A Tool for Approximation in Bivariate Periodic Sobolev Spaces

Frauke Sprengel

Abstract. We give a characterization of Sobolev spaces of bivariate periodic functions with dominating smoothness properties in terms of Sobolev spaces of univariate functions. The mixed Sobolev norm is proved to be a uniform crossnorm. This property can be used as a powerful tool in approximation theory.

§1. Introduction

Beside the approximation of functions from the usual isotropic periodic Sobolev spaces the approximation of bivariate periodic functions with dominating mixed smoothness properties attracted more and more attention. Spaces of functions with dominating mixed smoothness properties are especially well suited for the approximation or numerical integration using sparse grids or a hyperbolic approximation approach (e.g. [1,8]).

In this paper, we prove that the Sobolev spaces of bivariate periodic functions with dominating mixed smoothness properties are tensor products of Sobolev spaces of univariate periodic functions. Additionally, the corresponding mixed Sobolev norms turn out to be uniform cross norms.

In particular, that means that one can derive error estimates for functions from these spaces from error estimates for the univariate functions in the same simple way as in the Hilbert space case (see e.g. [5]).

§2. Periodic Sobolev Spaces

We recall the definitions of the periodic Sobolev spaces first. Let $\mathbb{T}^d$ denote the d-dimensional torus represented by the cube

$$\mathbb{T}^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \; ; \; |x_k| \le \pi, \; k = 1, \ldots, d\}.$$

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By $D'(\mathbb{T}^d)$ we denote the linear space of tempered periodic distributions (or generalized periodic functions) (see [3,6]). The Fourier coefficients of a distribution $g \in D'(\mathbb{T}^d)$ are $c_k(g) := g(e^{-ik\cdot})$ for $k \in \mathbb{Z}^d$.

Let $1 < p < \infty$ and $\alpha, \beta \in \mathbb{R}$. The Sobolev space $H_p^\alpha(\mathbb{T}^d)$ of order α is defined as

$$H_p^\alpha(\mathbb{T}^d) := \Big\{ f \in D'(\mathbb{T}^d) \; ; \; \|f \mid H_p^\alpha(\mathbb{T}^d)\| := \Big\| \sum_{k \in \mathbb{Z}^d} (1 + |k|_2^2)^{\alpha/2} c_k(f) e^{ik\cdot} \;\; \Big| \;\; L_p(\mathbb{T}^d) \Big\| < \infty \Big\}.$$

The Sobolev space $S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2)$ of order (α, β) of bivariate generalized periodic functions with dominating mixed smoothness properties is given by

$$S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2) := \Big\{ f \in D'(\mathbb{T}^2) \; ; \; \|f \mid S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2)\| := \Big\| \sum_{k \in \mathbb{Z}^2} (1 + k_1^2)^{\alpha/2} (1 + k_2^2)^{\beta/2} c_k(f) e^{ik\cdot} \;\; \Big| \;\; L_p(\mathbb{T}^2) \Big\| < \infty \Big\}.$$

§3. Tensor Products of Reflexive Banach Spaces

Because the Sobolev spaces considered here are reflexive, we restrict our following introduction to tensor products to this case. In [4], one can find a general introduction.

Given two reflexive Banach spaces X, Y. We form the linear space of formal linear combinations

$$F(X, Y) := \Big\{ \sum_{k=1}^{r} \alpha_k \, (x_k, y_k) \; ; \; \alpha_k \in \mathbb{C}, \; x_k \in X, \; y_k \in Y, \; k = 1, \ldots, r, \; r \in \mathbb{N} \Big\}$$

of pairs $(x_k, y_k) \in X \times Y$. Furthermore, we denote by G the subspace $F(X, Y)$ spanned by elements of the form

$$\sum_{k=1}^{r} \sum_{\ell=1}^{s} \alpha_k \, \beta_\ell \, (x_k, y_\ell) - \Big(\sum_{k=1}^{r} \alpha_k \, x_k \, , \; \sum_{\ell=1}^{s} \beta_\ell \, y_\ell \Big).$$

Then, the algebraic tensor product is the quotient space

$$X \otimes Y := F(X, Y)/G.$$

The equivalence class generated by (x, y) is denoted by $x \otimes y$. In general, the algebraic tensor product $X \otimes Y$ equipped with a suitable norm is not complete. We denote the completion of $X \otimes Y$ with respect to the norm λ by $X \otimes_\lambda Y$. Two Banach spaces will be identified if they are isomorphic to each other. A norm λ on $X \otimes Y$ with

$$\lambda(x \otimes y) = \|x \mid X\| \cdot \|y \mid Y\|$$

for all $x \otimes y \in X \otimes Y$, is termed a *crossnorm.*

A pair of crossnorms λ_1 on $X_1 \otimes Y_1$ and λ_2 on $X_2 \otimes Y_2$ is *uniform*, if for all operators $P \in \mathcal{L}(X_1, X_2)$ and $Q \in \mathcal{L}(Y_1, Y_2)$ and all $x_k \in X_1,\ y_k \in Y_1$, it holds that

$$\lambda_2\Big(\sum_{k=1}^r Px_k \otimes Qy_k\Big) \leq \ \|P \mid \mathcal{L}(X_1,X_2)\| \cdot \|Q \mid \mathcal{L}(Y_1,Y_2)\| \ \lambda_1\Big(\sum_{k=1}^r x_k \otimes y_k\Big).$$

