\in\mathbb{Z}} \boldsymbol{P}_n\, \boldsymbol{\Phi}(2t - 2l - n) \\ &= \sum_{\gamma\in\mathbb{Z}} (S_{\boldsymbol{P}}\boldsymbol{b})_\gamma^\star\, \boldsymbol{\Phi}(2t - \gamma), \end{aligned}$$

where (3) and (9) are used. On the other hand,

$$t^{k-1} = \frac{1}{2^{k-1}}(2t)^{k-1} = \frac{1}{2^{k-1}} \sum_{l\in\mathbb{Z}} \boldsymbol{b}_l^\star\, \boldsymbol{\Phi}(2t - l).$$

Comparison yields $S_{\boldsymbol{P}}\boldsymbol{b} = \frac{1}{2^{k-1}}\boldsymbol{b}$. By definition, $\boldsymbol{b}$ is a polynomial sequence and

$$\boldsymbol{b}_\alpha^\star\, \hat{\boldsymbol{\Phi}}(0) = \sum_{l=0}^{k-1} \binom{k-1}{l} \alpha^{k-1-l} (\boldsymbol{y}_s)_0^\star\, \hat{\boldsymbol{\Phi}}(0)$$

is of degree $k-1$ if and only if $(\boldsymbol{y}_0)_0^\star\, \hat{\boldsymbol{\Phi}}(0) \neq 0$. This is a direct consequence of the Poisson summation formula (applied to (12) for $n = 0$).

(v) $\Rightarrow$ (i): Let $\boldsymbol{b}$ satisfy the conditions of Theorem 6 (v). Set $p := \sum_{\alpha\in\mathbb{Z}} \boldsymbol{b}_\alpha^\star\, \boldsymbol{\Phi}(\cdot - \alpha)$. Then

$$p = \sum_{\alpha\in\mathbb{Z}} (S_{\boldsymbol{P}}\boldsymbol{b})_\alpha^\star\, \boldsymbol{\Phi}(2\cdot - \alpha) = \left(\frac{1}{2}\right)^{k-1} \sum_{\alpha\in\mathbb{Z}} \boldsymbol{b}_\alpha^\star\, \boldsymbol{\Phi}(2\cdot - \alpha) = \left(\frac{1}{2}\right)^{k-1} p(2\cdot).$$

Since $\boldsymbol{b}$ is a polynomial vector sequence of degree $k-1$, it follows that $\nabla^k_{(1/2)^n} p = 0$ for all positive integers n. Here $\nabla_h f := f - f(\cdot - h)$ and $\nabla^k_h f := \nabla^{k-1}_h (\nabla_h f)$. Using these properties of p one can find that $p(t) = ct^{k-1}$ with some constant $c \neq 0$ (see [30], Theorem 3.1). This means that $S_0(\boldsymbol{\Phi})$ contains the monomial t^{k-1}. Since $S_0(\boldsymbol{\Phi})$ is $\mathbb{Z}$-translation-invariant, it contains $1, \ldots, t^{k-1}$, i.e., $\boldsymbol{\Phi}$ has accuracy k. $\Box$

Let us give some further explanatory remarks.

1. In [26], Jia already characterized the L^∞-approximation order of FSI spaces $S_0(\boldsymbol{\Phi})$ in terms of Strang–Fix conditions implied on a single element $f \in S_0(\boldsymbol{\Phi})$. In the special case when $\boldsymbol{\Phi} = \phi$ is a single generator, Ron [46] showed that ϕ provides L^∞-approximation order k if and only if $S_0(\phi)$ contains Π_{k-1}. This statement was extended to the FSI spaces and L^p-approximation order in [27]. Observe that this result is not longer true for shift-invariant spaces on $\mathbb{R}^d$ with $d > 1$ (see [2]). There is a rich literature generalizing these results, including extensions to the multivariate case and to non compactly supported function vectors (see e.g. [3,4,5,6,15,20,28] and references therein).

2. The investigation of the approximation properties of FSI spaces in [4,5,6] was focused on the superfunction approach. In [6], de Boor, DeVore and Ron succeeded to construct a superfunction f of $S(\boldsymbol{\Phi}) \subset L^2(\mathbb{R}^d)$ with special properties. In the univariate case, this superfunction can be constructed directly (see Theorem 6). The symbol $\boldsymbol{A}$ of the superfunction is then used to derive direct conditions for the mask symbol $\boldsymbol{P}$ (see Theorem 6 (iv)). Conversely, the conditions (iv) on the mask symbol $\boldsymbol{P}$ can be successively applied to determine $\mathrm{D}^\mu \boldsymbol{A}(0)$ $\mu = 0, \ldots$ and hence to derive the accuracy of $\boldsymbol{\Phi}$ directly from the mask. Conditions of type (iv) were independently found in [22,37,42]. These approximation results can be generalized to the multivariate case with general dilation matrices (see [7]).

