On the *PBD*-Closure of Sets Containing 3

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Abstract

We describe a method used to prove nonexistence of pairwise balanced designs. We determine the exact closure of all subsets K of the set $\{3, 4, \ldots, 22\}$ with $K \cap \{11, 12, \ldots, 22\} \neq \emptyset$ and $3 \in K$.

1 Introduction

Let K be a set of positive integers. Then a *pairwise balanced design* ($PBD[v, K]$) of order v with block sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

(i) if $B \in \mathcal{B}$, then $|B| \in K$;

(ii) every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$.

A set S of positive integers is said to be *PBD-closed* if the existence of a $PBD[v, S]$ implies that v belongs to S. Let K be a set of positive integers and let $B(K) = \{v \mid \exists PBD[v, K]\}$. Then $B(K)$ is a *PBD*-closed set called the *closure* of K.

In [4] H.-D.Gronau, R.Mullin and Ch.Pietsch determined the complete closure of all subsets of the set $\{3, 4, \ldots, 10\}$ which include 3. In this paper we

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give a generalization of their proofs. To do this the concept of a prestructure is very useful.

Let K' be a set of positive integers. A *partial pairwise balanced design* (*PPBD*$[g, K']$) of order g with block sizes from K' is a pair $(G, \mathcal{F})$, where G is a point set of cardinality g and $\mathcal{F}$ is a family of subsets (called *blocks*) of G which satisfy the properties:

(i) if $F \in \mathcal{F}$, then $|F| \in K'$;

(ii) every pair of distinct elements of G occurs in at most one block of $\mathcal{F}$.

A *PPBD*$[v, K']$ with block set $\mathcal{F}$ is called a (K', K)-*expansible prestructure* of order v if there exists a *PBD*$[v, K]$ on the same pointset and with blockset $\mathcal{B}$, where $\mathcal{F} \subseteq \mathcal{B}$, $K' \subseteq K$ and if $B \in \mathcal{B} \setminus \mathcal{F}$ then $|B| \in K \setminus K'$.

Obviously nonexistence of a (K', K)-expansible prestructure of order v implies nonexistence of a $PBD[v, K]$. Particularly, if 3 occurs as block length then this is a good approach to prove nonexistence. In this paper we establish the exact closure of all subsets K of the set $\{3, 4, \ldots, 22\}$. The existence or nonexistence of a $PBD[v, K]$ is determined for all v if $K \cap \{11, 12, \ldots, 22\} \neq \emptyset$ and $3 \in K$. There is no case in doubt. Used construction methods are briefly described.

2 General necessary conditions

Let $\Pi = (V, \mathcal{B})$ be a pairwise balanced design and let $k_1, k_2, \ldots, k_n$ denote the block sizes occurring in Π. We say that $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ is the *block type* of Π if β_i counts the number of blocks of size k_i, $i = 1, 2, \ldots, n$. For the block type $\vec{\beta}$ of Π on v points the following is true:

Lemma 2.1

$$\sum_{i=1}^{n} \beta_i k_i (k_i - 1) = v(v-1).$$

The proof is done by counting the pairs of points in two ways.

Consider now a point $x \in V$ and let γ_i count the number of blocks through x with length k_i, $i = 1, 2, \ldots, n$. We call the vector $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ the *point type* of x in Π. If we want to refer to the point x we write $\vec{\gamma}(x)$. Now we have for all $x \in V$:

Lemma 2.2

$$\sum_{i=1}^{n} \gamma_i(x)(k_i - 1) = v - 1.$$

The proof is by double counting the pairs of elements of V which contain the point x.

If we know that there is a block B of a certain length k in Π then we can often restrict the possible point types for points not on B by the following lemma, see [4].

Lemma 2.3 *Let B be a block of length k in Π and let $\vec{\gamma}$ be the point type of a point $x \in V \setminus B$. Then the following inequality holds:*

$$\sum_{i=1}^{n} \gamma_i(x) \geq k.$$

Suppose that we have a $PBD[v, K]$. Let K' be a subset of K and $I(K') = \{i \in \{1, 2, \ldots, n\} \mid k_i \in K'\}$. We introduce the following notations:

$$\beta_{K'} = \sum_{i \in I(K')} \beta_i \quad \text{the number of blocks with size in } K',$$

$$\gamma_{K'}(x) = \sum_{i \in I(K')} \gamma_i(x) \quad \text{the number of blocks through } x \text{ with size in } K'.$$

An useful generalization of Lemma 6 in [4] is the following statement:

Lemma 2.4 *Let K' be a subset of $K = \{k_1, k_2, \ldots, k_n\}$. Then we have:*

$$\beta_{K'}(\beta_{K'} - 1) \geq \sum_{x \in V} \gamma_{K'}(x)(\gamma_{K'}(x) - 1).$$

Proof: We show this by counting the edges in the following bipartite graph $G = (V_1 \cup V_2, E)$. Subset V_1 contains as vertices all pairs $(B, B') \in \mathcal{B} \times \mathcal{B}$ with $B \neq B'$, $|B| \in K'$ and $|B'| \in K'$. Vertex subset V_2 has as elements all points from point set V. The edge set is defined as

$$E = \{((B, B'), x) \in V_1 \times V_2 \mid \{x\} = B \cap B'\}.$$

Two blocks intersect in at most one point hence the degree of any vertex in V_1 is at most one. By $|V_1| = \beta_{K'}(\beta_{K'} - 1)$ we have at most $\beta_{K'}(\beta_{K'} - 1)$ edges in E. Further each vertex $x \in V_2$ has degree $\gamma_{K'}(x)(\gamma_{K'}(x) - 1)$ and therefore the number of edges in E is $\sum_{x \in V} \gamma_{K'}(x)(\gamma_{K'}(x) - 1)$. □

A result on non-existence was given in the following lemma:

Lemma 2.5 *Let K be a set of positive integers whose smallest element is 3. Suppose that there exists a $PBD[v, K]$ which contains blocks B_h and B_k of sizes h and k respectively, where $h \geq k$. Then $v \geq 2h + k - 2$.*

Proof: This is a special case of a theorem proved by Drake and Larson [2] (or see Theorem III.3.1 in [5]). □

3 Necessary conditions for the existence of a prestructure

Let $\Pi = (V, \mathcal{B})$ be a (partial) pairwise balanced design of order v and let K' be a subset of K, where k_{min} is the smallest and k_{max} the greatest element of K'. Let $m = \left\lfloor \dfrac{v-1}{k_{min}-1} \right\rfloor$. Furthermore let

$$x_j \stackrel{\text{def}}{=} |\{x \in V \mid \gamma_{K'}(x) = j\}| \quad (0 \leq j \leq m)$$

denote the number of points of V which are lying on exactly j blocks with length from K'. We call the vector $\vec{x} = (x_0, x_1, \ldots, x_m)$ the *K'-prestructure type* of Π.

