ON ORTHOGONAL COVERS OF $\vec{K}_n$

Hans-Dietrich O.F. Gronau* and Volker Leck*
Universität Rostock, Fachbereich Mathematik
18051 Rostock, Germany
and
Ronald C. Mullin* and Paul J. Schellenberg*
University of Waterloo, Dept. of Combinatorics & Optimization
Waterloo, Ontario, N2L 3G1, Canada

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Abstract

An orthogonal cover of the complete digraph $\vec{K}_n$ is a collection of n spanning subgraphs $\vec{G}_1$, $\vec{G}_2$, ..., $\vec{G}_n$ of $\vec{K}_n$ such that
- every directed edge of $\vec{K}_n$ belongs to exactly 1 of the $\vec{G}_i$'s and
- for every two digraphs $\vec{G}_i$ and $\vec{G}_j$ $(i \neq j)$ there is a unique two-element set $\{a, b\}$ of vertices such that $\vec{G}_i$ contains the directed edge (a, b) and $\vec{G}_j$ contains the directed edge (b, a).

It is proven that there exists an orthogonal cover of $\vec{K}_n$, where the $\vec{G}_i$'s consist of short directed cycles, for all $n \geq 4$, with a few possible exceptions. We conclude by establishing the existence of an orthogonal cover of $\vec{K}_n$, where the $\vec{G}_i$'s have maximum outdegree and maximum indegree one, for all $n \geq 2$. This gives another proof of the directed version of a problem of Chung and West.

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1 Introduction

Let $n \geq 2$ be an integer and let $\vec{K}_n$ be the complete digraph on the n-element vertex set V. We consider collections $\vec{\mathcal{G}} = \{\vec{G}_1, \vec{G}_2, ..., \vec{G}_n\}$ of spanning subgraphs of $\vec{K}_n$, i.e. digraphs on V. We call $\vec{\mathcal{G}}$ an *orthogonal cover of* $\vec{K}_n$, if
(i) every directed edge of $\vec{K}_n$ belongs to exactly one of the $\vec{G}_i$'s and
(ii) for every two digraphs $\vec{G}_i$ and $\vec{G}_j$ $(i \neq j)$ there is a unique two-element set $\{a, b\}$ of vertices such that $\vec{G}_i$ contains the directed edge (a, b) and $\vec{G}_j$ contains the directed edge (b, a).
We call the i-th digraph $\vec{G}_i$ in our cover the *i-th page* of the cover. Furthermore, we call a digraph $\vec{G}_i$ *idempotent*, if it contains i as an isolated point. A cover $\vec{\mathcal{G}}$ will be called *idempotent*, if the i-th page is an idempotent digraph for all $i = 1, 2, ..., n$.

Note that every page must have exactly $n - 1$ directed edges.

In the present paper we consider pages from the class of idempotent digraphs consisting of directed cycles only. We will derive results for a larger class of digraphs, namely the class of digraphs having maximum outdegree one and maximum indegree one, i.e. digraphs consisting of disjoint directed cycles and at most one directed path. The last problem corresponds to a related question for undirected graphs asked by Chung and West [4].

The problem, where the pages are chosen from the class of idempotent digraphs consisting of vertex disjoint directed triangles only (possible only if $n \equiv 1 \bmod 3$), was first asked and partially solved by Demetrovics, Füredi, and Katona in [6] and [5] (affirmative answer for $n \equiv 1$ *or* $4 \bmod 12, n = 7$). Rausche [13] proved that there is no solution for $n = 10$. The completion was given by Ganter and Gronau [7] and independently by Bennett and Wu [3]. The result is

Theorem 1 *There exists an orthogonal cover of $\vec{K}_n$, where each page is an idempotent digraph consisting of disjoint directed 3-cycles, if and only if* $n \equiv 1 \bmod 3, n \neq 10$.

More recently Granville, Gronau and Mullin [9] proved

Theorem 2 *There exists an orthogonal cover of $\vec{K}_n$, where each page is an idempotent digraph consisting of disjoint directed 4-cycles, if and only if* $n \equiv 1 \bmod 4$.

