An upper bound for the size of the largest antichain in the poset of partitions of an integer

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Abstract

Let Pi_n be the poset of partitions of an integer n, ordered by refinement. Let $b(Pi_n)$ be the largest size of a level and $d(Pi_n)$ be the largest size of an antichain of Pi_n. We prove that

$$\frac{d(Pi_n)}{b(Pi_n)} \leq e + o(1) \text{ as } n \to \infty.$$

The denominator is determined asymptotically. In addition, we show that the incidence matrices in the lower half of Pi_n have full rank, and we prove a tight upper bound for the ratio from above if Pi_n is replaced by any graded poset P.

1 Introduction

Let P be a *graded poset*, i.e. a partially ordered set which can be partitioned into *levels* $N_0, \dots, N_{r(P)}$ such that N_0 (resp. $N_{r(P)}$) is the set of all minimal (resp. maximal) elements of P and $p \in N_i, p \lessdot q$ imply $q \in N_{i+1}$. Here $p \lessdot q$ means that $p < q$ and there is no element q' with $p < q' < q$. We say that in this case *q covers p*. Note that the partition of P into levels is unique if it exists. The number $r(P)$ is called the *rank of* P.

Let $b(P)$ be the largest size of a level of the graded poset P. An *antichain* in P is a set of pairwise incomparable elements of P. Let $d(P)$ be the largest size of an antichain in P. Obviously, for each graded poset P,

$$\frac{d(P)}{b(P)} \geq 1.$$

After Sperner [9], it was proven for many interesting classes of graded posets that the inequality is in fact an equality, cf. [5].

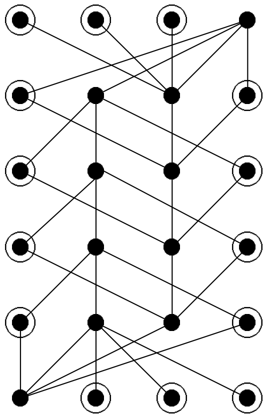

Figure 1

But there exist graded posets where the ratio is arbitrarily large. E.g., for the class of graded posets which is illustrated in Figure 1 for $r(P) = 5$ we have

$$\frac{d(P)}{b(P)} = \frac{|P|}{8} + \frac{1}{2}.$$

We will show that there is no graded poset with a larger ratio if $|P| \geq 12$.

Theorem 1 *Let P be a graded poset. Then*

$$\frac{d(P)}{b(P)} \leq \max\left\{\frac{|P|}{8}+\frac{1}{2}, 2\right\}.$$

Some similar results have been obtained in [6].

Let Π_n be the (graded) poset (lattice) of partitions of $[n] := \{1, \ldots, n\}$, ordered by refinement. From [2] and [4] we know (all logarithms are natural):

Theorem 2 *Let $a := (2 - e\log 2)/4$. Then for suitable constants c_1, c_2, and $n > 1$*

$$c_1 n^a (\log n)^{-a-1/4} \leq \frac{d(\Pi_n)}{b(\Pi_n)} \leq c_2 n^a (\log n)^{-a-1/4}.$$

Moreover, corresponding limit theorems (cf. [5, p. 316]) imply:

Theorem 3 *We have*

$$b(\Pi_n) \sim \frac{\sqrt{\log n}}{\sqrt{2\pi}} \frac{|\Pi_n|}{\sqrt{n}} \text{ as } n \to \infty.$$

In this paper we will study a quotient of the partition lattice Π_n, namely the poset Pi_n of unordered partitions of an integer n: A *partition of the integer n into k parts*, $k = 1, \ldots, n$, is an integral solution to the system

$$n = x_1 + \cdots + x_k, \ x_1 \geq \cdots \geq x_k > 0.$$

We obtain all partitions in Pi_n which are covered by this partition by taking one summand x_l $(1 \leq l \leq k)$ and partitioning x_l into exactly two parts and finally ordering the two new parts together with the old unpartitioned parts in a nonincreasing way. The Hasse diagram of the poset Pi_7 is illustrated in Figure 2. The main result of the paper is the following:

Theorem 4 *We have*

$$1 \leq \frac{d(Pi_n)}{b(Pi_n)} \leq e + o(1) \text{ as } n \to \infty.$$

We will give a proof of the following theorem, since it follows by the same method we use to prove Theorem 9; it was first shown by Auluck, Chowla, and Gupta [1].