Theorem 1 *Let K be a set of positive integers and let K' be a subset of K. If there exists a $PBD[v, K]$, say Π, then we have an integer solution $(\vec{\beta}; \vec{x}) = (\beta_1, \beta_2, \ldots, \beta_n;\ x_0, x_1, \ldots, x_m)$ of the following system* (S)*:*

$$(\mathrm{S}) \quad \left\{ \begin{array}{cclr} \displaystyle\sum_{i=1}^{n} \beta_i k_i (k_i - 1) & = & v(v-1) & (1) \\ \displaystyle\sum_{j=0}^{m} x_j & = & v & (2) \\ \displaystyle\sum_{i \in I(K')} k_i \beta_i & = & \displaystyle\sum_{j=0}^{m} j\, x_j & (3) \\ \beta_{K'}(\beta_{K'} - 1) & \geq & \displaystyle\sum_{j=0}^{m} j(j-1)\, x_j & (4) \\ x_j & \geq & 0 \qquad j = 0, 1, \ldots, m & \\ \beta_i & \geq & 0 \qquad i = 1, 2, \ldots, n & \end{array} \right.$$

Proof: Let $\vec{x}$ be the K'-prestructure type and let $\vec{\beta}$ be the block type of Π. Equation (1) is clear by Lemma 2.1. There is no point x who lies on more than m blocks with length form K', this gives us $0 \leq \gamma_{K'}(x) \leq m$ and therefore equation (2) is true. Equation (3) follows from counting the points on blocks with block length in K' in two different ways. By Lemma 2.4 and

$$\sum_{x\in V}\gamma_{K'}(x)(\gamma_{K'}(x)-1) = \sum_{j=0}^{m}\sum_{\substack{x\in V,\\ \gamma_{K'}(x)=j}} j(j-1) = \sum_{j=0}^{m} j(j-1) \underbrace{\sum_{\substack{x\in V,\\ \gamma_{K'}(x)=j}} 1}_{x_j}$$

we have inequality (4). $\square$

Example: We get a $PBD[10,\{3,4\}]$ by adjoining a new point to every block of a parallel class of the unique $2-(9,3,1)$ block design.

0	3	6	0	0	0	1	1	1	2	2	2
1	4	7	3	4	5	3	4	5	3	4	5
2	5	8	6	8	7	8	7	6	7	6	8
9	9	9									

This pairwise balanced design has block type $\vec{\beta} = (9,3)$ and $\{4\}$-prestructure type $\vec{x} = (0,9,0,1)$. The vector $(\vec{\beta};\vec{x})$ satisfies all conditions in (S).

Remark 3.1 *In order to prove nonexistence of a pairwise balanced design of order v we have to show that there exists a subset K' of K such that there is no integer solution of system* (S)*, this means there is no (K',K)-expansible prestructure .*

From now on we always assume that $v \notin B(K')$ if $K' \subset K$, since $B(K') \subseteq B(K)$. In the next section we concentrate on a special case of the subset K'.

3.1 Prestructures with all blocks of even size

Theorem 2 *Let v be a positive integer and K be a set of positive integers. Further let*

$$\beta_{min} = \max\{k \in K \mid k \equiv v-1 \pmod 2\} + (v \pmod 2)\ .$$

If the subset K' contains all even elements of K then the new system (S1) *has an integer solution $(\vec{\beta};\vec{x})$ if a $PBD[v,K]$ exists.*

$$
(\mathrm{S1}) \quad \left\{ \begin{array}{lcll}
\sum_{i=1}^{n} \beta_i k_i(k_i-1) & = & v(v-1) & (1)\\
\sum_{j=0}^{m} x_j & = & v & (2)\\
\sum_{i\in I(K')} k_i\beta_i & = & \sum_{j=0}^{m} j\, x_j & (3)\\
\beta_{K'}(\beta_{K'}-1) & \geq & \sum_{j=0}^{m} j(j-1)\, x_j & (4)\\
x_j & \geq & 0 \qquad j\equiv v-1 \pmod 2 & \\
x_j & = & 0 \qquad j\equiv v \pmod 2 & (5)\\
\beta_{K'} & \geq & \beta_{min} & (6)\\
\beta_i & \geq & 1 \qquad\quad i=1,2,\ldots,n & (7)
\end{array} \right.
$$

Proof: We know that there is a solution $(\vec{\beta};\vec{x})$ of system (S). Hence equation (1), (2), (3) and inequality (4) are satisfied by this solution. This solution satisfies also equation (5) since by Lemma 2.2 we have $\forall x\in V$ that

$$\sum_{i\in I(K\setminus K')} \gamma_i(x)(k_i-1) + \sum_{i\in I(K')} \gamma_i(x)(k_i-1) = v-1$$

this indicate that

$$\sum_{i\in I(K')} \gamma_i(x)(k_i-1) \equiv \sum_{i\in I(K')} \gamma_i(x) = \gamma_{K'} \equiv v-1 \pmod 2$$

and therefore we have no point who lies on exactly j blocks with size from K' if $j\equiv v \pmod 2$, even that means $x_j=0$.
If v is even then through every point on a block of odd length goes at least one block of even length, since $x_0=0$. Blocks through distinct points on one block are disjoint hence $\beta_{K'}\geq k$ is true for all odd $k\in K$. Suppose v is odd, there exists at least one block of length k_{max} and through every point of this block goes at least one other block of even length, since $x_1=0$. Therefore we have at least $k_{max}+1$ blocks of even size. Hence $\vec{\beta}$ holds inequality (6). We know $\beta_i\geq 1$ since by assumption $v\notin B(K\setminus\{k_i\})$. $\square$

Corollary 3.1 *If $\beta_{K'}$ is even then we can replace* (4) *in system* (S1) *by the following:*

$$\beta_{K'}(\beta_{K'}-2) \geq \sum_{j=0}^{m} j(j-1)\, x_j \ .$$

This is true since every block of even size intersects an even number of blocks with length from K' and therefore every block intersects at most $\beta_{K'}-2$ other blocks.

Remark 3.2 *We call system* (S1) *without equation* (1) *system* (S1').

How can we easily check whether a solution exists or not?
For the sake of brevity we write $\Sigma^{\beta}_{K'}$ instead of $\sum\limits_{i\in I(K')} k_i\beta_i$.

Lemma 3.2 *Suppose that $(\vec{\beta};\vec{x}')$ is solution of* (S1) *and j^* be the greatest integer with $j^* \leq \left\lfloor \frac{\Sigma^{\beta}_{K'}}{v} \right\rfloor$ and $j^* \equiv v-1 (\mathrm{mod}\, 2)$ then $(\vec{\beta};\vec{x}^* = (x^*_0, x^*_1, \ldots, x^*_m))$, where*

$$x^*_j = \begin{cases} 0 & j \neq j^*, j \neq j^*+2 \\ \dfrac{(j^*+2)v - \Sigma^{\beta}_{K'}}{2} & j = j^* \\ \dfrac{\Sigma^{\beta}_{K'} - j^* v}{2} & j = j^*+2, \end{cases}$$

is also a solution of (S1).

Proof: Clearly (1) is satisfied. We observe that $j^* \geq 0$ respectively $j^* \geq 1$ if v is even.
Furthermore we have $j^*+2 \leq m$ or $x_{j^*+2} = 0$. First if $m \equiv v-1 (\mathrm{mod}\, 2)$ then owing to $(\vec{\beta};\vec{x}')$ is a solution of (S1) we know that $m \geq \frac{\Sigma^{\beta}_{K'}}{v}$. If $m = \frac{\Sigma^{\beta}_{K'}}{v} = j^*$ then follows $x_{j^*+2} = 0$, otherwise $m \geq j^*+2$. In the case of $m \equiv v (\mathrm{mod}\, 2)$ we have $m-1 \geq \frac{\Sigma^{\beta}_{K'}}{v}$ which proves the fact $x_{j^*+2} = 0$ or $m \geq j^*+2$.
Moreover $\vec{x}^*$ satisfies equations (2) and (3) of system (S1):

$$\sum_{j=0}^{m} x^*_j = x^*_{j^*} + x^*_{j^*+2} = \frac{j^*v + 2v - \Sigma^{\beta}_{K'} + \Sigma^{\beta}_{K'} - j^*v}{2} = v$$

and

$$\sum_{j=0}^{m} j\, x^*_j = \frac{j^*(j^*+2)v - j^*\Sigma^{\beta}_{K'} + (j^*+2)\Sigma^{\beta}_{K'} - (j^*+2)j^*v}{2} = \Sigma^{\beta}_{K'}.$$

We now have to prove that $\vec{x}^*$ holds also inequality (4). It is easy to verify that $\vec{x}^*$ and j^* are the unique vector and number respectively which satisfy:

(i) $j^* \in \{1, 2, \ldots, m\}$, $j^* \equiv v - 1 (\bmod 2)$,

(ii) $x_{j^*} > 0$, $x_{j^*+2} \geq 0$, $\quad x_j = 0 \ \forall j \mid j \neq j^*, j \neq j^* + 2$,

(iii) $\sum_{j=0}^{m} x_j = v$ and $\sum_{j=0}^{m} j x_j = \Sigma_{K'}^{\beta}$.

