The Numbers of Nonisomorphic Drawings of the Connected Graphs with 6 vertices

Volker Leck
Universität Rostock, Germany

February 5, 1998

Abstract

We determine all nonisomorphic drawings of the connected graphs with 6 vertices. Beside the numbers this also gives new information about the numbers of crossings that are realizable by a drawing of such a graph.

Introduction

Many properties of drawings of graphs in the plane have been investigated (see [5]). Several papers deal with extremal drawings, e.g. with those whose number of crossings is maximal or minimal [3],[8]. But not much is known about the number of drawings a given graph has. In [1] a first step was done in determining the nonisomorphic drawings of all graphs with at most 5 vertices. Also the numbers of such drawings for first infinite classes of graphs were given. In this paper we extend the known drawings to the connected graphs with 6 vertices, where only few were previously known [2],[7].

A drawing $D(G)$ of a graph G is a realization of G in the plane such that the vertices are mapped onto different points (called vertices of $D(G)$) and the edges are mapped onto Jordan curves or lines between their endpoints (called edges of $D(G)$). We only consider good drawings, e.g. no two edges cross more than once, no edge crosses itself, no two incident edges cross, no more than two edges cross in one point and no edge meets a vertex which is not one of its endpoints.

Two drawings of G are *isomorphic* if and only if there is an incidence preserving one-to-one correspondence between their vertices, edges, parts of edges and regions.

Algorithm

We now present and algorithm to derive all nonisomorphic drawings of a graph G from all drawings of a graph $G-e$ which arises from G by deleting an arbitrary edge e. The idea of this method comes from [1] where the authors checked all the graphs by hand. Here we automate this procedure. But we still check all possibilities, so the method is costly and slow. Nevertheless we do not know how to do better.

The algorithm splits into two parts. Firstly, all drawings of G are constructed by adding edge e to each drawing of $G-e$ in every possible way. Secondly, a standard

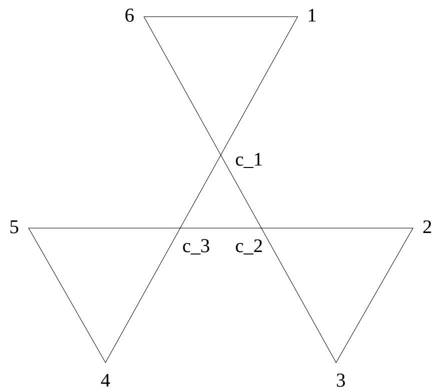

Figure 1: A Drawing of the 6–Cycle

form is made up for every derived drawing in such a way that two drawings have the same standard form if and only if they are isomorphic.

Consider a drawing $D(G)$ of a graph G with n vertices and k crossings. Let these vertices be labelled with $\{1,\ldots,n\}$ and the crossings with $\{c_1,\ldots,c_k\}$. Choose a direction, either from left to right or from right to left. Now $D(G)$ can be described by its regions, starting with an arbitrary vertex of each region (or with a crossing if there is no vertex) and walking along the edges or parts of edges in the chosen direction until returning to the starting point. The drawing in figure 1 for example could be described by $16c_1c_354c_3c_232c_2c_1$, $1c_16$, $23c_2$, $45c_3$, $c_1c_2c_3$.

Suppose we want to insert a new edge ab to a given drawing $D(G)$ (with $a, b \in V(G)$) in all possible ways. This can be done recursively by the function `new_edge`. Given the edges that are not allowed to be cut or are already cut by ab (`forbidden`) the function connects a and b if b is in the given region (this gives a new drawing), cuts the next possible edge in the given region and jumps into the opposite region.

```
function new_edge(drawing D, forbidden, region, cross_number)
BEGIN
  IF b is in region BEGIN
    D_new := connect c(cross_number) and b in region
    output D_new
  END
  FORALL edge-segments e of region DO BEGIN
    IF e is not forbidden BEGIN
     D_new := cut e of region
     new_edge(D_new,forbidden+edge(e),region behind e,cross_number+1)
    END
  END
END
```

Starting `new_edge` with `crossing_number` $:= 0$ and $c_0 := a$ for all regions which include a gives us all drawings of $D(G+ab)$ which contain $D(G)$. On the other hand if we delete edge ab in a drawing of $G+ab$ we get a drawing of G. Hence, if we apply `new_edge` to each drawing of G we derive all drawings of $G+ab$. Of course we derive many of them more than once. To exclude isomorphisms we transform them into standard form.