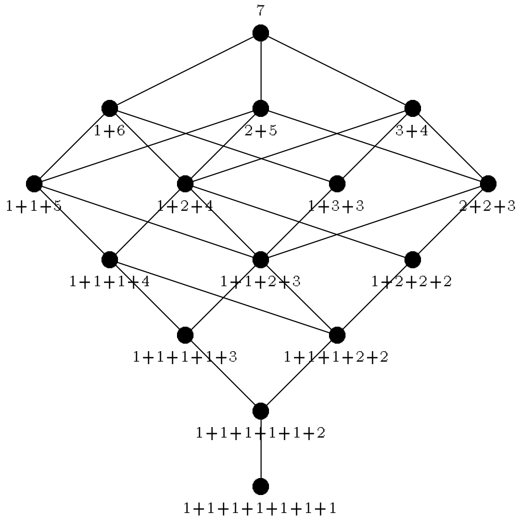

Figure 2

Theorem 5 *We have*

$$b(Pi_n) \sim \frac{\pi}{e\sqrt{6}} \frac{|Pi_n|}{\sqrt{n}} \text{ as } n \to \infty.$$

For a graded poset P, the *incidence matrix* M_k, $k = 0, \dots, r(P) - 1$, is an $(|N_k| \times |N_{k+1}|)$ 0–1–matrix whose rows and columns are indexed by the elements of N_k and N_{k+1}, respectively, and whose element in row $p \in N_k$ and column $q \in N_{k+1}$ equals 1 iff $p \lessdot q$. The following result is due to Kung [8] (see also [8] for further background):

Theorem 6 *Let* $P = \Pi_n$ *and* $k < \frac{n-1}{2}$. *Then*

$$\operatorname{rank}(M_k) = |N_k|.$$

We will prove that the theorem remains true for the poset of partitions of an integer:

Theorem 7 *Let* $P = Pi_n$ *and* $k < \frac{n-1}{2}$. *Then*

$$\operatorname{rank}(M_k) = |N_k|.$$

2 Proof of the general ratio bound

Proof of Theorem 1. We proceed by induction on $r(P)$. The case $r(P) = 0$ is trivial, thus consider the step $< r(P) \to r(P)$. Let briefly $b := b(P)$ and let A be a maximum antichain in P.

Case 1. There is some $k \in \{0, \dots, r(P)\}$ such that $|A \cap N_k| = |N_k|$. Since P is graded, we have $A = N_k$ and thus

$$\frac{d(P)}{b(P)} = 1 \le \max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}.$$

Case 2. There is some $k \in \{1, \dots, r(P)-1\}$ such that $|A \cap N_k| = |N_k| - 1$. Let

$$A_l := \bigcup_{i=0}^{k-1} (A \cap N_i) \text{ and } A_u := \bigcup_{i=k+1}^{r(P)} (A \cap N_i).$$

Let p be the (unique) element of $N_k \setminus A$. Since P is graded, all elements of A_l and A_u are comparable with p, hence $A_l = \emptyset$ or $A_u = \emptyset$. Let w.l.o.g. $A_u = \emptyset$. Let

$$P' := \bigcup_{i=0}^{k} N_i.$$

Clearly, P' is also graded and

$$\begin{aligned} &d(P) = |A| \le d(P') \le d(P),\\ &b(P') \le b(P). \end{aligned}$$