Let j^- be the smallest and j^+ the greatest component where $\vec{x}'$ is unequal 0. If $j^- + 2 \geq j^+$ holds then $\vec{x}' = \vec{x}^*$ and therefore $(\vec{\beta}; \vec{x}^*)$ is solution of system (S1). Otherwise we define a new vector $\vec{x}^1$ by

$$x_j^1 = \begin{cases} x_j' - 1 & j = j^-, j = j^+ \\ x_j' + 1 & j = j^- + 2, j = j^+ - 2 \\ x_j' & \text{else} \end{cases}$$

It holds $\beta_{K'}(\beta_{K'} - 1) > \sum_{j=0}^{m} j(j-1) x_j^1$ since $\sum_{j=0}^{m} j(j-1) x_j' \geq 8 + \sum_{j=0}^{m} j(j-1) x_j^1$. By its definition $\vec{x}^1$ satisfies (2), (3) and (5) hence $(\vec{\beta}; \vec{x}^1)$ is solution of (S1). We determine j^- and j^+ of $\vec{x}^1$ and (pairwise distinct) solution vectors $\vec{x}^2, \vec{x}^3, \ldots$. After finite many steps we receive $\vec{x}^t$ which has at most two components unequal 0, which are consecutive and $\vec{x}^t$ is solution of (S1). Hence $\vec{x}^t = \vec{x}^*$ which completes the proof. $\square$

Remark 3.3 *If $\vec{\beta}$ is fixed then $\vec{x}^*$ minimize $\sum_{j=0}^{m} j(j-1) x_j$ over all $\vec{x}$ satisfying* (S1).

In order to have as less intersections between blocks of even size as possible, we want to have $\sum_{i \in I(K')} k_i \beta_i$ as small as possible. If the number of blocks is fixed then this means that we want as much as possible blocks of smallest even block size and as less as possible blocks of other even block lengths.

Lemma 3.3 *Let v be a positive integer. Suppose that K' is the subset of K which contains all even elements. If a $PBD[v,K]$ with β blocks of even length exists then $(\vec{\beta}^*;\vec{x}^*)$ is an integer solution of the system* (S1)*, where $\vec{\beta}^*$ is a solution of the following optimizing problem* (O)*:Suppose that K' is the subset of K which contains all even elements.*

$$
(\mathrm{O}) \quad \left\{ \begin{array}{rcll}
\sum_{i\in I(K')} k_i\beta_i & = & min! & \\
\multicolumn{4}{l}{\text{under the conditions}} \\
\sum_{i=1}^{n}\beta_i k_i(k_i-1) & = & v(v-1) & \\
\sum_{i\in I(K')} k_i\beta_i & \geq & v & \text{(only if v is even)} \\
\beta & = & \sum_{i\in I(K')}\beta_i & \\
\beta_i & \geq & 1 & i=1,2,\dots,n
\end{array} \right.
$$

and $\vec{x}^* = (0,\dots,0,x^*_{j^*},0,x^*_{j^*+2},0,\dots,0)$ is defined as in Lemma 3.2 where j^* is the greatest integer with $j^* \leq \left\lfloor \frac{\Sigma_{K'}^{\beta^*}}{v} \right\rfloor$ and $j^* \equiv v-1(\mathrm{mod}\,2)$.

Proof: Equation (1) is true by the definition of $\vec{\beta}^*$. Equations (2), (3) and (5) are satisfied by the definition of $\vec{x}^*$.

Again we have to show that inequality (4) holds. Let $\vec{\beta}'$ be the block type of the $PBD[v,K]$ and let $\vec{x}'^* = (0,\dots,0,x'^*_{j'^*},0,x'^*_{j'^*+2},0,\dots,0)$ belong to $\vec{\beta}'$, where $j'^* \equiv v-1(\mathrm{mod}\,2)$ is the greatest integer satisfying $j'^* \leq \left\lfloor \frac{\Sigma_{K'}^{\beta'}}{v} \right\rfloor$.
Then $(\vec{\beta}';\vec{x}'^*)$ is a solution of (S1) and we know that

$$\sum_{j=0}^{m} j\,x'^*_j = \sum_{i\in I(K')} k_i\beta_i{}' \geq \sum_{i\in I(K')} k_i\beta_i{}^* = \sum_{j=0}^{m} j\,x^*_j.$$

This implies that $j'^* \geq j^*$ and therefore $\sum_{j=0}^{m} j(j-1)\,x'^*_j \;\geq\; \sum_{j=0}^{m} j(j-1)\,x^*_j$.
Then we have

$$\beta_{K'}{}'(\beta_{K'}{}'-1) \;=\; \beta_{K'}{}^*(\beta_{K'}{}^*-1) \geq \sum_{j=0}^{m} j(j-1)\,x'^*_j \;\geq\; \sum_{j=0}^{m} j(j-1)\,x^*_j$$

Inequalities (6) and (7) are true since $\vec{\beta}^*$ is solution of (O). $\square$

Remark 3.4 *If for* $\beta = \beta_{min}$ *up to* $\beta = \beta_{max} := \left\lfloor \frac{m\,v}{k_{min}} \right\rfloor$ *the vector* $(\vec{\beta}^*; \vec{x}^*)$ *derived as in Lemma 3.3 is not a solution of* (S1)*, then there exists no* $PBD[v, K]$*. We call* $(\vec{\beta}^*; \vec{x}^*)$ *the* (K', β)*-optimal vector. The optimizing problem* (O) *without equation* (1) *is called* (O')*.*

In special cases we can restrict β by the following lemma.

Lemma 3.4 *Let* v *be a positive even integer and let* K *be a set of positive even numbers. Suppose that there is no solution of system* (S1') *with* $K' = K$*, then there is no* $PBD[v, H]$ *if* H *satisfies the properties:*

(i) $K \subseteq H$*,*

(ii) *if* $h \in H \setminus K$ *is an even integer then* $h > \max\{k \in K\}$*,*

(iii) $v \notin B(H \setminus \{k\})$ *for all* $k \in K$ *and*

(iv) *there is no solution of system* (S1) *where* H' *is the set of all even integers of* H *and* $\beta = \beta_{H'} < \left\lceil \frac{v}{k_{min}} \right\rceil$*.*

Proof: Assume, by way of contradiction, that there exists a $PBD[v, H]$ then we have a solution $(\vec{\beta}'; \vec{x}')$ of system (S1), where H' contains all even elements of H. We define $\vec{\beta}'' = (\beta_1'', \ldots, \beta_{|H|}'')$ as:

$$\beta_i'' = \begin{cases} 0 & k_i \in H' \setminus K' \\ \beta_i' + \sum\limits_{i \in I(H' \setminus K')} \beta_i' & k_i = k_{max} \\ \beta_i' & \text{else .} \end{cases}$$

Hence

$$\sum_{j=0}^{m} j\, x_j' = \sum_{i \in I(H')} k_i \beta_i' \geq \sum_{i \in I(K')} k_i \beta_i'' \geq \sum_{i \in I(K')} k_i \beta_i^* = \sum_{j=0}^{m} j\, x_j^*,$$

where $(\vec{\beta}^*; \vec{x}^*)$ is the $(K', \beta_{H'})$-optimal vector of (O') in Lemma 3.3. The condition (iv) guarantees the existence of an optimal vector since it implies that $\sum\limits_{i \in I(K')} k_i \beta_i'' \geq v$ and therefore that $\vec{\beta}''$ is solution of (O'). Moreover we know by the same argumentation as in Lemma 3.3 that

$$\beta_{H'}(\beta_{H'} - 1) \geq \sum_{j=0}^{m} j(j-1)\, x_j' \geq \sum_{j=0}^{m} j(j-1)\, x_j^*.$$

