Asymptotic results on suborthogonal $\vec{\mathfrak{G}}$-decompositions of complete digraphs

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Abstract. A $\vec{\mathfrak{G}}$-decomposition of a complete digraph $\vec{\mathfrak{D}}_n$ is a partition of $\vec{\mathfrak{D}}_n$ into isomorphic copies (called pages) of $\vec{\mathfrak{G}}$. A $\vec{\mathfrak{G}}$-decomposition is said to be suborthogonal if the union of any two pages contains at most one pair of reverse arcs.
R. M. Wilson [5] proved in 1975 that a $\vec{\mathfrak{G}}$-decomposition exists for almost all integers n satisfying certain necessary conditions. In this paper we shall prove that under the same conditions there exists even a suborthogonal $\vec{\mathfrak{G}}$-decomposition.

1 Introduction

For any positive integer n, let $\vec{\mathfrak{D}}_n$ denote the complete digraph on n vertices. A *decomposition* of $\vec{\mathfrak{D}}_n$ is a family $\mathcal{G} = \{\vec{\mathfrak{G}}_1, \dots, \vec{\mathfrak{G}}_s\}$ of subdigraphs of $\vec{\mathfrak{D}}_n$ (called *pages*), such that each arc of $\vec{\mathfrak{D}}_n$ belongs to exactly one page of $\mathcal{G}$. If, in addition, all pages are isomorphic to a given digraph $\vec{\mathfrak{G}}$, we call $\mathcal{G}$ a $\vec{\mathfrak{G}}$-*decomposition* of $\vec{\mathfrak{D}}_n$.

$\vec{\mathfrak{G}}$-decompositions have been considered widely in literature, see e.g. [1] and its references. Necessary conditions for the existence of a $\vec{\mathfrak{G}}$-decomposition are well known. For any vertex w in $\vec{\mathfrak{G}} = (W, B)$, let $\deg^-(w)$ and $\deg^+(w)$ denote the indegree and the outdegree of w in $\vec{\mathfrak{G}}$, respectively. Further, let d be the solution of the integer linear program

$$d = \min Y = \min\{y \in \mathbb{Z} : y \geq 1 \text{ and } \forall\, w \in W\ \exists\ x_w \in \mathbb{Z} \text{ such that } \sum_{w \in W} \deg^-(w) x_w = \sum_{w \in W} \deg^+(w) x_w = y\},$$

or $d = 0$ if $\vec{\mathfrak{G}}$ has no arcs.

Necessary conditions for the existence of a $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ are

$$n - 1 \equiv 0 \mod d \tag{1}$$
$$n(n-1) \equiv 0 \mod b, \tag{2}$$

where b denotes the number of arcs in $\vec{\mathfrak{G}}$. Also, by a result of Wilson [5], these conditions are known to be sufficient whenever n is large enough.

Gronau, Mullin and Schellenberg introduced in [2] the concept of orthogonal $\vec{\mathfrak{G}}$-decompositions (or orthogonal $\vec{\mathfrak{G}}$-covers). A decomposition $\mathcal{G}$ of $\vec{\mathfrak{D}}_n$ is said to be *orthogonal* if the union of any two pages contains precisely one digon, i.e. a digraph isomorphic to $\vec{\mathfrak{D}}_2$. This precondition happens to be rather restrictive: Every non-trivial orthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ consists of n pages, and $\vec{\mathfrak{G}}$ has exactly $n - 1$ arcs.

Our objective is to study the larger class of suborthogonal $\vec{\mathfrak{G}}$-decompositions. We call a decomposition $\mathcal{G}$ of $\vec{\mathfrak{D}}_n$ *suborthogonal* if the union of any two pages contains at most one digon.

Obviously, for every digraph $\vec{\mathfrak{G}}$ there exists a suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathcal{G}$ of the complete digraph $\vec{\mathfrak{D}}_1$. (Just choose the empty family $\mathcal{G}$.) On the other hand, if $\vec{\mathfrak{G}}$ has no arcs, $\vec{\mathfrak{D}}_1$ is the only complete digraph with this property. If $\vec{\mathfrak{G}}$ contains a digon, then every suborthogonal $\vec{\mathfrak{G}}$-decomposition consists of at most one page. Hence, $\vec{\mathfrak{G}}$ has to be a complete digraph itself.

In the sequel, let $\vec{\mathfrak{G}}$ always be a digraph with non-empty arc set and without digons.