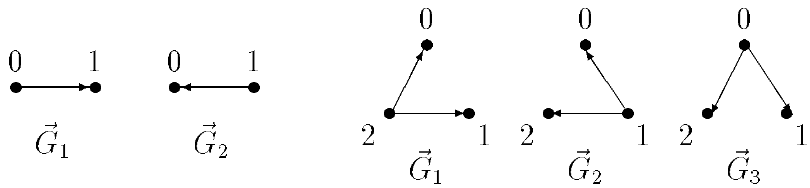


Figure 1: Solutions for $n = 2$ and $n = 3$

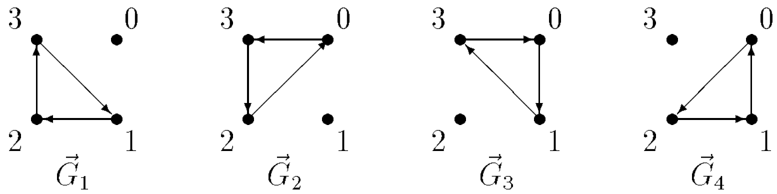


Figure 2: Solution for $n = 4$

Moreover, in that paper it was proved, that for every $k \geq 3$, there exists a number $n_0(k)$, such that there is an orthogonal cover of $\vec{K}_n$, where the pages are chosen from the class of idempotent digraphs consisting of vertex disjoint directed k-cycles whenever $n \equiv 1 \bmod k$ (a necesary condition) and $n \geq n_0(k)$. This question was asked in the undirected case by Hering [10]. For a survey on related results see Alspach, Heinrich, and Liu [1].

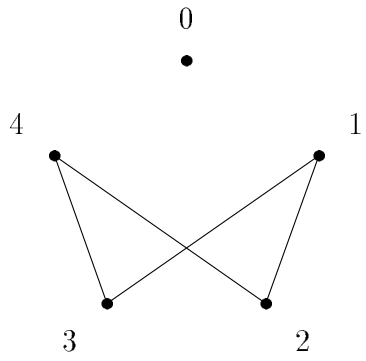
In Figures 1, 2, 3, and 4 we illustrate orthogonal covers of $\vec{K}_n$ for $n = 2, 3, 4, 5, 7$.

The examples show that it is useful to consider group-generated solutions, if those exist.

Let Γ be an additive group of order n. A spanning subgraph G of K_n, having vertex set Γ, is called a *starter* for a Γ-generated orthogonal cover if the subgraphs $\{G + x : x \in \Gamma\}$, obtained from G under the action of Γ, constitute an orthogonal cover of $\vec{K}_n$. If Γ is the cyclic group $\mathbb{Z}_n$, then G is called a *cyclic starter*. For example, the solutions for $n = 2, 3, 5, 7$ in Figures 1,3, and 4 are cyclic ones, whereas the solution for $n = 4$ in Figure 2 is not cyclic, but generated by the four group.

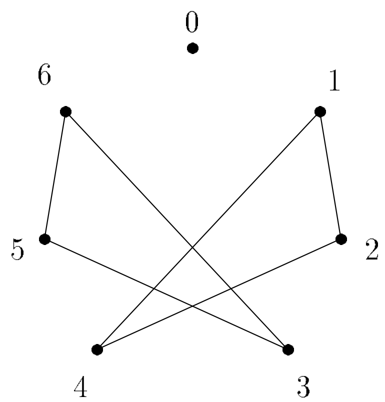
It is not too hard to check the following

Lemma 1 *$\vec{G}$ is a starter for a Γ-generated orthogonal cover of $\vec{K}_n$, if and only if:*
i) $\{x - y : (x, y) \in E(\vec{G})\} = \Gamma - \{0\}$.
ii) For any $c \in \Gamma - \{0\}$ let $(x, x + c)$ the unique directed edge of distance c



$\vec{G}\ \mathbb{Z}_5 - generated$

Figure 3: Solution for $n = 5$



$\vec{G}\ \mathbb{Z}_7 - generated$

Figure 4: Solution for $n = 7$

and let $(y, y-c)$ the unique directed edge of distance $-c$. Furthermore, let $R(c) = \{(x+c-y), -(x+c-y)\}$. $R(c) \neq R(c')$ for any c, c' with $c \neq c'$ and $c \neq -c'$.

An orthogonal cover of $\vec{K_n}$ implies an *orthogonal double cover* of K_n, if the orientations are omitted.

By looking for solutions it turned out that the cases $n \equiv 2 \bmod 4$ were difficult to handle. For example, $n = 30$ took us about two years to settle. The reason behind the scene is the following result given in Ganter, Gronau, and Mullin [8].

Lemma 2 *If G is a cyclic starter for an orthogonal double cover of K_n, then $n \not\equiv 2 \bmod 4$.*

This Lemma even holds for arbitrary abelian groups.