On the other hand $(\vec{\beta}^*;\vec{x}^*)$ is not a solution of (S1'). Since (2), (3), (5), (6) and (7) are all satisfied, we thus have obtained

$$\beta_{H'}(\beta_{H'}-1) < \sum_{j=0}^{m} j(j-1)\,x_j^*,$$

the desired contradiction. $\square$

Example: Let $K=\{12\}$. There is no solution of (S1') where $v=52$ and $K'=\{12\}$.
We know $v \notin B(\{3,5\})$ and therefore holds $v \notin B(\{3,5,12\})$.
We know $v \notin B(\{3,14\})$ and therefore holds $v \in B(\{3,12,14\})$ only if $\beta_{H'}=4$. Such a solution exists, but no $(\{12,14\},\{3,12,14\})$-expansible prestructure, see section 3.3. We thus have $v \notin B(\{3,12,14\})$.

Lemma 3.5 *Let v^0 be an odd integer and K' be the subset of K which contains all even elements. If there exists an integer solution $(\vec{\beta}';\vec{x}')$ of* (S1') *then there exists a solution for all odd $v \geq v^0$.*

Proof: We define

$$x''_j = \begin{cases} x'_0 + v - v^0 & j = 0 \\ x'_j & 1 \leq j \leq m \end{cases}$$

Clearly $(\vec{\beta}';\vec{x}'')$ is an integer solution of (S1'). $\square$

Corollary 3.6 *Let v^0 be an odd integer and K' be the subset of K which contains all even elements. If there exists no integer solution of* (S1') *then there exists no integer solution for all odd $v \leq v^0$ and therefore there is no $PBD[v,K]$ for all odd $v \leq v^0$ if $v \notin B(\underline{K})$ for all $\underline{K} \subset K$.*

Example: Let $K=\{3,8\}$. We have no solution of (S1') where $v=35$ and $K'=\{8\}$. It is well know that $v \notin B(\{3\})$ if $v \equiv 5 (\mathrm{mod}\, 6)$ and hence $v \notin B(\{3,8\})$ for all $v \equiv 5 (\mathrm{mod}\, 6)$ with $v \leq 35$.

Remark 3.5 *Assume that $v \equiv v_0 (\mathrm{mod}\, 6)$ and K contains 3. Then we can replace* (S1') *by* (S1) *in Lemma 3.5 and Corollary 3.6. Since $(\vec{\beta}'';\vec{x}'')$ is solution of* (S1)*, with $\vec{\beta}''$ is defined for $i=1,2,\ldots,n$ as*

$$\beta''_i = \begin{cases} \beta'_i + \frac{v(v-1)-v_0(v_0-1)}{6} & k_i = 3 \\ \beta'_i & else. \end{cases}$$

3.2 Adding blocks of odd length to the prestructure

In this subsection we assume that we have both, blocks of odd and even length. Let v be a positive even integer and k be the greatest odd block size of K. To make sure that we can include a block of size k into a prestructure whose prestructure type we derived in (S1) we establish the following claim.

Theorem 3 *Assume the existence of a $PBD[v,K]$, say Π. Let K' be the set of all even block lengths and let $\vec{x}$ be the K'-prestructure type of Π. Further let $\tilde{x} = \max\{0,\ k - \left\lfloor \frac{\beta_{K'}-k}{2} \right\rfloor\}$. Then $x_1 \geq \tilde{x}$.*

Proof: Let B be a block in Π of size k. Let us designate the number of blocks with even length which intersect B by β_B. Furthermore we denote by b_1, b_2 respectively the number of points on B which lie on exactly one or at least three blocks of size from K'. Then holds $b_1+b_2 = k$ and $\beta_B \geq b_1+3b_2$. It follows that $b_1 \geq k - \frac{\beta_B-k}{2}$. Thus, observation that $x_1 \geq b_1$ and $k - \frac{\beta_B-k}{2} \geq \tilde{x}$ concludes the proof. $\square$

Example: Let $K = \{3,6,11\}$, $K' = \{6\}$ and $v = 32$. Assume that Π is a $PBD[v,K]$. By Remark 3.4 we know $11 \leq \beta_{K'} \leq 26$.
If $\beta_{K'} \leq 16$ then we have for the $(\{6\}, \beta_{K'})$-optimal solution $x_5^* = 0$. Hence $x_1^*+x_3^* = 32$ and $6\beta_{K'} = x_1^*+3x_3^* = 96-2x_1^*$. This implies $2x_1^* = 96-6\beta_{K'} \geq 2\tilde{x} \geq 2(11 - \frac{\beta_{K'}-11}{2})$ and therefore $\beta_{K'} \leq 12$.
Thus, we need only to consider $\beta_{K'} = 11, 12$ or $\beta_{K'} \geq 17$.

Remark 3.6 *With respect to Theorem 3 we can replace in* (S1) *equations* (2) *and* (3) *by*

$$\sum_{j=0}^{m} x_j = \tilde{v} := v - \tilde{x} \qquad (\tilde{2})$$

$$\sum_{i\in I(K')} k_i\beta_i - \tilde{x} = \sum_{j=0}^{m} j\, x_j \qquad (\tilde{3})$$

to obtain system $(\widetilde{\mathrm{S1}})$*. For easy proving that there exists a solution of* $(\widetilde{\mathrm{S1}})$ *or not, we replace v by $\tilde{v}$ and $\sum_{i\in I(K')} k_i\beta_i$ resp. $\Sigma^{\beta}_{K'}$ by $\sum_{i\in I(K')} k_i\beta_i - \tilde{x}$ or $\Sigma^{\beta}_{K'} - \tilde{x}$ in Lemma 3.2 and Lemma 3.3.*

Proposition 3.7 *Let v be a positive even integer. Suppose that there is no solution of system $(\widetilde{\mathrm{S1}})$, where $3 \in K$ and k is the greatest odd block length. Then there exists no solution of $(\widetilde{\mathrm{S1}})$ with block size set $H = K \cup \{l\}$ and $k < l, k \equiv l (\bmod 6)$.*

Proof: Assume the existence of a solution of $(\widetilde{\mathrm{S1}})$ with block size set H. Obviously $K' = H'$ and we thus have a solution of $(\widetilde{\mathrm{S1'}})$ with block size set K, since $\tilde{x}(l) \geq \tilde{x}(k)$. Block length 3 establish that we can hold equation (1). Therefore the existence of a solution of $(\widetilde{\mathrm{S1}})$ with block size set K is proved, a contradiction. $\square$

3.3 Unexpansible prestructures

In this section we consider solution vectors which lead to partial pairwise balanced designs but not to an expansible prestructure.

Example: Let $K = \{3, 6\}$, $K' = \{6\}$ and let $v = 22$. There exists a unique solution of (S1) : $(57, 4;\ 0, 21, 0, 1, 0)$. But $22 \notin B(\{3, 6\})$. Suppose that there is $PBD[v, K]$ then there exists a point $p \in V$ who lies on exactly 3 blocks of length 6, say D_1, D_2, D_3, and on exactly 3 blocks of length 3. Let $\{p, x, y\}$ be such a block of size 3 then x and y are not points on D_1, D_2 or D_3, respectively. Thus $\{x, y\} \subset D_4$ the 4-th block of length 6, contradicting the fact that every pair of points occurs on exactly one block.