In this paper we shall give a Wilson-type result for suborthogonal $\vec{\mathfrak{G}}$-decompositions, i.e. we prove that even such a decomposition of $\vec{\mathfrak{D}}_n$ exists whenever the necessary conditions (1) and (2) hold and n is sufficiently large. Actually, we will show how to alter Wilson's original work to obtain the claimed result.

2 Eventual periodicity of the sets $\mathcal{S}_{\vec{\mathfrak{G}}}$

Let $\vec{\mathfrak{G}} = (W, B)$ be a digraph with k vertices and $b \geq 1$ arcs. By $\mathcal{D}_{\vec{\mathfrak{G}}}$ and $\mathcal{S}_{\vec{\mathfrak{G}}}$ we denote the sets of all integers n such that there is a $\vec{\mathfrak{G}}$-decomposition, resp. a suborthogonal $\vec{\mathfrak{G}}$-decomosition of $\vec{\mathfrak{D}}_n$.

Further, we need the notion of a pairwise balanced design (PBD). Let v be a positive integer and K a set of positive integers. A *pairwise balanced design* PBD(v, K) is a pair $(V, \mathcal{F})$, where V is a v-set and $\mathcal{F}$ is a family of subsets of V (called *blocks*), such that every pair of distinct elements of V lies in exactly one block of $\mathcal{F}$ and all block sizes belong to K.

A set K of positive integers is said to be *PBD-closed* if the existence of a PBD(v, K) implies that v lies in K. In [5] Wilson proved that $\mathcal{D}_{\vec{\mathfrak{G}}}$ is PBD-closed for all digraphs $\vec{\mathfrak{G}}$. The crucial observation is that the sets $\mathcal{S}_{\vec{\mathfrak{G}}}$ have the same property.

Theorem 1 *Let $\vec{\mathfrak{G}}$ be a digraph without digons. The set*

$$\mathcal{S}_{\vec{\mathfrak{G}}} = \{n \in \mathbb{Z} : \text{ there exists a suborthogonal } \vec{\mathfrak{G}}\text{-decomposition of } \vec{\mathfrak{D}}_n\}$$

is PBD-closed.

Proof: Let n be an integer in the PBD-closure of $\mathcal{S}_{\vec{\mathfrak{G}}}$. Hence, there is a pairwise balanced design $(V, \mathcal{F})$ of order n and block sizes from $\mathcal{S}_{\vec{\mathfrak{G}}}$. For each block $F \in \mathcal{F}$, let $\mathcal{G}_F$ be a suborthogonal $\vec{\mathfrak{G}}$-decomposition of the complete digraph $\vec{\mathfrak{D}}_{|F|}$ with vertex set F. Let $\mathcal{G}$ be the union of all the families $\mathcal{G}_F$ ($F \in \mathcal{F}$). Obviously, $\mathcal{G}$ is a $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$.

We claim that $\mathcal{G}$ is suborthogonal, too. Assume, there are two pages whose union contains two or more digons. If these pages belong to different families $\mathcal{G}_F$ and $\mathcal{G}_{F'}$, this implies that the intersection of F and F' contains at least 3 vertices in contradiction to $(V, \mathcal{F})$ being a pairwise balanced design.

On the other hand, the two pages may not belong to the same family $\mathcal{G}_F$, since each of these families is a suborthogonal $\vec{\mathfrak{G}}$-decomposition of a complete digraph. This concludes the proof. ∎

A set K of positive integers is *eventually periodic* with period π if, for any $s \in K$, we have $n \in K$ for all sufficiently large integers n satisfying $n \equiv s \mod \pi$. A deep theorem of Wilson [4] states that a PBD-closed set K is eventually periodic with period $\beta(K) = \gcd\{n(n-1) : n \in K\}$, and hence, with period $\pi = \lambda\beta(K)$ for any positive integer λ.

By Theorem 1, the sets $\mathcal{S}_{\vec{\mathfrak{G}}}$ are eventually periodic. In the following sections, we shall investigate eventual periods of $\mathcal{S}_{\vec{\mathfrak{G}}}$.

3 Edge colorings of complete graphs

To find $\beta(\mathcal{S}_{\vec{\mathfrak{G}}})$ or at least a multiple of this value, we need some results about edge colorings in complete graphs. Let $\mathfrak{K}_k = (W, E)$ be the complete graph on k vertices. An edge coloring of $\mathfrak{K}_k$ with b colors is a mapping $\gamma : E \to \{1, \dots, b\}$.