Fix a drawing. Every crossing can be decribed as an unordered pair of edges. So every one-to-one correspondence $w : \binom{E}{2} \longleftrightarrow \left\{1, \dots \binom{|E|}{2}\right\}$ gives an ordering of the crossings. Let be given a vertex labelling and a correspondence w. Choose an arbitrary direction and describe the regions as above. Rotate each region until the greatest vertex is in first position. If there is no vertex take the greatest crossing (with respect to w). Setting the crossings to zero you get a $(n+1)$–adic number for every region. Sort the regions with respect to this value. If two regions have the same value consider the crossings. Regions which only consist of crossings (say m of them) get the value $(n+1)^{m-1} - 1$. The same procedure using the opposite direction gives a second ordering of the regions. The greater one is called *maximal*. Now the greatest maximal ordering of all labellings is the unique *standard form* of the drawing (with respect to w).

```
function standard_form
BEGIN
  FORALL vertex labellings DO BEGIN
    compute maximal ordering ℓ
    ℓ_max  :=  max {ℓ_max, ℓ}
  END
  output ℓ_max
END
```

The example in figure 1 has the standard form $65c_1c_243c_2c_321c_3c_1$, $6c_15$, $4c_23$, $2c_31$, $c_1c_3c_2$, assuming that $w(c_1) > w(c_2) > w(c_3)$.

Clearly, if two drawings are isomorphic then they have the same standard form since there is an isomorphism mapping one drawing onto the other. But if two drawings have the same standard form this gives an isomorphism, namely mapping the regions with equal ordering onto eachother. Hence, two drawings are isomorphic if and only if they have the same standard form.

Results

We implemented the described algorithm. The 112 connected graphs with 6 vertices given in figure 2 were generated by `makeg`, part of *nauty* [6]. Enumerating them from left to right row-wise the drawing number of graph G and the numbers for a fixed number of crossings are given in the following table.

The program confirms the results for small graphs from [1] and the already known numbers $C(K_{3,3}) = 102$, $C(C_6) = 114$ and $C(K_6) = 121$. It also finds 555 drawings of the octahedron, in agreement with an unpublished result of van Haandel [4]. For the path on 6 vertices it only finds 182 drawings, in contradiction to [2], where 185 is given. But it is already known [4], that 182 is the right number.

Concluding we remark that this method is already for these small graphs very lengthy. So it can only be a first approach and seems not to be the right way to treat bigger graphs.

References

[1] H.D.O.–F. Gronau and H. Harborth *Numbers of Nonisomorphic Drawings for small Graphs*, Congressus Numerantium **71** (1990), 105–114.

[2] H. Harborth *Drawings of the Cycle Graph*, Congressus Numerantium **66** (1988), 15–22.

[3] H. Harborth *Maximum Number of Crossings for the Cube Graph*, Congressus Numerantium **82** (1991), 117–122.

[4] H. Harborth private communication.

[5] N. Hartsfield and R. Ringel *Pearls in Graph Theory*, Academic Press, Inc. (1990).

[6] B.D. McKay *Nauty User's Guide*, Australian National University, Comp.Sc.Techn.Rep TR-CS-84-05 (1984).

[7] I. Mengersen *Darstellungen des vollständigen Graphen ohne kreuzungsfreie Kanten*, Diplomarbeit, TU Braunschweig, 1973.