3. Already in 1994, Strang and Strela [50] had observed that accuracy k of $\boldsymbol{\Phi}$ implies that the double infinite matrix $\boldsymbol{L} = (\boldsymbol{P}_{\alpha-2\beta})_{\alpha,\beta\in\mathbb{Z}}$ has eigenvalues $1, \frac{1}{2}, \ldots, \left(\frac{1}{2}\right)^{k-1}$ with corresponding eigenvectors of special structure. Since $\boldsymbol{L}^\star$ is the representing matrix of the subdivision operator $S_{\boldsymbol{P}}$, their results are closely related to the conditions of Theorem 6 (v).

4. Jia, Riemenschneider and Zhou [30] showed that it suffices to consider the eigenvalue $\left(\frac{1}{2}\right)^{k-1}$ of $S_{\boldsymbol{P}}$ and its eigenvector sequence $\boldsymbol{b}$ only (see Theorem 6 (v)). Furthermore, [30] generalizes the result to the case of non-stable function vectors; then $S_{\boldsymbol{P}}\boldsymbol{b} - \left(\frac{1}{2}\right)^{k-1} \boldsymbol{b}$ has to be contained in the linear space $\{\boldsymbol{a} \in l(\mathbb{Z})^r : \sum_{\alpha\in\mathbb{Z}} \boldsymbol{a}^\star_\alpha \, \boldsymbol{\Phi}(\cdot - \alpha) = 0\}$.

In the scalar case $r = 1$, the conditions (iv) of Theorem 6 simplify to

$$\begin{aligned} \mathrm{D}^n \boldsymbol{P}(\pi) &= 0 \quad (n = 0, \ldots, k-1) \\ \boldsymbol{P}(0) &= 1. \end{aligned} \tag{18}$$

They directly imply a factorization of $\boldsymbol{P}(\omega)$ in the form

$$P(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^k Q(\omega). \tag{19}$$

In the vector case, the accuracy of $\boldsymbol{\Phi}$ also implies a factorization, but it is more complicated.

Theorem 7. *Let $\boldsymbol{\Phi}$ be a compactly supported, L^2-stable scaling vector with (finite) mask symbol $\boldsymbol{P}$. If $\boldsymbol{\Phi}$ provides approximation order k then there exists an $r \times r$ matrix $\boldsymbol{C}(\omega)$ of the form $\boldsymbol{C}(\omega) = \sum_{l=0}^{k} \boldsymbol{C}_l \, e^{-i\omega l}$ satisfying*

$$\det \boldsymbol{C}(\omega) = \text{const}\,(1 - e^{-i\omega})^k$$

such that

$$\boldsymbol{P}(\omega) = \frac{1}{2^k}\, \boldsymbol{C}(2\omega)\, \boldsymbol{Q}(\omega)\, \boldsymbol{C}(\omega)^{-1}, \tag{20}$$

where $\boldsymbol{Q}$ is a matrix of trigonometric polynomials. In particular,

$$\det \boldsymbol{P}(\omega) = \left(\frac{1+e^{-i\omega}}{2^r}\right)^k \det \boldsymbol{Q}(\omega).$$

For the proof we refer to [42]. As shown in [42], a matrix $\boldsymbol{C}$ can be constructed explicitly, and it can be factored into $\boldsymbol{C} = \boldsymbol{C}_1 \ldots \boldsymbol{C}_{k-1}$, where each $\boldsymbol{C}_l$ (corresponding to a change of approximation order by 1) has an analogous structure. If $\boldsymbol{A}(\omega)$ is a symbol of trigonometric polynomials of the form (16) satisfying (15) for $n = 0, \ldots, k-1$, then

$$\mathrm{D}^n [\boldsymbol{A}(\omega)^\star \, \boldsymbol{C}(\omega)]|_{\omega=0} = \mathbf{0}^{\mathrm{T}} \qquad (n = 0, \ldots, r-1) \tag{21}$$

(cf. [45]). The factorization matrix $\boldsymbol{C}$ is not unique. A general characterization of factorization matrices is presented in [44,45,52].