For a given edge coloring γ of $\mathfrak{K}_k$, we call a 4-cycle $\mathfrak{C}$ in $\mathfrak{K}_k$ *alternating* if its edges $e_1 = \{w_1, w_2\}$, $e_2 = \{w_2, w_3\}$, $e_3 = \{w_3, w_4\}$ and $e_4 = \{w_4, w_1\}$ satisfy $\gamma(e_1) = \gamma(e_3)$ and $\gamma(e_2) = \gamma(e_4)$. We shall prove the following

Lemma 2 *Let $\mathfrak{G} = (W, F)$ be a spanning subgraph of $\mathfrak{K}_k = (W, E)$ with edges $e_1, \dots, e_b$. There exists an edge coloring γ of $\mathfrak{K}_k$ such that*

$$\gamma(e_i) = i$$

holds for each of the edges $e_1, \dots, e_b$, and such that the vertices of every alternating 4-cycle in $\mathfrak{K}_k$ induce in $\mathfrak{K}_k$ at most one of the edges $e_1, \dots, e_b$.

The proof requires some preparations.

Claim 3 *Let $\mathfrak{B} = (W, F)$ be a tree and z a vertex of $\mathfrak{B}$. There is a bijection $\phi : W \backslash \{z\} \to F$ satisfying $w \in \phi(w)$ for each vertex $w \neq z$ in $\mathfrak{B}$.*

Proof: We use induction on $|W| = k$. For $k = 1$ the statement is trivial. Let $k \geq 2$. Assume, z is a leaf in $\mathfrak{B}$ and let y denote its neighbor in $\mathfrak{B}$. Deleting z from $\mathfrak{B}$, we again obtain a tree $\mathfrak{B}'$ with $k-1$ vertices. By the induction hypothesis, there is a bijection $\phi' : W \backslash \{y, z\} \to F \backslash \{e\}$, where e denotes the edge $\{y, z\}$, such that $w \in \phi'(w)$ holds for all $w \neq y, z$. Define ϕ by

$$\phi(w) = \begin{cases} \phi'(w) & \text{if } w \neq y, z, \\ e & \text{if } w = y. \end{cases}$$

Obviously, ϕ is the claimed bijection.

Now, let z be an arbitrary vertex in $\mathfrak{B}$ and let y be a leaf in $\mathfrak{B}$. Then, there is a bijection $\phi' : W \backslash \{y\} \to F$ satisfying $w \in \phi'(w)$ for all $w \neq y$. Let $y = y_0, y_1, \ldots, y_t = z$ be a path from y to z in $\mathfrak{B}$. Define ϕ by

$$\phi(w) = \begin{cases} \phi'(y_{i+1}) & \text{if } w = y_i (i = 0, \ldots, t-1), \\ \phi'(w) & \text{if } w \neq y_0, \ldots, y_t \end{cases}.$$

It is easy to see, that ϕ has the claimed property. ∎

Claim 4 *Let $\mathfrak{G} = (W, F)$ be a connected graph, but not a tree. There is an injection $\phi : W \to F$ satisfying $w \in \phi(w)$ for each vertex w in $\mathfrak{G}$.*

Proof: Let $\mathfrak{B}$ be a spanning subtree of $\mathfrak{G}$. Since $\mathfrak{G}$ is not a tree, there is an edge $e = \{y, z\}$ in $\mathfrak{G}$, which is not in $\mathfrak{B}$. By Claim 3, there is an injection $\phi' : W \backslash \{z\} \to F \backslash \{e\}$ such that $w \in \phi'(w)$ holds for all vertices $w \neq z$. Define ϕ by

$$\phi(w) = \begin{cases} e & \text{if } w = z, \\ \phi'(w) & \text{otherwise.} \end{cases}$$

Obviously, ϕ is the claimed mapping. ∎

Now, we are ready to prove Lemma 2.

Proof: (of Lemma 2) In every tree component of $\mathfrak{G} = (W, F)$ choose a leaf or a singleton, and let Z denote the set of all these vertices. Using Claims 3 and 4, it is easy to verify that there exists an injection $\phi : W \backslash Z \to F$ satisfying $w \in \phi(w)$ for all vertices $w \in W$.

Initially, let all edges of $\mathfrak{K}_k$ be uncolored. First, color the edges $e_1, \ldots, e_b$ by $\gamma(e_i) = i$ $(i = 1, \ldots, b)$. Next, for each vertex $w \in W \backslash Z$, we color every uncolored edge e where $w \in e$ by $\gamma(e) = \gamma(\phi(w))$. So far, we have colored all edges with at least one vertex in $W \backslash Z$. Finally, the remaining edges of $\mathfrak{K}_k$ may be colored arbitrarily.