In the case of uniform crossnorms the tensor product $P \otimes Q$ has a unique continuous extension $P\hat{\otimes}Q$ with

$$\|P\hat{\otimes}Q \mid \mathcal{L}(X_1 \otimes_{\lambda_1} Y_1, X_2 \otimes_{\lambda_2} Y_2)\| = \|P \mid \mathcal{L}(X_1,X_2)\| \cdot \|Q \mid \mathcal{L}(Y_1,Y_2)\|.$$

Using this property of uniform crossnorms, one is able to obtain results for the tensor product spaces directly from the results for the underlying single spaces. An example for spaces with uniform crossnorms is a pair of tensor product spaces both equipped with the p-nuclear norm (see [7]).

For $1 \leq p \leq \infty$, $1/p + 1/q = 1$, the *p-nuclear norm* of $z \in X \otimes Y$ is given by

$$\tilde{\lambda}_p(z) := \inf\Big\{ \Big(\sum_{j=1}^n \|x_j \mid X\|^p\Big)^{1/p} \theta_q(y_1,\ldots,y_n)\ ;\ z = \sum_{j=1}^n x_j \otimes y_j \Big\},$$

where

$$\theta_q(y_1,\ldots,y_n) := \sup\Big\{ \Big\|\sum_{j=1}^n \eta_j\, y_j \ \Big|\ Y\Big\|\ ;\ \|\{\eta_j\}_{j=1,\ldots,n} \mid \ell_p(\{1,\ldots,n\})\| = 1\Big\}.$$

The infimum is taken with respect to all representations of z.

§4. Tensor Products of Sobolev Spaces

After all these preparations we are able to characterize the Sobolev spaces of bivariate functions with dominating mixed smoothness properties.

Theorem 1. *Let $1 < p < \infty$, $\alpha, \beta \in \mathbb{R}$. Then it holds that*

$$S^{\alpha,\beta}_{p,p}H(\mathbb{T}^2) = H^\alpha_p(\mathbb{T}) \otimes_{\tilde{\lambda}_p} H^\beta_p(\mathbb{T})$$

with the p-nuclear norm $\tilde{\lambda}_p$ and

$$\tilde{\lambda}_p(f) = \|f \mid S^{\alpha,\beta}_{p,p}H(\mathbb{T}^2)\|$$

for all $f \in S^{\alpha,\beta}_{p,p}H(\mathbb{T}^2)$.

Since the p-nuclear norms are uniform, error estimates for functions with higher order dominating mixed smoothness in a mixed norm of lower orders can be derived in a simple manner from the error estimates for the univariate functions. This can be done for all approaches using tensor products of linear operators, for instance interpolation on full or on sparse grids.

The proof of the theorem requires the following two lemmata. Let here and in the sequel μ denote the Lebesgue measure.

Lemma 2. *Let $f_1,\dots,f_n \in H_p^\alpha(\mathbb{T})$, $1<p<\infty$, $\alpha\in\mathbb{R}$,*

$$\operatorname{supp}\Big(\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f_j)e^{ik\cdot}\Big) = E_j \subset \mathbb{T}$$

with $\mu(E_j\cap E_k)=0$ for $j\neq k$, then

$$\theta_q(f_1,\dots,f_n) = \max_j \|f_j \mid H_p^\alpha(\mathbb{T})\|.$$

Proof: In the following, we take the supremum over all $\|\eta \mid \ell_p\{1,\dots,n\}\| = 1$ and compute

$$\begin{aligned}
\theta_q(f_1,\dots,f_n) &= \sup_\eta\Big\{\Big\|\sum_{j=1}^n \eta_j f_j \;\Big|\; H_p^\alpha(\mathbb{T})\Big\|\Big\}\\
&= \sup_\eta\Big\{\int_{\mathbb{T}}\Big|\sum_{j=1}^n\eta_j\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f_j)e^{ikx}\Big|^p\,dx\Big\}^{1/p}\\
&= \sup_\eta\Big\{\sum_{\ell=1}^n\int_{E_\ell}\Big|\sum_{j=1}^n\eta_j\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f_j)e^{ikx}\Big|^p\,dx\Big\}^{1/p}\\
&= \sup_\eta\Big\{\sum_{j=1}^n\int_{E_j}\Big|\eta_j\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f_j)e^{ikx}\Big|^p\,dx\Big\}^{1/p}\\
&= \sup_\eta\Big\{\sum_{j=1}^n|\eta_j|^p\,\|f_j \mid H_p^\alpha(\mathbb{T})\|^p\Big\}^{1/p}\\
&= \sup_{\sum|\kappa_j|=1}\Big\{\sum_{j=1}^n|\kappa_j|\;\|f_j \mid H_p^\alpha(\mathbb{T})\|^p\Big\}^{1/p}\\
&= \max_j\|f_j \mid H_p^\alpha(\mathbb{T})\|. \quad\square
\end{aligned}$$