Using a reverse version of the factorization theorem, a procedure for construction of multi-scaling functions with special properties (symmetry, compact support, arbitrary approximation order) is given in [44]. We remark that there is a close connection between the spectrum of the symbol $\boldsymbol{P}(0)$ and that of the inner matrix $\boldsymbol{Q}(0)$. More exactly, $\boldsymbol{P}(0)$ possesses the spectrum $\{1, \mu_1, \ldots, \mu_{r-1}\}$, if $\boldsymbol{Q}(0)$ possesses the spectrum $\{1, 2^k\mu_1, \ldots, 2^k\mu_{r-1}\}$ (see [9], Lemma 4.3, [52], Lemma 2.2).

The factorization can be transfered into time domain, using the representing matrix of the subdivision operator $\boldsymbol{L}^\star$ (see [38]).

The inner matrix $\boldsymbol{Q}$ can be considered as a mask symbol of a distribution vector $\boldsymbol{\Psi}$ such that

$$\hat{\boldsymbol{\Phi}}(\omega) = \frac{1}{(i\omega)^k}\, \boldsymbol{C}(\omega)\, \hat{\boldsymbol{\Psi}}(\omega).$$

In the scalar case $r = 1$, when $C(\omega) = (1 + e^{-i\omega})^k$, we observe that $\hat{\boldsymbol{\Phi}}(\omega) = \hat{B}_k(\omega)\, \hat{\boldsymbol{\Psi}}(\omega)$, i.e., $\boldsymbol{\Phi} = B_k \star \boldsymbol{\Psi}$, where B_k is the cardinal B-spline of order k with support $[0, k]$. In the vector case, a similar convolution result is not obvious, however, if $\boldsymbol{\Phi}$ has accuracy k, a special linear combination $\hat{\boldsymbol{F}}(\omega) = \boldsymbol{M}(\omega)\hat{\boldsymbol{\Phi}}(\omega)$ (with some invertible matrix $\boldsymbol{M}$ of trigonometric polynomials) can be found, such that $\hat{\boldsymbol{F}}(\omega) = \hat{B}_k(\omega)\, \hat{\boldsymbol{\Psi}}_0(\omega)$, where $\boldsymbol{\Psi}_0$ is a refinable vector of distributions (see [45]).

§4. Regularity of Scaling Vectors

In this section we analyze the regularity of a scaling vector $\boldsymbol{\Phi}$. Again, let us set the symbol $\boldsymbol{P}(\omega)$ to be a matrix of trigonometric polynomials, and $\boldsymbol{\Phi}$ to be L^2-stable.

Throughout this section we assume that $\boldsymbol{\Phi}$ provides approximation order k. This assumption makes sense, since a stable scaling vector $\boldsymbol{\Phi}$ with all components in C^{k-1} always provides approximation order k (see [9], Lemma 2.2). Many papers dealing with the smoothness of scaling functions or scaling vectors rely on the given approximation order and the corresponding factorization of the mask symbol. Unfortunately, in the multivariate setting, no factorization properties are known, and the ideas based on factorization, can not be generalized to that case.

Similarly to the scalar case, we can use the product representation of $\hat{\boldsymbol{\Phi}}$,

$$\hat{\boldsymbol{\Phi}}(\omega) = \lim_{n\to\infty} \prod_{j=1}^{n} \boldsymbol{P}\left(\frac{\omega}{2^j}\right) \boldsymbol{r}, \tag{22}$$

where $\boldsymbol{r}$ is a right eigenvector of $\boldsymbol{P}(0)$ to the eigenvalue 1. By Theorem 2, this representation is unique up to multiplication by a constant.

The simplest method to find regularity estimates of $\boldsymbol{\Phi}$ is to consider the decay of its Fourier transform. Let us briefly recall the situation in the scalar case ($r = 1$). Let $P(\omega)$ be a trigonometric polynomial of the form

$$P(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right) Q(\omega),$$

where $Q(0) = 1$ and $Q(\pi) \neq 0$. Assume that the corresponding scaling function ϕ is given by $\hat{\phi}(\omega) = \prod_{j=1}^{\infty} P(\omega/2^j)$. Exploiting the factorization we find

$$|\hat{\phi}(\omega)| \leq C\,(1+|\omega|)^k \prod_{j=1}^{\infty} |Q(\omega/2^j)|.$$

Together with estimates of the type $\sup_\omega |Q(2^{k-1}\omega)\ldots Q(\omega)| \leq B^l$ we obtain

$$|\hat{\phi}(\omega)| \leq C\,(1+|\omega|)^{-k+\log B}$$

(see [13,14]). This idea can be generalized to the vector case (see [9]). Suppose that $\boldsymbol{\Phi} \in (L^2(\mathbb{R}))^r$ is compactly supported and

$$\|\hat{\Phi}(\omega)\| := (\sum_{\nu=1}^{r} |\hat{\phi}_\nu(\omega)|^2)^{1/2}.$$

For $\boldsymbol{H} = (h_{\mu,\nu})_{\mu,\nu=1}^r \in (L^2_{2\pi})^{r\times r}$ let $\|\boldsymbol{H}(\omega)\| = (\sum_{\mu,\nu=1}^r |h_{\mu,\nu}(\omega)|^2)^{1/2}$. Then the following theorem holds.