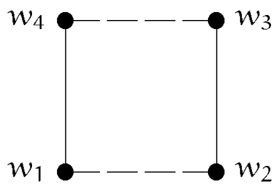

Figure 1: An alternating 4-cycle.

Obviously, whenever two edges of $\mathfrak{K}_k$ have the same color, then they join a vertex in $W \backslash Z$, or at least one of both edges completly lies in Z. Assume, there is an alternating 4-cycle in $\mathfrak{K}_k$ with vertices $w_1, \ldots, w_4$. This implies $\{w_1, w_2\} \subseteq Z$ or $\{w_3, w_4\} \subseteq Z$, as well as $\{w_2, w_3\} \subseteq Z$ or $\{w_4, w_1\} \subseteq Z$. W.l.o.g. we may assume, that the vertices w_1, w_2 and w_3 belong to Z. However, every vertex in $W \backslash Z$ has at most one neighbor in Z with respect to $\mathfrak{G}$. Hence, at most one of the pairs $\{w_1, w_4\}, \{w_2, w_4\}, \{w_3, w_4\}$ is an edge in $\mathfrak{G}$. This concludes the proof. ■

Note, that whenever the digraph $\mathfrak{G}$ has at most one tree component, the coloring γ constructed in the proof of Lemma 2 has a nice property, namely for any two edges e and e' in $\mathfrak{K}_k$ with $\gamma(e) = \gamma(e')$ we have $e \cap e' \neq \emptyset$. In some sense one may feel that this is an anti-concept to what one usually considers as a good edge coloring. However, it turned out that our edge colorings are helpful for direct constructions of suborthogonal $\mathfrak{G}$-decompositions.

4 A direct construction

Using Lemma 2, we are able to prove the following result which is the analogon to Wilson's Proposition 1 in [5].

Lemma 5 *Let $\vec{\mathfrak{G}} = (W, B)$ be a digraph without digons and with $b \geq 1$ arcs. Then, a prime power $q \equiv 1 \mod 2b$ belongs to $\mathcal{S}_{\vec{\mathfrak{G}}}$ whenever q is sufficiently large.*

Proof: Consider $V = \{0, \ldots, q-1\}$ to be the finite field $GF(q)$ of order q. Let ω be a generator of the multiplicative group $(V \backslash \{0\}, \cdot)$, and let T be its subgroup of index b and cardinality $\frac{q-1}{b}$ generated by ω^b. The cosets of T are $T_1, \ldots, T_b$, given by $T_i = \omega^i T$ $(i = 1, \ldots, b)$. Note, that the field element -1 lies in T, since $\frac{q-1}{b}$ is even.

Let $W = \{1, \ldots, k\}$ and $\mathfrak{K}_k$ the complete graph with vertex set W. Further, let $\mathfrak{G}$ be the undirected underlying graph of $\vec{\mathfrak{G}}$. Obviously, $\mathfrak{G}$ is a spanning subgraph of $\mathfrak{K}_k$. Let γ be the coloring proposed in Lemma 2. Denoting the edges of $\mathfrak{G}$ by $e_1, \ldots, e_b$, we have $\gamma(e_i) = i$ for all $i = 1, \ldots, b$.

A k-*choice* C is a mapping of $\{(i,j) \in W \times W : i \neq j\}$ into the family of cosets of T. Define the k-choice C by

$$C(i,j) = T_{\gamma(\{i,j\})} \tag{3}$$

for any two distinct elements i and j from W. Assume, q is sufficiently large. By [3], there is a k-set $X = \{x_1, \ldots, x_k\} \subseteq V$ such that $x_j - x_i \in C(i,j)$ holds for all pairs (i,j) with $1 \leq i < j \leq k$. Due to our choice of C, we also have $x_j - x_i \in C(i,j)$ if $1 \leq j < i \leq k$. In particular, the differences $x_j - x_i$ taken over the b arcs (i,j) in $\vec{\mathfrak{G}}$ form a system of representatives of the cosets of T.

Now, for every $v \in V$ and every $t \in T$, we construct a digraph $\vec{\mathfrak{G}}(v,t)$ as the isomorphic image of $\vec{\mathfrak{G}}$ under the mapping $w \to tx_w + v$ for all vertices $w \in W$.