We generalized this example to the following proposition:

Proposition 3.8 *Let v be a positive even integer, let K be a set of positive integers and K' the subset of K containing all even elements. Now let $(\vec{\beta}; \vec{x})$ be a solution of system* (S1). *Further let us denote $\sum_{i \in I(K')} x_i$ by β. If $x_{\beta - 1} = 1$ then there exists no (K', K)-expansible prestructure and hence no $PBD[v, K]$ with K'-prestructure type $\vec{x}$.*

Proof: The proof is done by the same argumentation as in the example above. $\square$

The following proposition is obvious.

Proposition 3.9 *Suppose that 3 is the smallest element in K and let K' be the subset of K containing all even elements. If $\beta_{K'} = 3$ then $|K'| = 1$.*

Proof: Every block of length 3 intersects the three blocks of length from K'. Thus, these blocks have same size. $\Box$

The next claim follows from observation that m defined in the beginning of section 3 as $\left\lfloor \frac{v-1}{k_{min}-1} \right\rfloor$ is only an upper bound for the maximal number of blocks of length from K' a point lies on.
We say $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a *possible point type* if $\vec{\gamma}$ satisfies the equation in Lemma 2.2 and holds for $i = 1, 2, \ldots, n$

$$\sum_{l=1}^{n} \gamma_l \geq k_i \qquad \text{if } \gamma_i = 0 \ .$$

Closed to the notation used in section 2 we define $\gamma_{K'} = \sum\limits_{i \in I(K')} \gamma_i$.

Proposition 3.10 *Let v be a positive integer and let K' be a subset of block size set K. Further let $m' = max\, \gamma_{K'}$ over all possible point types. If $(\vec{\beta}; \vec{x})$ is a solution of system* (S1) *and $x_j > 0$ for some $j > m'$ then there exists no (K', K)-expansible prestructure with K'-prestructure type $\vec{x}$.*

Proof: Assume the existence of a $PBD[v, K]$ then Lemma 2.2 and Lemma 2.3 imply that there is no point who lies on more than m' blocks of length from K'. $\Box$

Remark 3.7 *We used m only in the sence of the maximal number of blocks from the prestructure a point can lay on, therefore we can replace m by m' in all theorems above.*

Example: Let $K = \{3, 6, 11\}$ and $v = 32$. For brevity we write $3^3 6^5$ instead of $\vec{\gamma} = (3, 5, 0)$. Possible point types are $3^3 6^1 11^2$, $3^3 6^3 11^1$, $3^8 6^1 11^1$, $3^8 6^3$ and $3^{13} 6^1$. $3^3 6^5$ is not a possible point type since it satisfies the equation from Lemma 2.2 but $3 + 5 < 11$. Hence $m' = 3 < m = 5$.

Proposition 3.11 *Let v be a positive even integer and let K' be the subset of K which contains all even elements. Suppose that there exists no solution of $(\widetilde{\mathrm{S1}})$ with $\beta_{K'} \equiv 5 (\mathrm{mod}\, 6)$ and $x_1 + x_3 < \tilde{v}$. If $(\vec{\beta}; \vec{x})$ is a solution of system $(\widetilde{\mathrm{S1}})$ with $\beta_{K'} \equiv 5 (\mathrm{mod}\, 6)$ and $\beta_{K'}(\beta_{K'} - 1) - 6 < \sum\limits_{j=1}^{m} j(j-1) x_j$ then there exists no (K', K)-expansible prestructure with K'-prestructure type $\vec{x}$.*

Proof: Let Π be a $PBD[v, K]$ with $\beta_{K'} \equiv 5 (\bmod 6)$. Suppose that each block of even size intersects every other block with length from K'. Since $x_j = 0$ for all $j > 3$ and by the proof of Lemma 2.4 we know that $\beta_{K'}(\beta_{K'} - 1) = 6x_3$. Hence 6 divides $\beta_{K'}(\beta_{K'} - 1) \equiv 2 (\bmod 6)$, a contradiction. Thus there is one and therefore three blocks of even length which intersect at most $\beta_{K'} - 3$ other blocks from the prestructure. Now we have $(\beta_{K'} - 3)(\beta_{K'} - 1) + 3(\beta_{K'} - 3) = \beta_{K'}(\beta_{K'} - 1) - 6 \geq 6x_3$. $\square$

Example: Let $K = \{3, 11, 16\}$ and $v = 140$. There is no solution of $(\widetilde{\mathrm{S1}})$ with $\beta_{K'} \geq 12$ or $\beta_{K'} = 11$ and $x_j > 0$ for some $j > 3$. Then there is a unique solution $(x, y, 11;\ 0, 122, 0, 18, 0, 0, 0)$ which satisfies equation (4): $110 \geq 108$. But $110 - 6 < 108$, hence $140 \notin B(\{3, 11, 16\})$.

Proposition 3.12 *Let v be a positive even integer and K be a set of positive integers, including 3, but not 4. Let K' be the subset of K which contains all even integers. Suppose that $(\vec{\beta}; \vec{x})$ is a solution of system* (S1) *with $x_1 = v$ and $\beta_{K'} = 4$. Let us denote the four (not necessarly distinct) integers of K' for which $\beta_i > 0$ holds by $l_1, \ldots, l_4$. If $l_1 l_2 + l_3 l_4$ is unequal the number of blocks of length 3, then there exists no (K', K)-expansible prestructure with K'-prestructure type $\vec{x}$.*

Proof: Assume the existence of a $PBD[v, K]$ with four blocks $B_1, \ldots, B_4$ of size $l_1, \ldots, l_4$, respectively. Every block of length 3 has either two points on B_1 and B_2 or on B_3 and B_4, but not both. We have exactly $l_1 l_2$ blocks of length 3 with two points on B_1 and B_2 and exactly $l_1 l_2$ blocks of length 3 with two points on B_3 and B_4. $\square$

Example: Let $K = \{3, 14, 20\}$ and $v = 68$. There is no solution of (S1) with the unique exception of $(572, 2, 2;\ 0, 68, 0, 0)$. Let B_1 and B_2 are the blocks of length 20, B_3 and B_4 of size 14. We have $20^2 + 14^2 = 596 \neq 572$ and therefore $68 \notin B(\{3, 14, 20\})$.

4 Construction of PBDs

4.1 Recursive constructions

We used various recursive constructions and existence results from the *CRC Handbook of Combinatorial Designs* [5]. In particular, recursive constructions are obtained from chapter *III.2 PBDs: Recursive Constructions*, (pp.

193-202). All necessary definitions and methods are well described and examples are given. Therefore we renounce to give a complete description of the methods used.

Furthermore we considered known existence results in section *III.1.3 Some Families of Group Divisible Designs*, (pp. 189-191).
Let K and G be sets of positive integers. A *group divisible design* (GDD) of order v is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a finite set of cardinality v, $\mathcal{G}$ is a partition of V into parts (*groups*) whose sizes lie in G and $\mathcal{B}$ is a family of subsets (*blocks*) of V which satisfy the properties:

(i) if $B \in \mathcal{B}$, then $|B| \in K$;

(ii) every pair of distinct elements of V occurs in exactly one block or one group, but not both.

GDDs are useful for constructing PBDs. By regarding groups to be blocks a GDD can be viewed as a $PBD[v, K \cup G]$. For reducing the set of block sizes of a PBD we can use an operation called *breaking up blocks*. Let Π be a $PBD[v, K]$ and $k \in K$. Suppose that there exists a $PBD[k, K \setminus \{k\}]$, say Π'. By replacing every block of size k of Π by the blocks of Π' in each case defined on the point set of the block we obtain a $PBD[v, K \setminus \{k\}]$.

4.2 The Hill-climbing algorithm

From now on we consider only sets K which include 3. In order to start recursive constructions we need PBDs of small order as "ingredients". This ingredients are produced by Hill-climbing which is a very successful random algorithm for constructing combinatorial designs with block length 3. For an introduction to the Hill-climbing method see [6] and [3].