Theorem 8. *Let $\boldsymbol{P}$ be a finite mask symbol of a scaling vector $\boldsymbol{\Phi}$ such that*

$$\boldsymbol{P}(\omega) = \frac{1}{2^k}\, \boldsymbol{C}(2\omega)\, \boldsymbol{Q}(\omega)\, \boldsymbol{C}(\omega)^{-1}.$$

Here, $\boldsymbol{C}(\omega)$ and $\boldsymbol{Q}(\omega)$ are matrices of trigonometric polynomials satisfying the conditions of Theorem 7. Suppose that $\rho(\boldsymbol{Q}(0)) < 2$, and let for $l \geq 1$

$$\gamma_l := \frac{1}{l} \log_2 \sup_{\omega} \|\boldsymbol{Q}(2^{l-1}\omega) \ldots \boldsymbol{Q}(\omega)\|.$$

Then there exists a constant C such that for all $\omega \in \mathbb{R}$

$$\|\hat{\boldsymbol{\Phi}}(\omega)\| \leq C\,(1+|\omega|)^{-k-\gamma_l}. \tag{23}$$

Proof: The idea of the proof is based on the following observation. Using the factorization (20) in the infinite product (22), one can see that

$$\hat{\boldsymbol{\Phi}}(\omega) = \lim_{n\to\infty} \frac{1}{2^{kn}}\, \boldsymbol{C}(\omega)\, \boldsymbol{Q}(\frac{\omega}{2}) \ldots \boldsymbol{Q}(\frac{\omega}{2^n})\, \boldsymbol{C}(\frac{\omega}{2^n})\, \boldsymbol{r},$$

and with $\boldsymbol{E}(\omega) := (1-e^{-i\omega})^k\, \boldsymbol{C}(\omega)^{-1}$,

$$\hat{\boldsymbol{\Phi}}(\omega) = \lim_{n\to\infty} \left\{ \left(\frac{1}{2^n(1-e^{-i\omega/2^n})} \right)^k \boldsymbol{C}(\omega) \prod_{j=1}^{n} \boldsymbol{Q}(\frac{\omega}{2^j})\, \boldsymbol{E}(\frac{\omega}{2^n})\, \boldsymbol{r} \right\}.$$

Since $\lim_{n\to\infty} |2^{-n}(1-e^{-i\omega/2^n})^{-1}| = |\omega|^{-1}$ for $\omega \neq 0$, it follows that

$$\|\hat{\boldsymbol{\Phi}}(\omega)\| \leq C\,(1+|\omega|)^{-k}\, \|\boldsymbol{C}(\omega)\| \lim_{n\to\infty} \|\prod_{j=1}^{n} \boldsymbol{Q}(\frac{\omega}{2^j})\, \boldsymbol{E}(\frac{\omega}{2^n})\, \boldsymbol{r}\|. \tag{24}$$

We show that $\boldsymbol{E}(0)\,\boldsymbol{r}$ is a right eigenvector of $\boldsymbol{Q}(0)$: By assumption, there exists a polynomial vector $\boldsymbol{A}(\omega) = \sum_{l=0}^{k-1} \boldsymbol{a}_l\, e^{-i\omega l}$ with $\mathrm{D}^n[\boldsymbol{A}(\omega)^\star\, \boldsymbol{C}(\omega)]|_{\omega=0} = \boldsymbol{0}^{\mathrm{T}}$ for $n = 0, \ldots, k-1$. In particular, $\boldsymbol{A}(0)^\star\, \boldsymbol{C}(0) = \boldsymbol{0}^{\mathrm{T}}$. Since the eigenvalue 0 of $\boldsymbol{C}(0)$ has geometric multiplicity 1 and $\boldsymbol{C}(0)\,\boldsymbol{E}(0) = \boldsymbol{E}(0)\,\boldsymbol{C}(0) = \boldsymbol{0}$, we find $\boldsymbol{E}(0) = \boldsymbol{a}\,\boldsymbol{A}(0)^\star$ where $\boldsymbol{a}$ is a suitable right eigenvector of $\boldsymbol{C}(0)$ to the eigenvalue 0. Then $\boldsymbol{E}(0)\,\boldsymbol{P}(0) = \boldsymbol{E}(0)$, because $\boldsymbol{A}(0)^\star\, \boldsymbol{P}(0) = \boldsymbol{A}(0)^\star$. On the other hand, the factorization (20) also implies