First, we shall show that these digraphs form a $\vec{\mathfrak{G}}$-decomposition $\mathcal{G}$ of $\vec{\mathfrak{D}}_q$. Note, that we have got the right size for the family $\mathcal{G}$. There are just $|V||T| = q\frac{q-1}{b}$ pages in $\mathcal{G}$, each of them containing b arcs. On the other hand, the complete digraph $\vec{\mathfrak{D}}_q$ has exactly $q(q-1)$ arcs. Thus, $\mathcal{G}$ is a decomposition of $\vec{\mathfrak{D}}_q$ if every arc of $\vec{\mathfrak{D}}_q$ occurs in at least (hence exactly) one page. For the arc (y, z) in $\vec{\mathfrak{D}}_q$, let (i, j) be the unique arc in $\vec{\mathfrak{G}}$ such that the difference $x_j - x_i$ lies in the same coset of T as the difference $z - y$. Choose $t = (z - y)(x_j - x_i)^{-1}$, which lies in T, and $v = y - tx_i$, which belongs to V. It is easy to check, that (y, z) is contained in the uniquely determined page $\vec{\mathfrak{G}}(t, v)$.

It remains to show, that due to our choice of C, the $\vec{\mathfrak{G}}$-decomposition $\mathcal{G}$ is suborthogonal. Assume, there are two distinct pages $\vec{\mathfrak{G}}(v, t)$ and $\vec{\mathfrak{G}}(v', t')$ whose union contains two or more digons. Hence, there are two distinct arcs (i, j) and (l, m) in $\vec{\mathfrak{G}}$ such that

$$tx_i + v = t'x_j + v' \qquad tx_l + v = t'x_m + v'$$
$$tx_j + v = t'x_i + v' \qquad tx_m + v = t'x_l + v'.$$

This implies $t = -t'$, and

$$x_l - x_i = x_j - x_m \text{ as well as } x_m - x_i = x_j - x_l.$$

Hence, $i = l$ holds iff $j = m$ holds, and $i = m$ holds iff $j = l$ holds. Assume, the vertices i, j, l and m are not pairwise distinct. Then, the arcs (i, j) and (l, m) are either equal or form a digon. However, both cases contradict the assumptions.

Hence, $\{i, j, l, m\}$ is a 4-set. In particular, this implies

$$\gamma(\{i, l\}) = \gamma(\{j, m\}) \text{ as well as } \gamma(\{i, m\}) = \gamma(\{j, l\}),$$

i.e. the 4-cycle with vertices i, j, l, m in $\mathfrak{K}_k$ is alternating. By Lemma 2, at most one of the pairs $\{i, j\}$ and $\{l, m\}$ is an edge in $\mathfrak{G}$, which gives a contradiction. This concludes the proof. ∎

With the help of Lemma 5, we are able to determine an eventual period of $\mathcal{S}_{\vec{\mathfrak{G}}}$. Let $\mathcal{Q}_{\vec{\mathfrak{G}}}$ be the set of all prime powers $q \equiv 1 \mod 2b$ which are large enough in the sense of Lemma 5. By Dirichlet's Theorem on primes in arithmetic progressions, $\mathcal{Q}_{\vec{\mathfrak{G}}}$ is non-empty, and so is $\mathcal{S}_{\vec{\mathfrak{G}}} \supseteq \mathcal{Q}_{\vec{\mathfrak{G}}}$. Further, $\beta(\mathcal{Q}_{\vec{\mathfrak{G}}})$ is divisible by $2b$.

Let the prime p belong to $\mathcal{Q}_{\vec{\mathfrak{G}}}$, say $p = 2b\lambda + 1$ for some positive integer λ. Again by Dirichlet's Theorem, there is a prime q such that $q \equiv 1 + 2b \mod 2bp\lambda$, say $q = 2bp\lambda\mu + 2b + 1$ for some positive integer μ. Note, that $p \leq q$, and thus, q lies in $\mathcal{Q}_{\vec{\mathfrak{G}}}$, too. We have $\gcd\{p(p-1), q(q-1)\} =$

$\gcd\{p(p-1), q-1\} = \gcd\{2bp\lambda, 2bp\lambda\mu + 2b\} = 2b\ \gcd\{p\lambda, p\lambda\mu + 1\} = 2b$. Thus, $2b$ is divisible by $\beta(\mathcal{Q}_{\vec{\mathfrak{G}}}) = \gcd\{n(n-1) : n \in \mathcal{Q}_{\vec{\mathfrak{G}}}\}$. Since $\mathcal{Q}_{\vec{\mathfrak{G}}} \subseteq \mathcal{S}_{\vec{\mathfrak{G}}}$, $\beta(\mathcal{Q}_{\vec{\mathfrak{G}}})$ is divisible by $\beta(\mathcal{S}_{\vec{\mathfrak{G}}})$, which implies

Corollary 6 *$\mathcal{S}_{\vec{\mathfrak{G}}}$ is eventually periodic with period $2b$.*

Note, that using a similar argumentation, $\mathcal{D}_{\vec{\mathfrak{G}}}$ is eventually periodic with period $2b$, too.