Lemma 3. *Let $1<p<\infty$, $\alpha\in\mathbb{R}$. The functions of the form $\sum_{j=1}^N f_j$ with $N\in\mathbb{N}$, $f_j\in H_p^\alpha(\mathbb{T})$,*

$$\operatorname{supp}\Big(\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f_j)e^{ik\cdot}\Big) = E_j \subset \mathbb{T},$$

$\mu(E_j\cap E_k)=0$ for $j\neq k$, form a dense set in $H_p^\alpha(\mathbb{T})$.

Proof: Let $f\in H_p^\alpha(\mathbb{T})$, $\varepsilon>0$, then

$$\tilde f := \sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha/2}c_k(f)e^{ik\cdot} \in L_p(\mathbb{T}).$$

The simple functions form a dense set in $L_p(\mathbb{T})$ (cf. [2], p. 125). Therefore, a function

$$\tilde{f}_N = \sum_{j=1}^{N} a_j \, \chi_{E_j}$$

with $a_j \in \mathbb{C}$, $E_j \subset \mathbb{T}$, $\mu(E_j \cap E_k) = 0$ for $j \neq k$, has to exist, such that

$$\|\tilde{f} - \tilde{f}_N \mid L_p(\mathbb{T})\| < \varepsilon.$$

Now, we define

$$g_j := a_j \sum_{k \in \mathbb{Z}} (1+k^2)^{-\alpha/2} c_k(\chi_{E_j}) e^{ik\cdot} \in H_p^\alpha(\mathbb{T})$$

and conclude

$$\left\| f - \sum_{j=1}^{N} g_j \;\middle|\; H_p^\alpha(\mathbb{T}) \right\| = \|\tilde{f} - \tilde{f}_N \mid L_p(\mathbb{T})\| < \varepsilon.$$

This proves the lemma. $\square$

After having proved these two lemmata we are now able to proceed to the proof of the theorem.

Proof: (for Theorem 1). Denote $\lambda_{\alpha,\beta} := \| \cdot \mid S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2)\|$. Let

$$h = \sum_{j=1}^{n} f_j \otimes g_j \tag{1}$$

be a representation of $h \in H_p^\alpha(\mathbb{T}) \otimes H_p^\beta(\mathbb{T})$. Then

$$\begin{aligned} \tilde{f}_j &:= \sum_{k \in \mathbb{Z}} (1+k^2)^{\alpha/2} c_k(f_j) e^{ik\cdot} \in L_p(\mathbb{T}), \\ \tilde{g}_j &:= \sum_{\ell \in \mathbb{Z}} (1+\ell^2)^{\beta/2} c_\ell(g_j) e^{i\ell\cdot} \in L_p(\mathbb{T}). \end{aligned}$$

With the definition

$$\tilde{h} = \sum_{j=1}^{n} \tilde{f}_j \otimes \tilde{g}_j,$$

we obtain

$$\begin{aligned} \lambda_{\alpha,\beta}(h) &= \left\| \sum_{j=1}^{n} \sum_{k,\ell \in \mathbb{Z}} (1+k^2)^{\alpha/2} (1+\ell^2)^{\beta/2} c_k(f_j) c_\ell(g_j) e^{ikx} e^{i\ell y} \;\middle|\; L_p(\mathbb{T}^2) \right\| \\ &= \left\| \sum_{j=1}^{n} \tilde{f}_j \otimes \tilde{g}_j \;\middle|\; L_p(\mathbb{T}^2) \right\| = \|\tilde{h} \mid L_p(\mathbb{T}^2)\|. \end{aligned}$$

Now, we use the tensor product property of

$$L_p(\mathbb{T}^2) = L_p(\mathbb{T}) \otimes_{\tilde{\lambda}_p} L_p(\mathbb{T})$$

and the equality

$$\tilde{\lambda}_p = \| \cdot \mid L_p(\mathbb{T}^2)\|$$

of the p–nuclear norm and the usual L_p–Norm (cf. [4], Chap. 1). That leads to

$$\lambda_{\alpha,\beta}(h) = \inf\Big\{ \Big(\sum_{j=1}^{n} \|\hat{f}_j \mid L_p(\mathbb{T})\|\Big)^{1/p} \theta_q(\hat{g}_1, \dots, \hat{g}_n) \; ; \; \tilde{h} = \sum_{j=1}^{n} \hat{f}_j \otimes \hat{g}_j \Big\},$$

where the infimum is taken over all representations of $\tilde{h}$.