$$\boldsymbol{E}(0)\, \boldsymbol{P}(0) = \boldsymbol{Q}(0)\, \boldsymbol{E}(0).$$

Thus, $\boldsymbol{Q}(0)\boldsymbol{E}(0)\,\boldsymbol{r} = \boldsymbol{E}(0)\,\boldsymbol{r}$.

As in the scalar case, the problem is now reduced to the estimation of the infinite matrix product of the inner matrix $\boldsymbol{Q}$. For a complete proof see [9], Theorem 4.1. □

From (23) it follows that the components of $\mathbf{\Phi}$ are continuous if $\boldsymbol{P}$ satisfies the conditions of the Theorem with $\gamma_l < k-1$ for some l. Estimates similar to Theorem 8 were independently found in [52].

The brute force method presented above, usually does not provide sharp estimates for the smoothness of the components of the scaling vector. It gives only lower bounds.

In the second part of this section, we are going to consider a more refined method based on the transition operator.

Let the Sobolev exponent s of $\mathbf{\Phi}$ be defined by

$$s = \sup\{\delta : \int_{-\infty}^{\infty} \|\hat{\mathbf{\Phi}}(\omega)\|^2 \, (1+|\omega|^2)^{\delta} \, \mathrm{d}\omega < \infty \}.$$

Assume that the symbol $\boldsymbol{P}(\omega)$ satisfies the conditions of polynomial reproduction, i.e., there exists a vector $\boldsymbol{A}(\omega) = \sum_{l=0}^{k-1} \boldsymbol{a}_l \, e^{-i\omega l}$, such that for $n = 0, \ldots, k-1$

$$\begin{aligned} \mathrm{D}^n[\boldsymbol{A}(2\omega)^{\star} \, \boldsymbol{P}(\omega)]|_{\omega=0} &= \mathrm{D}^n \boldsymbol{A}(0)^{\star}, \\ \mathrm{D}^n[\boldsymbol{A}(2\omega)^{\star} \, \boldsymbol{P}(\omega)]|_{\omega=\pi} &= \mathbf{0}^T. \end{aligned}$$

According to Theorem 7, this assumption implies a factorization of $\boldsymbol{P}$:

$$\boldsymbol{P}(\omega) = \frac{1}{2^k} \, \boldsymbol{C}(2\omega) \, \boldsymbol{Q}(\omega) \, \boldsymbol{C}(\omega)^{-1},$$

such that $\mathrm{D}^n[\boldsymbol{A}(\omega)^{\star} \, \boldsymbol{C}(\omega)]|_{\omega=0} = \mathbf{0}^{\mathrm{T}}$ for $n = 0, \ldots, k-1$.

Consider the transition operator $T = T_{\boldsymbol{P}}$ as given in (7). We want to show the basic idea of how to estimate regularity by spectral properties of T restricted to special subspaces. Recall that for $\boldsymbol{P}(\omega)$ of the form (10) the space $\mathbb{H}_M$ of $r \times r$ matrices of trigonometric polynomials of degree at most M is for $M \geq N$ a finite-dimensional space invariant under the action of T.

We need some definitions. For each matrix of 2π-periodic functions $\boldsymbol{H}(\omega) = (h_{i,j}(\omega))_{i,j=1}^r \in (L^2_{2\pi})^{r\times r}$ let

$$\|\boldsymbol{H}\|_F^2 := \sum_{1\leq i,j\leq r} \|h_{i,j}\|_2^2$$

be the Frobenius norm of $\boldsymbol{H}$, where $\|\cdot\|_2$ denotes the usual norm in $L^2_{2\pi}$. Further, let the norm of a linear operator $T : (L^2_{2\pi})^{r\times r} \to (L^2_{2\pi})^{r\times r}$, restricted to a subspace $\mathbb{H}$ of $(L^2_{2\pi})^{r\times r}$, be defined by

$$\|T|_{\mathbb{H}}\| := \sup_{H\in\mathbb{H}\setminus\{0\}} \frac{\|TH\|_F}{\|H\|_F}.$$