2 The main results

The aim of this paper is to prove the following results.

Theorem 3 *There exists an idempotent orthogonal cover $\{\vec{G}_1, \vec{G}_2, \ldots, \vec{G}_n\}$ of $\vec{K_n}$, where each page $\vec{G}_i$ $(i = 1, 2, \ldots, n)$ consists of the isolated point i and vertex disjoint directed cycles of length 3 and 4 only, for all $n \geq 4$ with the exceptions $n = 6$, 8, 10, and 11, and the possible exceptions $n = 14$ and 54.*

The proof will be given in Section 4, but for the four definite exceptions let us make the following remarks. That there is no solution for $n = 6$ and $n = 8$ follows from Remark 1. In cases $n = 10$ or $n = 11$ the pages of a solution would consists of 3 directed 3-cycles and one isolated point or 1 directed 4-cycle, 2 directed 3-cycles and one isolated point, respectively. The nonexistence was proved for $n = 10$ by Rausche [13] and for $n = 11$ by Leck and Leck [11].

Remark 1 *A complete computer search showed that there is no solution in case $n = 6$ with idempotent pages having necessarily one directed 5-cycle. Note that there is such a solution in the undirected case, see* Ganter, Gronau, and Mullin *[8]. A complete computer search showed that there* *is no solution in case $n = 8$ with idempotent pages having necessarily one directed 4-cycle and one directed 3-cycle. There is an idempotent solution for $n = 8$ with one directed 7-cycle, see Theorem 5.*

Let us allow idempotent orthogonal covers with directed cycles of length 3, 4 and 5. Then the following theorem holds.

Theorem 4 *There exists an idempotent orthogonal cover $\{\vec{G}_1, \vec{G}_2, ..., \vec{G}_n\}$ of $\vec{K}_n$, where each page $\vec{G}_i$ $(i = 1, 2, ..., n)$ consists of the isolated point i and vertex disjoint directed cycles of length 3, 4 and 5 only, for all $n \geq 4$ with the exceptions $n = 6$ and 8, and the possible exception $n = 14$.*

Theorem 4 is essentially a consequence from Theorem 3. For the nonexistence of solutions for $n = 6$ and $n = 8$ see Remark 1. Solutions of the orders $n = 10$, $n = 11$, and $n = 54$ are given in Section 5.

Weakening the problem we are able to establish the following

Theorem 5 *There exists an idempotent orthogonal cover $\{\vec{G}_1, \vec{G}_2, ..., \vec{G}_n\}$ of $\vec{K}_n$, where each page $\vec{G}_i$ $(i = 1, 2, ..., n)$ consists of the isolated point i and disjoint directed cycles (without length restriction), if and only if $n \geq 4, n \neq 6$.*

The proof follows from Theorem 4, Remark 1 and the solutions for $n = 8$ and 14, which are given in Table 1.

Finally we are able to state the result to the corresponding problem of Chung and West [4].

Theorem 6 *There exists an orthogonal cover $\{\vec{G}_1, \vec{G}_2, ..., \vec{G}_n\}$ of $\vec{K}_n$, where each page $\vec{G}_i$ $(i = 1, 2, ..., n)$ has maximum outdegree one and maximum indegree one, if and only if $n \geq 2$.*

The solutions for $n = 2$ and $n = 3$ are given in Figure 1. The digraph $\vec{G} = \{[0, 1, 3], (2, 5, 4)\}$ is a cyclic starter over $\mathbb{Z}_6$. (Here [0,1,3] represents the dipath from 0 to 3 via 1 and (2,5,4)[1] represents a dicycle of length 3.) All the other orthogonal covers follow from Theorem 5.

[1] $(a_1, a_2, ..., a_k)$ means the cyclically directed cycle consisting of the edges $(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k), (a_k, a_1)$