Given an expansible prestructure we used the Hill-climbing approach to complete the missing blocks of length 3. We applied a variation of the Hill-climbing algorithm first described in [4].

Let $\mathcal{P} = (V,\mathcal{F})$ be a (K', K)-expansible prestructure . Define the set F of forbidden pairs as $F = \{\{x, y\} \mid \exists B \in \mathcal{F}$ such that $\{x, y\} \in B\}$. We search for a decomposition of $G = K_v \setminus F$ into triangles. Let T denote the set of triangles we have found so far. Starting with input v and F, the algorithm works as follows:

Step 0 $T := \emptyset,\ G = K_v \setminus F$

Step 1 Pick randomly $p \in V$ such that $d_G(p) > 0$. If for all $p \in V$ the degree in G is 0 then Goto **End**.

Step 2 Pick randomly two edges xp, yp from $E(G)$. If $xy \in F$ then Goto **Step 1**.

Step 3 (a) There is no triangle $xyz \in T$, $z \in V$. Then take the triangle xyp.
$T := T \cup \{xyp\}$
$E(G) := E(G) \setminus \{xp, yp, xy\}$
(b) There is a triangle $xyz \in T$, $z \in V$. Then delete that triangle and add xyp.
$T := T \cup \{xyp\} \setminus \{xyz\}$
$E(G) := E(G) \cup \{xz, yz\} \setminus \{xp, yp\}$
Goto **Step 1**.

End

An important question is: How can we find an expansible prestructure ?

4.3 Determining an expansible prestructure

The more intersections between blocks of length unequal 3, the more complicated the prestructure becomes. We used the following approach to determine a (K', K)-expansible prestructure . First we computed the $(K', \beta_{K'})$-optimal solution of $(\widetilde{\text{S1}})$ with $\beta_{K'}$ as small as possible, say $(\vec{\beta}^*; \vec{x}^*)$. This gave us information about the number of blocks of each length and about intersections between these blocks.

Suppose we have a partial pairwise balanced design with all blocks of even length, whose prestructure type is $\vec{x}^*$. Let us now regard points to be blocks and blocks to be points, this means that we transpose the incidence matrix. We obtain a $PPBD[\beta_{K'}, H]$, say Σ, where $H = \{j \mid x_j^* > 0\}$. In particular, there are exactly x_j blocks of length j. Furthermore let $K' = \{k'_1, \ldots, k'_{|K'|}\}$ with $k'_i < k'_{i+1}$ for $i = 1, \ldots, |K'| - 1$ and β'_i denote the number of blocks of size k'_i. Then there are β'_1 points which occur in at most k'_1 blocks of Σ, there are $\beta'_1 + \beta'_2$ points which occur in at most $k'_1 + k'_2$ blocks of Σ and so on.

What we do is, trying to construct a $PPBD[\beta_{K'}, H]$, where all properties above are satisfied. It turns out that in all cases we considered we have $H = \{1, 3, 5\}$ or $H = \{2, 4, 6\}$ and the number of blocks of length 2 or 3 is not zero. We do not care about blocks of length one. We construct blocks of size greater than 3 by backtracking, which is quick if there are enough blocks of size 2 or 3. These blocks are inserted again by hillclimbing. (It is easy to find a Hill-climbing-like algorithm for block size 2.) If this is done we add without problems blocks of size 1 and transpose the incidence matrix to obtain a $PPBD[v, K']$.

Adding blocks of odd length is easy, with a few exceptions. We use particularly points which lie on only one or no block of size from K'.

Example: Let us return to our first example, $v = 10$ and $K = \{3, 4\}$. The optimal solution is: $(9, 3;\ 0, 9, 0, 1)$. We search for a $PPBD[3, \{1, 3\}]$ on the point set $\{a, b, c\}$ with 9 blocks of size one and 1 block of size 3. Here is one:

a	a	a	b	b	b	c	c	c	a
									b
									c

which leads to the original prestructure by changing points and blocks.

Sometimes, if the structure of the $PPBD[\beta_{K'}, H]$ is very special, constructing the prestructure with the Hill-climbing approach was not successful. But the main idea, changing points and blocks, leads to a result.

Example: Let $v = 182$, $K = \{3, 16, 17\}$ and we have solution $(4125, 33, 1;$ $0, 9, 0, 173, 0, 0)$. A $PPBD[33, \{1, 3\}]$ is obtained from a 3-GDD on 11 groups of size 3. Eight groups are considered to be blocks. The nine points from remaining groups are blocks of length 1. These $8 + 9$ blocks are the points of the block of length 17.

A list with all prestructures used can be found at:
ftp://ftp.math.uni-rostock.de/pub/members/mgruttm/prestruc.htm

5 The closure of sets containing 3

Let N denote the set of all positive integers, $N_{1(2)}$ denote the subset of all odd integers and $N_{0,1(3)}$ denote the set of positive integers which are congruent 0 or 1 (mod 3). We call two integer sets N_1 and N_2 coterminal when there is an integer n such that $\{x \in N_1 \mid x \geq n\} = \{x \in N_2 \mid x \geq n\}$. It is well known that $B(\{3\}) = N_{1,3(6)}$. By Wilson's theory [7] we have three cases for the closure of a set K which includes 3 and K is not a subset of $N_{1,3(6)}$.

1. If K contains only odd integers and is not a subset of $N_{1,3(6)}$, then $B(K)$ is coterminal with $N_{1(2)}$.

2. If K contains a positive even integer and all members of K lie in $N_{0,1(3)}$, then $B(K)$ is coterminal with $N_{0,1(3)}$.

3. If K contains a positive even integer and is not a subset of $N_{0,1(3)}$, then $B(K)$ is coterminal with N.

5.1 The closure of sets containing only odd integers including 3

Lemma 5.1 *Let $k \equiv 5 \pmod 6$ be a positive integer, then*

$$B(\{3,k\}) = N_{1(2)} \setminus \{v \in N \mid v \equiv 5(\mathrm{mod}\, 6), v < 2k+1, v \neq k\}.$$

Proof: We need only consider the case $v \equiv 5(\mathrm{mod}\, 6), v \neq k$, since $B(\{3\}) \subset B(\{3,k\})$. By a theorem due to Colburn, Hoffman and Rees [1] (Thm. III.1.24 in [5]) we know that there exists a 3-GDD of type $k^1 1^{v-k}$ if the following conditions are satisfied:

1. $k \equiv 1(\mathrm{mod}\, 2)$
2. $k + (v-k) \equiv 1(\mathrm{mod}\, 2)$
3. $v - k \geq k + 1$
4. $\binom{v-k}{2} + k(v-k) + \binom{1}{2}k^2 \equiv 0(\mathrm{mod}\, 3)$

We can easily check that all conditions are satisfied, if $v \equiv k \equiv 5(\mathrm{mod}\, 6)$ and $v \geq 2k+1$. In this case we have a 3-GDD which is a $PBD[v,\{3,k\}]$. If $v < 2k+1$ $(v \neq k)$ then we know by Lemma 2.5 that $v \notin B(\{3,k\})$. □

Theorem 4 *Let K contain only odd integers and let k be the smallest integer in K with $k \equiv 5(\mathrm{mod}\, 6)$, then*

$$B(K) = K \cup N_{1,3(6)} \cup \{v \in N \mid v \equiv 5(\mathrm{mod}\, 6), v \geq 2k+1\}.$$

Proof: Again we need only consider the case $v \equiv 5(\mathrm{mod}\, 6), v \neq k$. If $v \geq 2k+1$ we have by Lemma 5.1 $v \in B(\{3,k\}) \subseteq B(K)$. Suppose that there exists a $PBD[v,K]$ with $v < 2k+1$ and $v \notin K$, then we have two blocks of length k_1, k_2 with $k_1 \equiv 5(\mathrm{mod}\, 6)$. The following inequality is satisfied:

$$2k_1 + k_2 - 2 \geq 2k_1 + 1 \geq 2k + 1 > v$$

But by Lemma 2.5 we have $v \geq 2k_1 + k_2 - 2$, a contradiction. □

5.2 The closure of sets containing 3 and an even integer and all elements lie in $N_{0,1(3)}$

We used a program written in C to verify whether a solution of (S1) exists or not. If a solution exists then the program tries to find a recursive construction or a suitable prestructure to construct the PBD. Otherwise it is checked by the program or by hand that no solution leads to an expansible prestructure.