The spectral radius $\rho_{\mathbb{H}}$ of T restricted to $\mathbb{H}$ satisfies

$$\rho_{\mathbb{H}} = \lim_{n\to\infty} \|(T|_{\mathbb{H}})^n\|^{1/n}.$$

Introduce the smallest closed subspace $\mathbb{H}_0 \subseteq \mathbb{H}_N$ invariant under T and containing the matrix $\boldsymbol{C}\boldsymbol{C}^\star$, where $\boldsymbol{C}$ is the factorization matrix in (20) satisfying (21). Obviously, $\mathbb{H}_0$ is finite-dimensional. Further, observe that by (8), for each $n \in \mathbb{N}$,

$$T^n(\boldsymbol{C}\boldsymbol{C}^\star)(\omega) = \sum_{l=0}^{2^n-1} \boldsymbol{\Pi}_n(\omega+2\pi l)\,\boldsymbol{C}(\frac{\omega+2\pi l}{2^n})\,\boldsymbol{C}(\frac{\omega+2\pi l}{2^n})^\star\,\boldsymbol{\Pi}_n(\omega+2\pi l)^\star,$$

where $\boldsymbol{\Pi}_n(\omega) := \prod_{j=1}^n \boldsymbol{P}(2^{-j}\omega)$. Using that $\boldsymbol{Q}(\omega) = 2^k\,\boldsymbol{C}(2\omega)^{-1}\,\boldsymbol{P}(\omega)\,\boldsymbol{C}(\omega)$ we find $\boldsymbol{C}(\omega)^{-1}\,\boldsymbol{\Pi}_n(\omega)\,\boldsymbol{C}(2^{-n}\omega) = 2^{-kn}\,\prod_{j=1}^n \boldsymbol{Q}(2^{-j}\omega)$ and hence

$$\begin{aligned} T^n(\boldsymbol{C}\boldsymbol{C}^\star)(\omega) &= \frac{1}{4^{kn}}\boldsymbol{C}(\omega)\sum_{l=0}^{2^n-1}(\prod_{j=1}^n \boldsymbol{Q}(\frac{\omega+2\pi l}{2^j}))(\prod_{j=1}^n \boldsymbol{Q}(\frac{\omega+2\pi l}{2^j}))^\star\boldsymbol{C}(\omega)^\star \\ &= 4^{-kn}\,\boldsymbol{C}(\omega)\,T_{\boldsymbol{Q}}^n(\boldsymbol{I})(\omega)\,\boldsymbol{C}(\omega)^\star, \end{aligned}$$

where $T_{\boldsymbol{Q}}$ denotes the transition operator corresponding to $\boldsymbol{Q}(\omega)$, and $\boldsymbol{I}$ is the identity matrix. In particular, since $\mathrm{D}^n[\boldsymbol{A}(\omega)^\star\,\boldsymbol{C}(\omega)]|_{\omega=0} = \boldsymbol{0}^{\mathrm{T}}$, it follows that for all $\boldsymbol{H} \in \mathbb{H}_0$,

$$\mathrm{D}^n[\boldsymbol{A}(\omega)^\star\,\boldsymbol{H}(\omega)]|_{\omega=0} = \boldsymbol{0}^{\mathrm{T}} \quad (n = 0, \ldots, k-1).$$

Consider the smallest subspace $\mathbb{H}_1$ of $\mathbb{H}_N$ which is invariant under $T_{\boldsymbol{Q}}$ and contains the identity matrix.

Lemma 9. *Let $\boldsymbol{H} \in \mathbb{H}_1$ be an eigenmatrix of $T_{\boldsymbol{Q}}$ to the eigenvalue λ. Then $\tilde{\boldsymbol{H}} = \boldsymbol{C}\,\boldsymbol{H}\,\boldsymbol{C}^\star$ is an element of $\mathbb{H}_0$ and it is an eigenmatrix of $T_{\boldsymbol{P}}$ to the eigenvalue $4^{-k}\lambda$.*