[8] R.D. Ringeisen *Two old Extremal Graph Drawing Conjectures: Progress and Perspectives*, Congressus Numerantium **115** (1996), 91–103.

G	N	Number of Crossings															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1															
2	14	1	3	6	4												
3	14	2	4	5	3												
4	93	1	5	20	33	26	8										
5	21	1	3	7	6	4											
6	81	2	6	19	26	20	8										
7	120	3	15	33	36	25	8										
8	101	4	14	27	29	19	8										
9	116	2	6	18	31	35	17	7									
10	129	1	4	16	33	39	25	11									
11	142	4	15	29	36	34	17	7									
12	174	4	18	33	45	38	25	11									
13	40	1	1	4	6	12	9	7									
14	45	1	3	6	8	11	9	7									
15	117	1	6	19	41	38	12										
16	208	2	11	30	61	62	35	7									
17	88	2	5	14	23	24	13	7									
18	272	2	12	36	66	81	56	19									
19	349	1	8	29	66	87	86	51	17	4							
20	73	2	7	11	21	19	9	4									
21	960	3	27	88	185	234	222	144	43	14							
22	764	6	33	95	169	206	159	73	23								
23	666	5	26	71	130	153	141	93	37	10							
24	182	1	4	20	44	60	41	12									
25	373	1	5	25	67	112	104	46	13								
26	371	2	8	20	59	87	106	59	25	5							
27	560	4	20	66	120	161	127	50	12								
28	121	1	4	14	29	35	26	10	2								
29	348	1	5	17	43	70	107	52	44	6	3						
30	557	2	10	37	87	145	145	96	27	8							
31	514	2	10	37	84	119	126	85	36	15							
32	470	2	8	26	61	94	119	80	52	25	3						