Theorem 5 *Let K be a set of positive integers containing 3 and a positive even integer. Let K be a subset of $N_{0,1(3)}$ and $\max K \leq 22$. Then we have exactly one of the following three cases:*

(i) *There is no solution of system* (S1).

(ii) *Every solution of system* (S1) *is forbidden by a proposition given in subsection 3.3.*

(iii) *There exists a $PBD[v, K]$.*

5.3 The closure of sets containing 3 and an even integer and not all elements lie in $N_{0,1(3)}$

Lemma 5.2 $32 \notin B(\{3, 6, 11\})$.

Proof: Since $32 \notin B(\{3, 6\})$ we have at least one block of length 11, say B. Points from $V \setminus B$ are all of type $3^8 6^3$ or $3^{13} 6^1$ and therefore B is the unique block of size 11. We know by Proposition 3.10 that $m' = 3$ and hence $\beta_{K'} \leq 16$, together with example to Theorem 3 we obtain $\beta_{K'} = 11$ or $\beta_{K'} = 12$ with unique solutions $(92, 11, 1;\ 0, 15, 0, 17)$, $(87, 12, 1;\ 0, 12, 0, 20)$, respectively. Exactly $88 = 8{\cdot}11$ blocks of length 3 intersect B, hence $\beta_{K'} = 12$ is not possible. If $\beta_{K'} = 11$ then we have four 3-blocks which do not intersect B. Moreover every point on these four blocks is of type $3^{13} 6^1$, since every block which contains a point of type $3^8 6^3$ has a point in common with B. But four blocks of length 3 cover at least $3 + 2 + 1 = 6$ points, contradicting that there are exactly $15 - 11 = 4$ points of type $3^{13} 6^1$. □

Lemma 5.3 $32 \notin B(\{3, 6, 14\})$ *and* $35 \notin B(\{3, 6, 14\})$.

Proof: First we consider $v = 32$. There exists exactly one solution of (S1): $(120, 3, 1;\ 0, 32, 0, 0)$. The block of length 14 intersects $14 * 9 = 126$ blocks of length 3, a contradiction.

If $v = 35$ then the following point types are possible: $3^4 14^2$, $3^3 6^3 14^1$, $3^8 6^1 14^1$, $3^{12} 6^2$ and 3^{17}. By equation (6) of (S1) holds $\beta_{K'} \geq 15$. If $\beta_{K'} = 15$ then $x_4 \geq 20$, but all points on the 14-block are of type $3^8 6^1 14^1$ and there is no suitable point type outside. The same holds if $\beta_{K'} > 15$. □

Lemma 5.4 $44 \notin B(\{3, 6, 17\})$.

Proof: Let B be a block of length 17 in a $PBD[44, \{3, 6, 17\}]$. Every point outside of B has point type $3^{14} 6^3$ or $3^{19} 6^1$. We have at least 17 blocks of size 6. These blocks cover at least $\frac{17 \cdot 5}{3} > 28$ points outside of B. But $28 + 17 > 44$, a contradiction. □

Lemma 5.5 $44 \notin B(\{3, 6, 20\})$ *and* $47 \notin B(\{3, 6, 20\})$.

Proof: Let B be a block of length 20 in a $PBD[44, \{3, 6, 20\}]$. There is only one possible point type outside of B: $3^{19} 6^1$. Since five blocks of size 6 would cover more than 24 points we have exactly four 6-blocks, which are disjoint to B. Points on B have type $3^{12} 20^1$ which implies that B intersects $20 * 12 = 240$ 3-blocks. But according to Lemma 2.1 there are exactly 232 blocks of length 3.

If $v = 47$ then there are two point types possible for points from $V \setminus B$: $3^{18} 6^2$ and 3^{23}. Thus, no point outside of B lies on more than two blocks of length 6. B intersects at least 20 blocks of length 6, which cover at least $\frac{20 * 5}{2} = 50$ points, a contradiction. □

Lemma 5.6 $62 \notin B(\{3, 10, 20\})$.

Proof: There exists only one solution of system (S1): $(292, 5, 1;\ 0, 58, 0, 4)$. Assume the existence of a $PBD[62, \{3, 10, 20\}]$ with a block of length 20, say B. Then point types $3^{17} 10^3$ and $3^{26} 10^1$ are possible outside of B. Suppose there exists a point of type $3^{17} 10^3$ then every block through this point intersects B. Through each of this three intersection points goes an other 10-block, but we have no solution of (S1) on six 10-blocks. On the over hand we have now four points on B which intersect 8 distinct blocks of length 10, again a contradiction. □

Theorem 6 *Let K be a set of positive integers containing 3 and a positive even integer. Let K be not a subset of $N_{0,1(3)}$ and $\max K \leq 22$. Then we have exactly one of the following four cases:*

(i) *There is no solution of system $(\widetilde{\mathrm{S1}})$.*

(ii) *Every solution of system $(\widetilde{\mathrm{S1}})$ is forbidden by a proposition given in subsection 3.3.*

(iii) *There exists no $PBD[v, K]$ by a lemma from this subsection.*

(iv) *There exists a $PBD[v, K]$.*

A Table of closures

We present here the closure of subsets $K \subset \{3, 4, \ldots, 22\}$ with $K \cap \{11, 12, \ldots, 22\} \neq \emptyset$ and the cardinality of K is at most three. A table with complete results is located at

ftp://ftp.math.uni-rostock.de/pub/members/mgruttm/closure.htm

We do not mention the closure of K if $B(K) = B(K \setminus \{k\})$ and $k \in K$.

Subset	Closure	Genuine Exceptions
3 11	$N_{1(2)}$	5 17
3 12	$N_{0,1(3)}$	4 6 10 16 18 22 24 28 30 40 42 46 52 54
3 14	N	4 5 6 8 10 11 12 16 17 18 20 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 52 53 54 58 59 60 62 64 65 66 68 70 71 72 74 77 80 83 88 89 90 94 95 96 100 101 104 107 110
3 16	$N_{0,1(3)}$	4 6 10 12 18 22 24 28 30 34 36 40 42 52 54 58 60 66 70 72 82 84
3 17	$N_{1(2)}$	5 11 23 29
3 18	$N_{0,1(3)}$	4 6 10 12 16 22 24 28 30 34 36 40 42 46 48 58 60 64 66 70 76 78 82 84 94 96
3 20	N	4 5 6 8 10 11 12 14 16 17 18 22 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 46 47 48 50 52 53 54 56 59 62 64 65 66 68 70 71 72 74 76 77 78 82 83 84 86 88 89 90 92 94 95 96 98 100 101 102 104 106 107 108 110 113 116 119 122 124 125 126 130 131 132 136 137 138 142 143 144 146 148 149 150 152 154 155 158 161 164 167 170 173 179 184 185 186 190

191 192 196 197 198 203 206 209 215

3 22 $\mathrm{N}_{0,1(3)}$ 4 6 10 12 16 18 24 28 30 34 36 40 42 46 48 52 54 58 60 70 72 76 78 82 84 90 94 96 100 102 112 114 118 120 136 138 156