Consequently, by the induction hypothesis

$$\frac{d(P)}{b(P)} \le \frac{d(P')}{b(P')} \le \max\left\{\frac{|P'|}{8} + \frac{1}{2}, 2\right\} \le \max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}.$$

Case 3. Not Case 1 and not Case 2. Then

$$d(P) = |A| \le |P| - 2(r(P) - 1) - 2 = |P| - 2(r(P) + 1) + 2.$$

Obviously,

$$|P| \le b(r(P) + 1), \text{ i.e., } r(P) + 1 \ge \frac{|P|}{b}.$$

Hence

$$d(P) \le |P| - 2\frac{|P|}{b} + 2 = |P|\frac{b-2}{b} + 2$$

and consequently (since $\frac{b-2}{b^2}$ attains its maximum at $b = 4$)

$$\frac{d(P)}{b(P)} \leq \frac{b-2}{b^2}|P| + \frac{2}{b} \leq \max\left\{\frac{|P|}{8} + \frac{1}{2}, 2\right\}.$$

■

3 Estimation of the size of the largest antichain in Pi_n

Let $Pi_{2,n}$ be the set of all unordered partitions of n into parts which are all greater than 1.

Theorem 8 *We have*

$$d(Pi_n) \leq |Pi_{2,n}|.$$

Proof. Let $\varphi : \ Pi_n \setminus Pi_{2,n} \to Pi_n$ be the mapping that assigns to the partition p (having a summand 1) the partition p' that can be obtained from p by combining a summand 1 and the largest summand of p. Clearly, for all $p \in Pi_n \setminus Pi_{2,n}$

$$p \lessdot \varphi(p).$$

The mapping φ is injective since p can be recovered from $\varphi(p)$ (partition the largest summand s of $\varphi(p)$ into $(s-1)+1$). Let $l(p)$ be the first natural number for which $\varphi^{l(p)}(p) \in Pi_{2,n}$. In addition, let for $p \in Pi_{2,n}$, $\varphi^0(p) := p$. If p and q are incomparable elements in Pi_n, then

$$\varphi^{l(p)}(p) \neq \varphi^{l(q)}(q)$$

since otherwise (say for $l(p) \geq l(q)$) by the injectivity of φ

$$\varphi^{l(p)-l(q)}(p) = q,$$

i.e., $p \leq q$. Hence, for any antichain A in Pi_n,

$$|A| = |\{\varphi^{l(p)}(p) : p \in A\}| \leq |Pi_{2,n}|.$$

■

Theorem 9 *We have*

$$|Pi_{2,n}| \sim \frac{\pi}{\sqrt{6}} \frac{|Pi_n|}{\sqrt{n}} \ \textit{as } n \to \infty.$$

Note that Theorem 4 follows from Theorems 5, 8, and 9. Thus it remains to prove Theorems 5 and 9. We will prove them almost simultaneously. Let $P(n,k)$ (resp. $p(n,k)$) be the number of partitions of n into k or fewer (resp. into exactly k) parts and let $p(n) := P(n,n) = |Pi_n|$. We need the following result of Szekeres [10, 11] which was reproved in [3] with a new recursion method in a more or less elementary way:

Theorem 10 *Let $\epsilon > 0$ be given. Then, uniformly for $k \geq n^{1/6}$*

$$P(n,k) = \frac{f(u)}{n} e^{\sqrt{n}g(u) + O(n^{-1/6+\epsilon})}.$$

Here, $u = k/\sqrt{n}$, and the functions $f(u), g(u)$ are:

$$f(u) = \frac{v}{\sqrt{8\pi u}} \left(1 - e^{-v} - \frac{1}{2}u^2 e^{-v}\right)^{-1/2}, \tag{1}$$