continued on next page

G	N	Number of Crossings															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
33	170	3	8	25	40	47	29	14	4								
34	82	2	6	14	19	21	12	6	2								
35	429	1	4	19	51	84	95	88	54	23	7	3					
36	1205	1	9	42	116	208	287	260	176	72	26	8					
37	432	3	11	29	62	101	104	79	33	10							
38	1579	6	37	117	240	345	352	282	129	57	14						
39	561	2	10	29	70	102	120	121	65	28	10	4					
40	1515	3	20	65	164	260	320	317	213	107	36	10					
41	916	2	16	48	126	185	221	179	88	39	12						
42	342	4	17	39	60	80	74	41	21	6							
43	1151	4	27	84	172	232	247	211	104	54	16						
44	509	1	4	14	35	69	104	129	79	48	17	9					
45	616	2	9	23	56	86	113	144	90	58	25	10					
46	308	1	5	20	48	82	86	50	16								
47	991	2	14	51	131	200	249	181	112	36	15						
48	372	1	7	18	52	64	86	69	46	20	9						
49	114	1	2	6	17	22	20	29	11	3	3						
50	475	1	6	18	54	88	117	99	71	12	9						
51	669	2	12	40	93	151	176	110	63	16	6						
52	886	1	8	30	78	147	175	202	145	67	20	13					
53	260	3	8	23	42	65	64	35	16	4							
54	1200	2	19	68	169	266	309	221	100	36	10						
55	2729	2	20	80	208	403	594	587	467	241	84	33	10				
56	2085	4	25	93	220	381	501	420	276	111	45	9					
57	1311	2	11	43	104	197	251	258	240	121	51	26	7				
58	1950	3	15	58	133	254	389	378	337	231	94	40	18				
59	426	1	3	8	36	53	118	70	110	10	17						
60	606	1	6	19	54	102	127	164	74	40	13	6					
61	2581	1	8	33	121	262	435	563	512	359	171	80	22	14			
62	1144	1	7	23	69	121	243	220	241	128	65	14	12				
63	3400	1	13	51	154	315	533	662	657	511	297	130	54	22			
64	1678	2	17	50	154	258	338	396	243	139	57	24					
65	1462	4	21	69	157	254	328	274	217	83	45	10					
66	1123	1	14	34	118	163	220	235	175	92	52	19					
67	4330	2	22	92	266	537	783	884	808	504	272	107	40	13			
68	622	2	6	22	49	88	114	113	105	72	24	20	3	4			
69	2875	3	18	69	173	337	501	536	522	382	185	109	26	14			
70	2362	1	9	29	94	194	323	422	465	373	229	161	27	35			
71	102	0	1	0	9	0	33	0	48	0	11						
72	771	0	2	3	28	30	120	112	192	144	79	39	10	12			
73	1882	1	5	20	66	162	248	390	383	289	179	97	19	23			
74	1051	0	2	5	25	52	103	164	213	206	141	92	14	34			
75	469	0	1	3	11	24	41	62	92	86	60	63	0	26			
76	692	2	8	29	80	143	179	149	71	23	8						
77	231	1	1	10	12	45	35	65	31	21	5	5					
78	678	6	7	19	44	94	128	142	126	63	32	13	4				
79	386	1	5	13	34	65	91	93	51	24	6	3					
80	356	5	11	9	22	43	66	65	61	43	17	10	4				
81	836	2	5	23	39	109	132	177	162	101	46	23	13	4			
82	2582	1	7	40	96	231	384	454	539	406	212	127	49	30	6		
83	933	2	10	27	67	129	180	222	151	93	29	20	3				
84	3505	2	12	51	136	282	513	621	640	601	335	154	117	24	17		
85	1229	1	10	26	69	123	210	247	241	157	86	33	21	3	2		
86	3505	2	12	51	136	282	513	621	640	601	335	154	117	24	17		
87	595	1	1	8	15	33	67	94	107	117	60	50	26	4	12		
88	761	1	3	10	21	39	85	127	116	144	99	46	51	0	19		
89	1581	1	8	34	111	224	299	340	268	172	81	30	10	3			
90	873	1	6	18	53	104	140	176	168	113	52	27	12	3			
91	2736	1	9	37	117	261	393	501	530	400	275	123	48	37	2	2	
92	860	1	3	10	27	67	98	146	154	125	123	57	18	28	0	3	
93	579	1	2	10	18	54	68	108	109	89	51	38	19	4	7	0	1
94	4711	1	7	38	108	290	520	724	901	850	565	369	187	83	53	9	6
95	6537	1	9	43	123	317	619	911	1145	1156	975	612	309	196	75	38	8
96	3937	2	10	46	124	294	501	650	762	650	447	249	111	64	23	4	
97	4841	1	4	23	64	154	356	576	755	836	801	573	355	154	124	47	18
98	4064	1	11	39	135	296	511	710	821	640	477	246	83	77	9	8	
99	906	1	5	15	41	74	145	165	163	144	94	16	37	2	4		

continued on next page

G	N	Number of Crossings															
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
100	5553	1	11	48	146	335	631	875	998	975	735	406	226	101	53	12	
101	905	1	8	18	60	109	153	201	166	106	45	31	7				
102	321	0	3	3	26	33	52	65	62	37	22	14	4				
103	1386	0	3	6	29	66	122	189	249	239	196	159	58	45	16	9	
104	2464	0	1	5	17	49	113	215	336	396	417	393	234	145	64	53	26
105	3639	1	2	21	49	143	296	451	634	675	536	414	208	94	88	15	12
106	1066	0	3	5	24	60	90	169	206	188	140	114	18	42	0	7	
107	1910	0	3	6	38	63	172	232	337	375	296	177	111	57	25	12	6
108	5359	0	3	12	53	140	316	552	792	980	929	702	408	262	104	72	34
109	555	1	0	3	6	13	30	47	80	94	81	90	50	18	29	0	13
110	2140	0	1	4	15	30	85	157	249	342	377	320	264	138	71	37	50
111	1022	0	0	1	5	9	25	52	100	143	166	159	162	102	27	24	47
112	121	0	0	0	1	1	3	3	9	14	19	12	25	19	0	0	15