The functions $\hat{f}_j, \hat{g}_j$ need not necessarily have a form like $\tilde{f}_j, \tilde{g}_j$ which stems from the representation (1) of h. Therefore, we obtain

$$\lambda_{\alpha,\beta}(h) \le \inf\Big\{ \Big(\sum_{j=1}^{n} \|\tilde{f}_j \mid L_p(\mathbb{T})\|\Big)^{1/p} \theta_q(\tilde{g}_1, \dots, \tilde{g}_n) \; ; \; \tilde{h} = \sum_{j=1}^{n} \tilde{f}_j \otimes \tilde{g}_j \Big\},$$

where we take the infimum over all representations of $\tilde{h}$ with particular functions $\tilde{f}_j, \tilde{g}_j$ from a representation of h. Using the definitions of $\tilde{f}_j, \tilde{g}_j$ yields

$$\begin{aligned}\lambda_{\alpha,\beta}(h) &\le \inf\Big\{ \Big(\sum_{j=1}^{n} \|f_j \mid H_p^{\alpha}(\mathbb{T})\|\Big)^{1/p} \theta_q(g_1, \dots, g_n) ; h = \sum_{j=1}^{n} f_j \otimes g_j \Big\} \\ &= \tilde{\lambda}_p(h).\end{aligned}$$

It remains to prove that $\tilde{\lambda}_p(h) \le \lambda_{\alpha,\beta}(h)$.

Let h have the representation (1) where the functions g_j satisfy

$$\operatorname{supp} \Big(\sum_{k \in \mathbb{Z}} (1 + k^2)^{\alpha/2} c_k(g_j) e^{ik\cdot} \Big) = E_j \subset \mathbb{T},$$

$\mu(E_j \cap E_k) = 0$ for $j \ne k$, and $\|g_j \mid H_p^{\beta}(\mathbb{T})\| = 1$. By Lemma 2,

$$\theta_q(g_1, \dots, g_n) = 1$$

and hence,

$$\tilde{\lambda}_p(h) \le \Big(\sum_{j=1}^{n} \|f_j \mid H_p^{\alpha}(\mathbb{T})\|\Big)^{1/p}.$$

On the other hand,

$$\begin{aligned}
\lambda_{\alpha,\beta}(h) &= \Big\| \sum_{j=1}^{n} \tilde{f}_j \otimes \tilde{g}_j \;\Big|\; L_p(\mathbb{T}^2) \Big\| \\
&= \Big(\sum_{k=1}^{n} \int_{\mathbb{T}} \int_{E_k} \Big| \sum_{j=1}^{n} \tilde{f}_j(x) \tilde{g}_j(y) \Big|^p \, dx dy \Big)^{1/p} \\
&= \Big(\sum_{j=1}^{n} \int_{\mathbb{T}} \int_{E_j} |\tilde{f}_j(x) \tilde{g}_j(y)|^p \, dx dy \Big)^{1/p} \\
&= \Big(\sum_{j=1}^{n} \|\tilde{f}_j \mid L_p(\mathbb{T})\| \cdot \|\tilde{g}_j \mid L_p(\mathbb{T})\| \Big)^{1/p} \\
&= \Big(\sum_{j=1}^{n} \|f_j \mid H_p^{\alpha}(\mathbb{T})\| \Big)^{1/p}.
\end{aligned}$$

That means, for functions h of this particular form we proved $\tilde{\lambda}_p(h) \le \lambda_{\alpha,\beta}(h)$. Because of Lemma 3 and the first part of this proof it holds for all functions $h \in H_p^{\alpha}(\mathbb{T}) \otimes H_p^{\beta}(\mathbb{T})$ that

$$\tilde{\lambda}_p(h) = \lambda_{\alpha,\beta}(h).$$

Since $S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2)$ is complete, we finally have

$$S_{p,p}^{\alpha,\beta} H(\mathbb{T}^2) = H_p^{\alpha}(\mathbb{T}) \otimes_{\lambda_{\alpha,\beta}} H_p^{\beta}(\mathbb{T}) = H_p^{\alpha}(\mathbb{T}) \otimes_{\tilde{\lambda}_p} H_p^{\beta}(\mathbb{T}),$$

where the first equality was proved in [7] and isomorphic spaces are identified. □

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Frauke Sprengel
FB Mathematik, Universität Rostock
18051 Rostock
Germany
`frauke.sprengel@mathematik.uni-rostock.de`