$$g(u) = \frac{2v}{u} - u\log(1 - e^{-v}), \tag{2}$$

where $v(= v(u))$ is determined implicitly by

$$u^2 = v^2 \Big/ \int_0^v \frac{t}{e^t - 1}\, dt. \tag{3}$$

With standard calculus one may verify that the RHS of (3), and thus also u is an increasing (continuous) function of v, hence the inverse function exists. We know from [3] (using $(e^t - 1)^{-1} = \sum_{m=1}^{\infty} e^{-mt}$ and $\sum_{m=1}^{\infty} m^{-2} = \pi^2/6$) that, with $C := \frac{\pi}{\sqrt{6}}$,

$$\int_0^{\infty} \frac{t}{e^t - 1}\, dt = C^2 \tag{4}$$

which implies that with u also v tends to infinity (and vice versa) and that

$$\lim_{u \to \infty} \frac{v}{u} = C. \tag{5}$$

Lemma 1 *We have for $u \to \infty$ (or $v \to \infty$)*

$$\frac{v}{u} = C - \frac{v+1}{2C} e^{-v} + O(v^2 e^{-2v}).$$

Proof. It is easy to verify that for $t \geq 1$

$$te^{-t} \leq \frac{t}{e^t - 1} \leq te^{-t} + 2te^{-2t}.$$

Taking the integral from $v \geq 1$ to infinity yields

$$(v+1)e^{-v} \leq \int_v^\infty \frac{t}{e^t - 1}\, dt = C^2 - \left(\frac{v}{u}\right)^2 \leq (v+1)e^{-v} + \frac{e^{-2v}(2v+1)}{2},$$

and hence

$$\left(\frac{v}{u}\right)^2 = C^2 - (v+1)e^{-v} + O(ve^{-2v}).$$

Consequently,

$$\begin{aligned}\frac{v}{u} &= C\left(1 - \frac{v+1}{C^2}e^{-v} + O(ve^{-2v})\right)^{1/2} \\ &= C - \frac{v+1}{2C}e^{-v} + O(v^2 e^{-2v}).\end{aligned}$$

■

Lemma 2 *We have for $u \to \infty$ (or $v \to \infty$)*

$$g(u) = 2C - \frac{1}{C}e^{-v} + O(v^2 e^{-2v}).$$

Proof. We have

$$-u \log(1 - e^{-v}) = ue^{-v} + O(ue^{-2v}),$$

and consequently by (2) and Lemma 1

$$g(u) = 2C - \frac{v+1}{C}e^{-v} + O(v^2 e^{-2v}) + ue^{-v} + O(ue^{-2v}).$$

Moreover, by Lemma 1

$$v = Cu + O(uve^{-v}).$$

Hence

$$\frac{v}{C}e^{-v} = ue^{-v} + O(v^2 e^{-2v}),$$

and finally

$$g(u) = 2C - \frac{1}{C}e^{-v} + O(v^2 e^{-2v}).$$

■

Lemma 3 *Let $0 < \delta < \frac{1}{4C}$ and $I = [(\frac{1}{2C} - \delta)\sqrt{n}\log n, (\frac{1}{2C} + \delta)\sqrt{n}\log n]$. Then, uniformly for $k \in I$ as $n \to \infty$*

$$
\begin{aligned}
P(n,k) &\sim p(n) e^{-\frac{\sqrt{n}}{C} e^{-Cu}}, \\
p(n,k) &\sim p(n) e^{-Cu - \frac{\sqrt{n}}{C} e^{-Cu}}, \\
p(n-k,k) &\sim p(n) e^{-2Cu - \frac{\sqrt{n}}{C} e^{-Cu}}.
\end{aligned}
$$

Here $u := k/\sqrt{n}$.