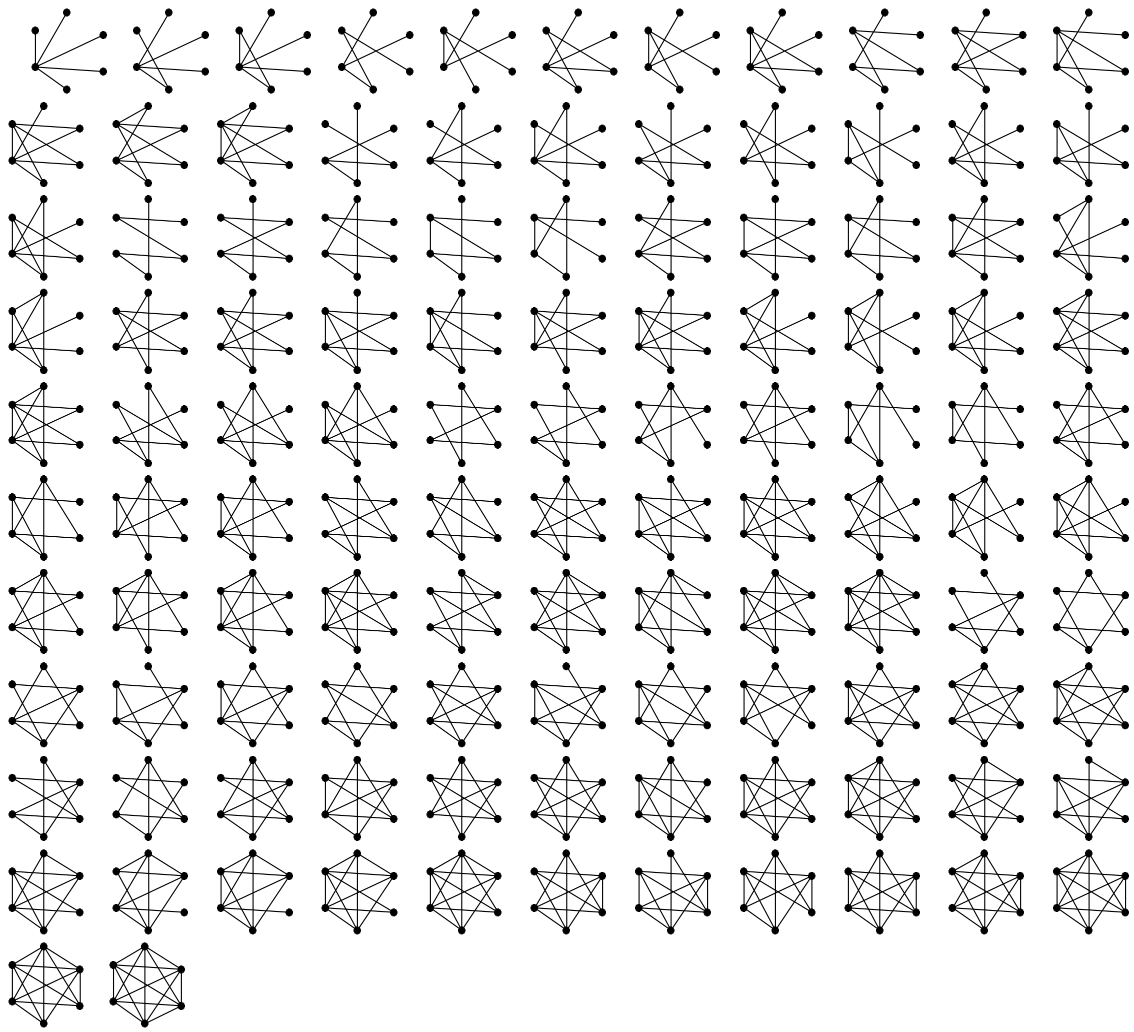

Figure 2: the connected graphs with 6 vertices