5 A recursive construction

According to Corollary 6, it remains to prove that $\mathcal{S}_{\vec{\mathfrak{G}}}$ contains a representative from each admissible residue class modulo $2b$. The following lemma is the analogon to Wilson's Proposition 3 in [5].

Lemma 7 *Let $\vec{\mathfrak{G}} = (W, B)$ be a digraph without digons and with $b \geq 1$ arcs. For every integer s satisfying the necessary conditions (1) and (2), there is an integer $n \equiv s \mod 2b$ such that $n \in \mathcal{S}_{\vec{\mathfrak{G}}}$.*

Proof: By Wilson's original claim [5], there is an integer $r \equiv s \mod 2b$ such that $r \in \mathcal{D}_{\vec{\mathfrak{G}}}$, i.e. such that there exits a $\vec{\mathfrak{G}}$-decomposition $\mathcal{H}$ of the complete digraph $\vec{\mathfrak{D}}_r$. However, $\mathcal{H}$ will usually not be suborthogonal.

Starting with the $\vec{\mathfrak{G}}$-decomposition $\mathcal{H}$ we shall construct a suborthogonal $\vec{\mathfrak{G}}$-decomposition of a larger complete digraph $\vec{\mathfrak{D}}_n$.

First, we present some notations to be used in the sequel. Let $p \geq r^2$ be a prime such that $p \equiv 1 \mod 2b$ and $q = p^\delta \in \mathcal{S}_{\vec{\mathfrak{G}}}$, where $\delta = \binom{r}{2}$. Choose $n = rq$ with $n \equiv r \equiv s \mod 2b$. Note, that n satisfies the necessary conditions (1) and (2). Our objective is to present a suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$.

Put $U = \{1, \ldots, r\}$, $Y = \{0, \ldots, p-1\}$, $X = Y^\delta$ and $V = U \times X$. Suppose Y to be the finite field $GF(p)$ of order p, and X to be the δ-dimensional vectorspace over $GF(p)$.

Consider the sequence

$$(1,2), \ldots, (1,r), (2,3), \ldots, (2,r), \ldots, (r-1,r)$$

of all pairs (i,j) from $U \times U$ with $1 \le i < j \le r$. The position (starting with 0) of a fixed pair (i,j) in this sequence is given by the function

$$\rho(i,j) = r(i-1) - \binom{i}{2} + (j-i) - 1.$$

For any vector $\underline{x} \in X$, let its coordinates be indexed by the pairs in this sequence, i.e. let $\underline{x} = (x_{(1,2)}, \ldots, x_{(1,r)}, x_{(2,3)}, \ldots, x_{(r-1,r)})$. Further, denote by T the $(\delta - 1)$-dimensional subspace $\{\underline{x} \in X : x_{(1,2)} + \ldots + x_{(r-1,r)} = 0\}$ of X.

Let ω be a generator of the multiplicative group $(Y \backslash \{0\}, \cdot)$ of $GF(p)$. We define a mapping $\zeta : U \times X \to X$ by $\zeta(u, \underline{x}) = \underline{z}$ with coordinates

$$z_{(i,j)} = \begin{cases} x_{(i,j)} & \text{if } u = i \text{ or } u = j, \\ \omega^u x_{(i,j)} & \text{otherwise.} \end{cases}$$

Further, for every page $\vec{\mathfrak{H}}$ in $\mathcal{H}$, we define a mapping $\phi_{\vec{\mathfrak{H}}} : U \times Y \to X$ by $\phi_{\vec{\mathfrak{H}}}(u, y) = \underline{f}$ with coordinates

$$f_{(i,j)} = \begin{cases} y & \text{if } u = j \text{ and } (i,j) \text{ is an arc in } \vec{\mathfrak{H}}, \\ y + \rho(i,j) & \text{if } u = j \text{ and } (j,i) \text{ is an arc in } \vec{\mathfrak{H}}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we shall show how to obtain the pages of the claimed suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathcal{G}$. First, on each of the sets $\{u\} \times X$ $(u \in U)$, we construct a suborthogonal $\vec{\mathfrak{G}}$-decomposition $\mathcal{G}_u$ of the complete digraph $\vec{\mathfrak{D}}_q$ with vertex set $\{u\} \times X$. The existence of $\mathcal{G}_u$ is ensured by our assuption $q \in \mathcal{S}_{\vec{\mathfrak{G}}}$.