Proof. Obviously (subtract from each part a one)

$$
p(n,k) = P(n-k,k), \tag{6}
$$
$$
p(n-k,k) = P(n-2k,k). \tag{7}
$$

All the following estimates are uniform for $k \in I$ and taken for $n \to \infty$. Let $i \in \{0,1,2\}$. Let $u_i := k/\sqrt{n-ik}$. Since $u_i \to \infty$ we have

$$
f(u_i) \sim \frac{C}{\sqrt{8}\pi}.
$$

Moreover, by Theorem 10

$$
P(n-ik,k) \sim \frac{C}{\sqrt{8}\pi n} e^{\sqrt{n-ik}\, g(u_i)}. \tag{8}
$$

We have

$$
\sqrt{n-ik} = \sqrt{n}\left(1 - \frac{ik}{n}\right)^{1/2} = \sqrt{n} - \frac{iu}{2} + o(1), \tag{9}
$$
$$
u_i = u\left(1 - \frac{ik}{n}\right)^{-1/2} = u + O(\log^2 n/\sqrt{n}). \tag{10}
$$

Let $\delta < \delta_1 < \delta_2 < \frac{1}{4C}$. Then, for large n,

$$
\left(\frac{1}{2C} - \delta_1\right)\log n < u_i < \left(\frac{1}{2C} + \delta_1\right)\log n.
$$

Let $v_i := v(u_i)$. From (5) it follows

$$
\left(\frac{1}{2} - C\delta_2\right)\log n < v_i < \left(\frac{1}{2} + C\delta_2\right)\log n.
$$

Consequently,

$$
e^{-v_i} < \frac{1}{n^{1/2 - C\delta_2}}.
$$

From Lemma 1 we obtain (noting (10))

$$v_i = Cu_i + O\left(\frac{\log^2 n}{n^{1/2-C\delta_2}}\right) = Cu + O\left(\frac{\log^2 n}{n^{1/2-C\delta_2}}\right),$$
$$e^{-v_i} = e^{-Cu}\left(1 + O\left(\frac{\log^2 n}{n^{1/2-C\delta_2}}\right)\right).$$

Obviously,

$$e^{-Cu} = O\left(\frac{1}{n^{1/2-C\delta}}\right),$$

and thus

$$e^{-v_i} = e^{-Cu} + O\left(\frac{\log^2 n}{n^{1-C(\delta+\delta_2)}}\right) = e^{-Cu} + o(1/\sqrt{n}).$$

Lemma 2 yields

$$g(u_i) = 2C - \frac{1}{C}e^{-Cu} + o(1/\sqrt{n}),$$

and from (9) we derive

$$\sqrt{n - ik}g(u_i) = \left(\sqrt{n} - \frac{iu}{2}\right)\left(2C - \frac{1}{C}e^{-Cu}\right) + o(1).$$

Note that by the Hardy–Ramanujan formula [7] (put in Theorem 10 $u := \sqrt{n}$)

$$p(n) \sim \frac{C}{\sqrt{8}\pi n}e^{\sqrt{n}2C}. \tag{11}$$

Now we obtain from (8) – (11)

$$P(n - ik, k) \sim p(n)e^{-iCu - \frac{\sqrt{n}}{C}e^{-Cu} + o(1)},$$

and the assertion follows from (6) and (7). ■

In the following let only $i \in \{1, 2\}$. Note that

$$U_i := \frac{1}{2C}\log n - \frac{1}{C}\log iC$$

is the unique point at which the function

$$h_i(u) := -iCu - \frac{\sqrt{n}}{C}e^{-Cu}$$

achieves its maximum. For $u = U_i + t$ we have

$$e^{h_i(u)} = \frac{(iC)^i}{n^{i/2}} e^{-iCt - ie^{-Ct}}. \tag{12}$$

Let $0 < \delta < \frac{1}{4C}$ and let $\underline{U}_i := U_i - \delta \log n, \overline{U}_i := U_i + \delta \log n$. Further let $\underline{k}_i := \lfloor \underline{U}_i \sqrt{n} \rfloor, \overline{k}_i := \lfloor \overline{U}_i \sqrt{n} \rfloor, k_i^* = \lfloor U_i \sqrt{n} \rfloor$ and $\underline{u}_i := \underline{k}_i / \sqrt{n}, \underline{v}_i := v(\underline{u}_i)$.