Proof: From $\boldsymbol{C}(2\omega)\,\boldsymbol{Q}(\omega) = 2^k\,\boldsymbol{P}(\omega)\,\boldsymbol{C}(\omega)$ we find

$$\begin{aligned} &T_{\boldsymbol{P}}\tilde{\boldsymbol{H}}(2\omega) \\ &= \boldsymbol{P}(\omega)\,\boldsymbol{C}(\omega)\,\boldsymbol{H}(\omega)\,\boldsymbol{C}(\omega)^\star\,\boldsymbol{P}(\omega)^\star + \\ &+ \boldsymbol{P}(\omega+\pi)\,\boldsymbol{C}(\omega+\pi)\,\boldsymbol{H}(\omega+\pi)\,\boldsymbol{C}(\omega+\pi)^\star\,\boldsymbol{P}(\omega+\pi)^\star \\ &= 4^{-k}\,\boldsymbol{C}(2\omega)\;[\boldsymbol{Q}(\omega)\,\boldsymbol{H}(\omega)\,\boldsymbol{Q}(\omega)^\star + \boldsymbol{Q}(\omega+\pi)\,\boldsymbol{H}(\omega+\pi)\,\boldsymbol{Q}(\omega+\pi)^\star]\;\boldsymbol{C}(2\omega)^\star \\ &= 4^{-k}\,\boldsymbol{C}(2\omega)\,(T_{\boldsymbol{Q}}\boldsymbol{H})(\omega)\,\boldsymbol{C}(2\omega)^\star \\ &= 4^{-k}\,\boldsymbol{C}(2\omega)\,\lambda\,\boldsymbol{H}(\omega)\,\boldsymbol{C}(2\omega)^\star = 4^{-k}\lambda\,\tilde{\boldsymbol{H}}(2\omega). \quad \blacksquare \end{aligned}$$

Indeed, there is a close connection between $\mathbb{H}_0$ and $\mathbb{H}_1$; $\boldsymbol{H} \in \mathbb{H}_1$ if and only if $\boldsymbol{C}\,\boldsymbol{H}\,\boldsymbol{C}^\star \in \mathbb{H}_0$.

Theorem 10. *Let $\boldsymbol{\Phi}$ be a compactly supported, L^2-stable scaling vector with mask symbol $\boldsymbol{P}(\omega)$. Further, let $\boldsymbol{\Phi}$ provide approximation order k leading to the factorization (20) of $\boldsymbol{P}$ with $\boldsymbol{C}$ and $\boldsymbol{Q}$ as in Theorem 7. Denote by $T_{\boldsymbol{P}}$ and $T_{\boldsymbol{Q}}$ the transition operators corresponding to $\boldsymbol{P}$ and $\boldsymbol{Q}$, and let the spaces $\mathbb{H}_0$*

and $\mathbb{H}_1$ *be defined as above. Let* ρ_0 *be the spectral radius of* $T_{\boldsymbol{P}}$ *restricted to* $\mathbb{H}_0$ *and* ρ_1 *the spectral radius of* $T_{\boldsymbol{Q}}$ *restricted to* $\mathbb{H}_1$. *Then the Sobolev exponent* s *is:*
(1) $s = \frac{-\log \rho_0}{2 \log 2}$.
(2) $s = k - \frac{\log \rho_1}{2\log 2}$.

Proof:

For the proof of (2) we can follow the lines of the proof of Theorem 5.1 in [9]. As was observed in (24), the factorization (20) implies

$$\|\hat{\boldsymbol{\Phi}}(\omega)\| \leq C\,(1+|\omega|)^{-k}\,\|\boldsymbol{G}(\omega)\|$$

with $\boldsymbol{G}(\omega) = \lim_{n\to\infty} \boldsymbol{G}_n(\omega)$ and $\boldsymbol{G}_n(\omega) := \boldsymbol{Q}(2^{-1}\omega)\ldots\boldsymbol{Q}(2^{-n}\omega)\,\boldsymbol{E}(0)\,\boldsymbol{r}$.

Using the definition of s it follows that $s \leq k-\gamma$ if the estimate

$$\int_{-2^n\pi}^{2^n\pi} \|\boldsymbol{G}_n(\omega)\|^2 \mathrm{d}\omega \leq C\,2^{2n\gamma}$$

is true. Really,

$$\begin{aligned}\int_{-\pi}^{\pi} (T^n_{\boldsymbol{Q}}\boldsymbol{I})(\omega)\,\mathrm{d}\omega &= \int_{-2\pi}^{2\pi} \boldsymbol{Q}(\frac{\omega}{2})\,(T^{n-1}\boldsymbol{I})(\frac{\omega}{2})\,\boldsymbol{Q}(\frac{\omega}{2})^\star\,\mathrm{d}\omega = \ldots\\ &= \int_{-2^n\pi}^{2^n\pi} \boldsymbol{Q}(\frac{\omega}{2})\ldots\boldsymbol{Q}(\frac{\omega}{2^n})\,\boldsymbol{Q}(\frac{\omega}{2^n})^\star\ldots\boldsymbol{Q}(\frac{\omega}{2})^\star\,\mathrm{d}\omega\end{aligned}$$