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3 4 11 N 5 6 8 14 17 20

3 4 14 N 5 6 8 11 17 20 23 26 29

3 4 17 N 5 6 8 11 14 20 23 26 29 32

3 4 20 N 5 6 8 11 14 17 23 26 29 32 35 38 41

3 5 12 N 4 6 8 10 14 16 18 20 22 24 26 28 30 32 38 40 42 44 46 50 52 54 62

3 5 14 N 4 6 8 10 12 16 18 20 22 24 26 28 30 32 34 36 38 44 46 48 50 52 54 58 60 62 64 72 74

3 5 16 N 4 6 8 10 12 14 18 20 22 24 26 28 30 32 34 36 38 40 42 44 50 52 54 56 58 60 62 66 68 70 72 74 82 84 86

3 5 18 N 4 6 8 10 12 14 16 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 92 94 96 98 110

3 5 20 N 4 6 8 10 12 14 16 18 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 62 64 66 68 70 72 74 76 78 82 84 86 88 90 92 94 102 104 106 108 110 122 124 142

3 5 22 N 4 6 8 10 12 14 16 18 20 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 58 60 62 68 70 72 74 76 78 80 82 84 86 90 92 94 96 98 100 102 104 112 114 116 118 120 122 134 136 138 156 158

3 6 11 N 4 5 8 10 12 14 17 20 22 26 32

3 6 12 $\mathrm{N}_{0,1(3)}$ 4 10 22

3 6 14 N 4 5 8 10 11 12 17 20 22 23 26 29 32 35

3 6 17 N 4 5 8 10 11 12 14 20 22 23 26 29 32 38 44

3 6 20 N 4 5 8 10 11 12 14 17 22 23 26 29 32 35 38 41 44 47

3 6 22 $\mathrm{N}_{0,1(3)}$ 4 10 12

3 8 11 N 4 5 6 10 12 14 16 17 18 20 26 28 30 34 36 38

3 8 12 N 4 5 6 10 11 14 16 17 18 20 23 26 28 29 30 35 38

3 8 14 N 4 5 6 10 11 12 16 17 18 20 23 26 28 29 30 34 35 36

3 8 16 N 4 5 6 10 11 12 14 17 18 20 23 26 28 29 30 34 35 36 38

3 8 17 N 4 5 6 10 11 12 14 16 18 20 23 26 28 29 30 34 36 38

3 8 18 N 4 5 6 10 11 12 14 16 17 20 23 26 28 29 30 34 35 36 38

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$\boxed{3\ 10\ 12}$ $\boxed{N_{0,1(3)}}$ 4 6 16 18 22 24

$\boxed{3\ 10\ 14}$ $\boxed{N}$ 4 5 6 8 11 12 16 17 18 20 22 23 24 26 29 32 34 35 36 38 41 47 53 59 65

$\boxed{3\ 10\ 16}$ $\boxed{N_{0,1(3)}}$ 4 6 12 18 22 24 34 36 42

$\boxed{3\ 10\ 17}$ $\boxed{N}$ 4 5 6 8 11 12 14 16 18 20 22 23 24 26 29 32 34 36 38 42 44 50 56 62 68 74

$\boxed{3\ 10\ 18}$ $\boxed{N_{0,1(3)}}$ 4 6 12 16 22 24 34 36 42

$\boxed{3\ 10\ 20}$ $\boxed{N}$ 4 5 6 8 11 12 14 16 17 18 22 23 24 26 29 32 34 35 36 38 41 42 44 47 53 59 62 65 71 77 83

$\boxed{3\ 10\ 22}$ $\boxed{N_{0,1(3)}}$ 4 6 12 16 18 24 34 36 42

$\boxed{3\ 11\ 12}$ $\boxed{N}$ 4 5 6 8 10 14 16 17 18 20 22 24 26 28 30 32 38 40 42 44 46 50 52 54 56 62 68 74 80 86 92

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$\boxed{3\ 11\ 16}$ $\boxed{N}$ 4 5 6 8 10 12 14 17 18 20 22 24 26 28 30 32 34 36 38 40 42 44 50 52 54 56 58 60 62 66 68 70 72 74 80 82 84 86 92 98 104 110 116 122 128 134 140

$\boxed{3\ 11\ 17}$ $\boxed{N_{1(2)}}$ 5

$\boxed{3\ 11\ 18}$ $\boxed{N}$ 4 5 6 8 10 12 14 16 17 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 86 92 94 96 98 104 110 116 122 128 134 140 146 152 158

$\boxed{3\ 11\ 20}$ $\boxed{N}$ 4 5 6 8 10 12 14 16 17 18 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50 52 54 56 62 64 66 68 70 72 74 76 78 82 84 86 88 90 92 94 96 98 100 102 104 106 108 110 116 122 124 126 130 132 136 138 142 144 146 148 150 152 154 158 164 170 184

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$\boxed{3\ 12\ 14}$ $\boxed{N}$ 4 5 6 8 10 11 16 17 18 20 22 23 24 26 28 29 30 32 35 38 41 44 46 47 52 53 59 65 71 77 83

$\boxed{3\ 12\ 16}$ $\boxed{N_{0,1(3)}}$ 4 6 10 18 22 24 28 30 40 42 54

$\boxed{3\ 12\ 17}$ $\boxed{N}$ 4 5 6 8 10 11 14 16 18 20 22 23 24 26 28 29 30 32 38 40 42 44 46 50 52 54 56 62 68 74 80 86 92 98 104

$\boxed{3\ 12\ 18}$ $\boxed{N_{0,1(3)}}$ 4 6 10 16 22 24 28 30 40 42 46

$\boxed{3\ 12\ 20}$ $\boxed{N}$ 4 5 6 8 10 11 14 16 17 18 22 23 24 26 28 29 30 32 35 38 40 41 42 44 46 47 50 52 53 54 59 62 65 71 77 83 89 95 101 107

$\boxed{3\ 12\ 22}$ $\boxed{N_{0,1(3)}}$ 4 6 10 16 18 24 28 30 40 42 46 52 54

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$\boxed{3\ 14\ 17}$ $\boxed{N}$ 4 5 6 8 10 11 12 16 18 20 22 23 24 26 28 29 30 32 34 36 38 44 46 48 50 52 54 58 60 62 64 66 68 70 72 74 80 88 90 94 96 100 104 110

$\boxed{3\ 14\ 18}$ $\boxed{N}$ 4 5 6 8 10 11 12 16 17 20 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 53 58 59 62 64 65 66 70 71 77 80 83 89 95 101 107

$\boxed{3\ 14\ 20}$ $\boxed{N}$ 4 5 6 8 10 11 12 16 17 18 22 23 24 26 28 29 30 32 34 35 36 38 41 44 46 47 48 50 52 53 54 59 64 65 66 68 70 71 72 77 83 89 95 101 107

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$\boxed{3\ 16\ 20}$ $\boxed{N}$ 4 5 6 8 10 11 12 14 17 18 22 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 47 50 52 53 54 56 59 62 65 66 70 71 72 74 77 82 83 84 86 89 95 101 107 113 119 125 131 137 143 149 155

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$\boxed{3\ 17\ 18}$ $\boxed{N}$ 4 5 6 8 10 11 12 14 16 20 22 23 24 26 28 29 30 32 34 36 38 40 42 44 46 48 50 56 58 60 62 64 66 68 70 74 76 78 80 82 84 86 92 94 96 98 104 110 116 122 128 134 140 146 152 158 164 170 176 182 188 194 200 206 212

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3 18 22	$N_{0,1(3)}$	4 6 10 12 16 24 28 30 34 36 40 42 46 48 58 60 70 78 82
3 20 22	N	4 5 6 8 10 11 12 14 16 17 18 23 24 26 28 29 30 32 34 35 36 38 40 41 42 44 46 47 48 50 52 53 54 56 59 62 65 68 70 71 72 74 76 77 78 83 84 89 90 92 94 95 96 101 107 110 113 119 125 131 137 143 149 155 161 167 173 179 185 191 197 203 209 215

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