Lemma 4 *We have for* $i \in \{1, 2\}$

$$P(n, \underline{k}_i) = o(p(n)/\sqrt{n}).$$

Proof. Since

$$\underline{u}_i = \left(\frac{1}{2C} - \delta\right) \log n + O(1),$$

we have

$$e^{-\frac{\sqrt{n}}{C} e^{-C\underline{u}_i}} = e^{-n^{\delta C} e^{O(1)}/C} = o(1/\sqrt{n}).$$

The assertion follows from Lemma 3. ∎

Lemma 5 *We have for* $i \in \{1, 2\}$

$$p(n - \overline{k}_i) = o(p(n)/\sqrt{n}).$$

Proof. Let $0 < \delta_1 < \delta$. Then, for large n,

$$n - \overline{k}_1 \leq n - \left(\frac{1}{2C} + \delta_1\right) \sqrt{n} \log n,$$

$$\sqrt{n - \overline{k}_1} \leq \sqrt{n} \left(1 - \left(\frac{1}{2C} + \delta_1\right) \frac{\log n}{\sqrt{n}}\right)^{1/2} = \sqrt{n} - \left(\frac{1}{4C} + \frac{\delta_1}{2}\right) \log n + o(1).$$

From (11) we derive

$$p(n - \overline{k}_i) \lesssim p(n) e^{2C\left(-\left(\frac{1}{4C} + \frac{\delta_1}{2}\right) \log n\right)} = \frac{p(n)}{\sqrt{n}} n^{-C\delta_1} = o(p(n)/\sqrt{n}).$$

∎

Proof of Theorem 5. By Lemma 3 and (12) (note $t = o(1)$)

$$p(n, k_1^*) \sim \frac{C}{e\sqrt{n}} p(n).$$

Because $h_1(U_1)$ is the maximum of $h_1(u)$ and again in view of Lemma 3 we have for $k \in [\underline{k}_1 + 1, \overline{k}_1 - 1]$

$$p(n, k) \lesssim p(n, k_1^*).$$

For $k \le \underline{k}_1$ Lemma 4 implies, for large n,

$$p(n,k) \le P(n,\underline{k}_1) = o(p(n)/\sqrt{n}) < p(n,k_1^*).$$

For $k \ge \overline{k}_1$ we have by Lemma 5, for large n,

$$p(n,k) = P(n-k,k) \le p(n-\overline{k}_1) = o(p(n)/\sqrt{n}) < p(n,k_1^*).$$

■

Proof of Theorem 9. Obviously (subtract from each part of a member of $Pi_{2,n}$ a one)

$$|Pi_{2,n}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k). \tag{13}$$

We divide the sum into 3 parts:

$$\sum = \sum_{k=1}^{\underline{k}_2} + \sum_{k=\underline{k}_2+1}^{\overline{k}_2-1} + \sum_{k=\overline{k}_2}^{\lfloor \frac{n}{2} \rfloor}.$$

By Lemma 3 and (12)

$$\sum_{k=\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) \sim \frac{4C^2}{n} p(n) \sum_{k=\underline{k}_2+1}^{\overline{k}_2-1} e^{-2C(k/\sqrt{n}-U_2)-2e^{-C(k/\sqrt{n}-U_2)}}.$$

The sum on the RHS can be considered as an integral approximation with step size $n^{-1/2}$. Since $\underline{k}_2 \to -\infty$ and $\overline{k}_2 \to \infty$ this sum multiplied by $\sqrt{n}$ converges for $n \to \infty$ to

$$\int_{-\infty}^{\infty} e^{-2Ct-2e^{-Ct}}\, dt = \frac{1}{4C}\left(2e^{-2e^{-Ct}-Ct} + e^{-2e^{-Ct}}\right)\Big|_{-\infty}^{\infty} = \frac{1}{4C}.$$