implies

$$\int_{-2^n\pi}^{2^n\pi} \|\boldsymbol{G}_n(\omega)\|^2\,\mathrm{d}\omega \leq C_\epsilon\,(\rho_1+\epsilon)^n \leq \tilde{C}_\epsilon\,2^{2n\gamma}$$

for $\gamma > \frac{\log\rho_1}{2\log 2}$. Hence, it follows that $s \leq k - \frac{\log\rho_1}{2\log 2}$. Moreover, analogously as e.g. in [31], the L^2-stability of $\boldsymbol{\Phi}$ implies (2). The equality (1) is a simple consequence of (2) since $\rho_0 = 4^{-k}\,\rho_1$. $\square$

Let us finish this section with several remarks.

1. Accuracy of order k implies eigenvalues 2^{-n} $(n = 0,\ldots,k-1)$ of the transition operator $T = T_{\boldsymbol{P}}$ (see e.g [33], Theorem 2.2.). By restriction of $\mathbb{H}_N$ to the subspace $\mathbb{H}_0$ we get rid of the eigenvalues of T that are related to polynomial reproduction. Another way to suppress these eigenvalues is to use the factorization of the symbol, and then consider the transition operator $T_{\boldsymbol{Q}}$ of the inner matrix $\boldsymbol{Q}$.

2. In [48] and [33], the subspace $\mathbb{H}_{\boldsymbol{A}}$,

$$\mathbb{H}_{\boldsymbol{A}} := \{\boldsymbol{H}\in\mathbb{H}_N : \boldsymbol{H} = \boldsymbol{H}^\star,\ \mathrm{D}^n[\boldsymbol{A}(\omega)^\star\,\boldsymbol{H}(\omega)]|_{\omega=0} = \boldsymbol{0}^{\mathrm{T}},\ n = 0,\ldots,k-1\},$$

of $\mathbb{H}_N$ was introduced. It can be shown that $\mathbb{H}_{\boldsymbol{A}}$ is invariant under the action of T and $\mathbb{H}_0 \subseteq \mathbb{H}_{\boldsymbol{A}}$. The use of $\mathbb{H}_{\boldsymbol{A}}$ has the advantage, that a matrix factorization of $\boldsymbol{P}$ needs not to be known, only the symbol vector $\boldsymbol{A}(\omega)$ of the superfunction is applied (see [34,48]). Moreover, instead of $\mathbb{H}_{\boldsymbol{A}}$, one can also consider the smallest $T_{\boldsymbol{P}}$ invariant subspace $\mathbb{H}_{\boldsymbol{I}}$ of $\mathbb{H}_M$ $(M = \max(N, k'))$ which contains the matrices $(1-\cos\omega)^{k'}\, \boldsymbol{e}_j\, \boldsymbol{e}_j^{\mathrm{T}}$ $(j = 1, \ldots r)$, where $\boldsymbol{e}_j$ are the usual unit vectors. This subspace $\mathbb{H}_{\boldsymbol{I}}$, applied in [31], Theorem 3.4, does not need any information on approximation properties of $\boldsymbol{\Phi}$ (up to an estimate of the approximation order k, since k' should be chosen $\geq k$).

In case of an L^2-stable scaling vector, Jia, Riemenschneider and Zhou [31] showed that $s = -\frac{\log \rho_{\boldsymbol{I}}}{2\log 2}$, where $\rho_{\boldsymbol{I}}$ is the spectral radius of T restricted to $\mathbb{H}_{\boldsymbol{I}}$.

Since the factorization is not involved when using the subspaces $\mathbb{H}_{\boldsymbol{A}}$ or $\mathbb{H}_{\boldsymbol{I}}$, the corresponding method can be transfered to the multivariate setting.

3. The computation of the eigenvalues of the transition operator of the magnitude smaller than the spectral radius is numerically unstable. Factorization greatly improves numerical stability!

4. Smoothness estimates can also be obtained in terms of the subdivision operator. This approach is taken in [38] using the factorization technique, and in [31] without using factorization. Results in [31] and [38] are also obtained in the general L^p-space (Besov-exponents) and in the L^∞-case (Hölder-exponents).

5. Unfortunately, all methods above do not allow to estimate the smoothness of each entry of $\boldsymbol{\Phi}$ separately, but only the common smoothness of all functions in $\boldsymbol{\Phi}$. A first attempt to tackle this problem can be found in [47].

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