In addition, for every page $\vec{\mathfrak{H}}$ in $\mathcal{H}$, every vector $\underline{x} \in X$, every vector $\underline{t} \in T$ and every integer $y \in Y$, we construct a digraph $\vec{\mathfrak{H}}(\underline{x}, \underline{t}, y)$ as the isomorphic image of $\vec{\mathfrak{H}}$ under the mapping $u \to (u, \underline{u})$ with $\underline{u} = \underline{x} + \zeta(u, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(u, y)$. Denote the family of all these digraphs by $\mathcal{G}_0$.

First, we shall show that the union $\mathcal{G}$ of $\mathcal{G}_0$ and the families $\mathcal{G}_u$ $(u \in U)$ is a $\vec{\mathfrak{G}}$-decomposition of the complete digraph $\vec{\mathfrak{D}}_n$. Note, that we have got the right size for $\mathcal{G}$. Each of the families $\mathcal{G}_u$ contains $q\frac{q-1}{b}$ pages, the family $\mathcal{G}_0$ consists of

$$|\mathcal{H}||X||T||Y| = r\frac{r-1}{b}p^{\delta}p^{\delta-1}p = r\frac{r-1}{b}p^{2\delta} = r\frac{r-1}{b}q^2$$

digraphs. The total number of digraphs in $\mathcal{G}$ is

$$rq\frac{q-1}{b} + r\frac{r-1}{b}q^2 = rq\frac{rq-1}{b} = n\frac{n-1}{b},$$

where each of them has exactly b arcs. On the other hand, the complete digraph $\vec{\mathfrak{D}}_n$ contains just $n(n-1)$ arcs.

Thus, we have to verify that every arc of $\vec{\mathfrak{D}}_n$ occurs in at least (and hence exactly) one of the digraphs in $\mathcal{G}$. Consider the arc $((i,\underline{i}),(j,\underline{j}))$. If $i = j$, both vertices of the arc lie in the same set $\{u\} \times X$. Hence, the arc belongs to a digraph in $\mathcal{G}_u$, which is a $\vec{\mathfrak{G}}$-decomposition of the complete digraph on $\{u\} \times X$.

If $i < j$, we are looking for a digraph $\vec{\mathfrak{H}}$ from $\mathcal{H}$, vectors $\underline{x}$, $\underline{t}$ and an integer y such that $((i,\underline{i}),(j,\underline{j}))$ lies in $\vec{\mathfrak{H}}(\underline{x},\underline{t},y)$. Let $\vec{\mathfrak{H}}$ be unique digraph in $\mathcal{H}$ which contains the arc (i,j). Further, let y be the coordinate indexed with (i,j) in the vector $\underline{j}-\underline{i}$, i.e. $y = (\underline{j}-\underline{i})_{(i,j)}$. Now, we are able to determine the vector $\underline{z} = \underline{j}-\underline{i}-\phi_{\vec{\mathfrak{H}}}(j,y)+\phi_{\vec{\mathfrak{H}}}(i,y)$. Choose for any coordinate indexed with $(u,v) \neq (i,j)$ in $\underline{t}$

$$t_{(u,v)} = \begin{cases} (\omega^j - 1)^{-1} z_{(u,v)} & \text{if } u = i \text{ or } v = i, \\ (1-\omega^i)^{-1} z_{(u,v)} & \text{if } u = j \text{ or } v = j, \\ (\omega^j - \omega^i) z_{(u,v)} & \text{otherwise.} \end{cases}$$

The remaining coordinate $t_{(i,j)}$ of $\underline{t}$ should be choosen such that $\underline{t}$ lies in T, i.e. such that $t_{(1,2)} + \ldots + t_{(i,j)} + \ldots + t_{(r-1,r)} = 0$ holds in $GF(p)$. Finally, let $\underline{x} = \underline{i} - \zeta(i,\underline{t}) - \phi_{\vec{\mathfrak{H}}}(i,y)$.

Similarly, we handle the case $j < i$. It is not difficult to check, that the arc $((i,\underline{i}),(j,\underline{j}))$ is contained in the uniquely determined digraph $\vec{\mathfrak{H}}(\underline{x},\underline{t},y)$.