Consequently,

$$\sum_{k=\underline{k}_2+1}^{\overline{k}_2-1} p(n-k,k) \sim \frac{C}{\sqrt{n}} p(n). \tag{14}$$

Moreover, by Lemma 4

$$\sum_{k=1}^{\underline{k}_2} p(n-k,k) \le P(n,\underline{k}_2) = o(p(n)/\sqrt{n}). \tag{15}$$

Finally, by Lemma 5

$$\sum_{k=\overline{k}_2}^{\lfloor \frac{n}{2} \rfloor} p(n-k,k) \le p(n-\overline{k}_2) = o(p(n)/\sqrt{n}). \tag{16}$$

With (13)–(16) the assertion is proved. ■

4 The proof of the incidence matrix result

We represent the elements of Pi_n as n-tuples of natural numbers $\boldsymbol{a} = (a_1, \ldots, a_n)$ where $\sum_{i=1}^{n} ia_i = n$ (a_i counts the number of summands i). We have $\boldsymbol{a} \lessdot \boldsymbol{b}$ iff there are $i, j \in [n]$ such that $b_{i+j} = a_{i+j} + 1$ as well as $b_i = a_i - 1, b_j = a_j - 1$ if $i \neq j$ and $b_i = a_i - 2$ if $i = j$. The kth level of Pi_n is given by

$$N_k = \{\boldsymbol{a} \in Pi_n : \ a_1 + \cdots + a_n = n - k\}, \ k = 0, \ldots, n-1.$$

Proof of Theorem 7. First note that for $\boldsymbol{a} \in N_k$ with $k < \frac{n-1}{2}$ necessarily $a_1 \geq 2$. Indeed:

$$\begin{aligned} n = a_1 + 2a_2 + \cdots + na_n \geq a_1 + 2(a_2 + \cdots + a_n) \geq 2(a_1 + \cdots + a_n) - a_1 \\ n \geq 2(n-k) - a_1 \\ a_1 \geq n - 2k > 1. \end{aligned}$$

Now order the elements of N_k lexicographically: Let for $\boldsymbol{a}, \boldsymbol{b} \in N_k, \boldsymbol{a} \prec \boldsymbol{b}$ if $a_i > b_i$ for the smallest index i for which $a_i \neq b_i$. Define $\psi : N_k \to N_{k+1}, k < \frac{n-1}{2}$, by

$$\psi(\boldsymbol{a}) := (a_1 - 2, a_2 + 1, \ldots, a_n).$$

In contrast to the proof of Theorem 8 we do not combine here one summand 1 and the largest summand, but two summands 1. Obviously, $\boldsymbol{a} \lessdot \psi(\boldsymbol{a})$ for every $\boldsymbol{a}$, and ψ is injective. Moreover, if $\boldsymbol{a} \prec \boldsymbol{b}$ then $\psi(\boldsymbol{a}) \prec \psi(\boldsymbol{b})$. Let $S := \{\psi(\boldsymbol{a}) : \boldsymbol{a} \in N_k\}$ and consider the minor A of M_k which is determined by all rows of M_k and those columns of M_k which are indexed by elements of S. Here we suppose that the rows and columns are ordered w.r.t. $\prec$. From above we know that A is square and that the diagonal elements of A are equal to 1. It is enough to show that A is lower triangular. Assume that there are elements $\boldsymbol{a}, \boldsymbol{b} \in N_k$ with $\boldsymbol{a} \prec \boldsymbol{b}$ and $\boldsymbol{a} \lessdot \psi(\boldsymbol{b})$. It is easy to see that $\psi(\boldsymbol{a})$ is the greatest element w.r.t. $\prec$ which covers $\boldsymbol{a}$ (for all other such elements the first coordinate is greater since at most one 1 is combined with another summand). Consequently,

$$\psi(\boldsymbol{b}) \prec \psi(\boldsymbol{a}) \prec \psi(\boldsymbol{b}),$$

a contradiction. ∎

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