It remains to verify the suborthogonality of $\mathcal{G}$. Assume, there are two pages in $\mathcal{G}$ whose union contains two or more digons. Obviously, both pages should belong to the family $\mathcal{G}_0$. Hence, denote them by $\vec{\mathfrak{H}}(\underline{x},\underline{t},y)$ and $\vec{\mathfrak{H}}'(\underline{x}',\underline{t}',y')$. Since their union contains two digons, there are two different arcs $((i,\underline{i}),(j,\underline{j}))$ and $((l,\underline{l}),(m,\underline{m}))$ in $\vec{\mathfrak{H}}(\underline{x},\underline{t},y)$ such that the reverse arcs $((j,\underline{j}),(i,\underline{i}))$ and $((m,\underline{m}),(l,\underline{l}))$ lie in $\vec{\mathfrak{H}}'(\underline{x}',\underline{t}',y')$. In particular, this implies that the digraph $\vec{\mathfrak{H}}$ contains arcs (i,j) and (l,m) as well as the digraph $\vec{\mathfrak{H}}'$ contains arcs (j,i) and (m,l).

Due to the construction of the pages in $\mathcal{G}_0$, we have

$$\begin{aligned}
\underline{i} =& \underline{x} + \zeta(i, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(i, y) = \underline{x}' + \zeta(i, \underline{t}') + \phi_{\vec{\mathfrak{H}}'}(i, y') \\
\underline{j} =& \underline{x} + \zeta(j, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(j, y) = \underline{x}' + \zeta(j, \underline{t}') + \phi_{\vec{\mathfrak{H}}'}(j, y') \\
\underline{l} =& \underline{x} + \zeta(l, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(l, y) = \underline{x}' + \zeta(l, \underline{t}') + \phi_{\vec{\mathfrak{H}}'}(l, y') \\
\underline{m} =& \underline{x} + \zeta(m, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(m, y) = \underline{x}' + \zeta(m, \underline{t}') + \phi_{\vec{\mathfrak{H}}'}(m, y').
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\underline{j} - \underline{i} &= \zeta(j, \underline{t}) - \zeta(i, \underline{t}) + \phi_{\vec{\mathfrak{H}}}(j, y) - \phi_{\vec{\mathfrak{H}}}(i, y) \\
&= \zeta(j, \underline{t}') - \zeta(i, \underline{t}') + \phi_{\vec{\mathfrak{H}}'}(j, y') - \phi_{\vec{\mathfrak{H}}'}(i, y').
\end{aligned}$$

W.l.o.g. let $i < j$. Considering the coordinate of $\underline{j} - \underline{i}$ indexed with (i, j), this implies

$$\begin{aligned}
t_{(i,j)} - t_{(i,j)} + y - 0 &= t'_{(i,j)} - t'_{(i,j)} + (y' + \rho(i,j)) - 0 \\
y &= y' + \rho(i,j).
\end{aligned} \tag{4}$$

Similarly, considering the coordinate indexed with (l, m) (or (m, l), respectively) of the vector $\underline{m} - \underline{l}$, we obtain

$$y = \begin{cases} y' + \rho(l, m) & \text{if } l < m, \\ y' - \rho(m, l) & \text{if } m < l. \end{cases}$$

Together with (4), this implies

$$\rho(i,j) = \begin{cases} \rho(l, m) & \text{if } l < m, \\ -\rho(m, l) & \text{if } m < l \end{cases} \tag{5}$$

in the finite field $GF(p)$.

However, by the definition of the function ρ, we have

$$0 \leq \rho(u, v) < \binom{r}{2} < \frac{p}{2}.$$

for every pair (u, v) of integers with $1 \leq u < v \leq r$. Therefore, equation (5) implies

$$(i,j) = \begin{cases} (l, m) & \text{if } l < m, \\ (m, l) & \text{if } m < l, \end{cases}$$

i.e. both cases give a contradiction to our assumption. This concludes the proof. ∎

As an immediate consequence of Lemma 7 we obtain our main theorem on the asymptotic existence of suborthogonal $\vec{\mathfrak{G}}$-decompositions.

Theorem 8 *Let $\vec{\mathfrak{G}}$ be a digraph without digons and with non-empty arc set. There exists a suborthogonal $\vec{\mathfrak{G}}$-decomposition of the complete digraph $\vec{\mathfrak{D}}_n$ for all sufficiently large integers n satisfying the necessary conditions (1) and (2).*

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