n	solution
8	$\vec{G} = \{((0,0,0)), ((0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,1,0),(1,1,1),(1,0,1))\}$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 - generated$
14	$\vec{G}_1 = \{(0),(1,9,2,11,5,3,6,10,12,4),(7,8,13)\}$ $\vec{G}_2 = \{(1),(0,5,2,10,3,12,6,4,7,11),(8,9,13)\}$ $\vec{G}_3 = \{(2),(0,7,5,8,12,1,6,3,11,4),(9,10,13)\}$ $\vec{G}_4 = \{(3),(0,2,7,4,12,5,1,8,6,9),(10,11,13)\}$ $\vec{G}_5 = \{(4),(0,6,2,9,7,10,1,3,8,5),(11,12,13)\}$ $\vec{G}_6 = \{(5),(0,13,12),(1,7,3,10,8,11,2,4,9,6)\}$ $\vec{G}_7 = \{(6),(0,1,13),(2,8,4,11,9,12,3,5,10,7)\}$ $\vec{G}_8 = \{(7),(0,4,6,11,8,3,9,5,12,10),(1,2,13)\}$ $\vec{G}_9 = \{(8),(0,11,1,5,7,12,9,4,10,6),(2,3,13)\}$ $\vec{G}_{10} = \{(9),(0,10,5,11,7,1,12,2,6,8),(3,4,13)\}$ $\vec{G}_{11} = \{(10),(0,3,7,9,1,11,6,12,8,2),(4,5,13)\}$ $\vec{G}_{12} = \{(11),(0,9,3,1,4,8,10,2,12,7),(5,6,13)\}$ $\vec{G}_{13} = \{(12),(0,8,1,10,4,2,5,9,11,3),(6,7,13)\}$ $\vec{G}_{14} = \{(13),(0,12,11,10,9,8,7,6,5,4,3,2,1)\}$

Table 1: Solutions for $n = 8, 14$

3 PBD-closure

The main idea for our construction is to show first that the set of orders for which idempotent solutions, i.e. i is an isolated point in the digraph $\vec{G}_i$, exist, is PBD-closed. Then the set of orders for which *pairwise balanced designs* (PBD) exist gives the desired solutions.

Theorem 7 *The set of integers n for which an idempotent orthogonal cover of $\vec{K}_n$ exists is PBD-closed.*

Proof. Let P be the set of blocks of a PBD on V, and suppose that for every $e \in P$ we have an idempotent solution

$$\vec{\mathcal{G}}^e = \{\vec{G}^e_{x_1}, \vec{G}^e_{x_2}, ..., \vec{G}^e_{x_{|e|}}\}, e = \{x_1, x_2, ..., x_{|e|}\}$$

Then for every $x \in V$ define

$$\vec{G}_x := \bigcup_{e \in P, x \in e} \vec{G}^e_x.$$

We show that $\{\vec{G}_x : x \in V\}$ is again an idempotent solution. The pages $\vec{G}_x$ are idempotent and have $n-1$ edges by construction. Moreover, any directed edge belongs to exactly one page. Consider two pages $\vec{G}_x$ and $\vec{G}_y$ for arbitrary $x, y \in V$ with $x \neq y$. Assume that $u, v \in V, u \neq v$ are vertices with $(u, v) \in E(\vec{G}_x)$ and $(v, u) \in E(\vec{G}_y)$. Then u and v must lie on a block $e_x \in P$ together with x by construction of $\vec{G}_x$. Analogously, u and v must lie on a block e_y together with y. Since P is a PBD, there is a unique block containing u and v, i.e. $e = e_x = e_y$. Since we used an orthogonal cover on e, the two pages $\vec{G}_x$ and $\vec{G}_y$ are orthogonal. $\square$

4 Proof of Theorem 3

For brevity let $\mathcal{B}(K)$ denote the set of orders for which a pairwise balanced design with block sizes from K exists, where K is a set of integers. We will use the following theorem.

Theorem 8 ([2], [12])
$\mathcal{B}(\{4, 5, 7, 9\}) = \{4, 5, 7, 9, 13, 16, 17, 20, 21, 22, 24, 25, 28, 29\} \cup \{n : n \geq 30\} - M$, *where* $M \subseteq \{30, 51, 54\}$.

Contributions (partly overlapping ones) to the above theorem were given by several authors, but the present status is summarized in BENNETT, GRONAU, LING, AND MULLIN [2] and completed in MULLIN, LING, ABEL AND BENNETT [12].

In order to prove Theorem 3, we first give solutions for $n = 4, 5, 7, 9$ in Table 2

n	solution	fig.
4	$\vec{G} = \{((0,0)),((1,0),(0,1),(1,1))\}\ \mathbb{Z}_2 \times \mathbb{Z}_2 - generated$	2
5	$\vec{G} = \{(0),(1,2,4,3)\}\mathbb{Z}_5 - generated$	3
7	$\vec{G} = \{(0),(1,2,4),(3,5,6)\}\ \mathbb{Z}_7 - generated$	4
9	$\vec{G} = \{((0,0)),((0,1),(1,0),(0,2),(2,0)),((1,1),(1,2),(2,2),(2,1))\}$ $\mathbb{Z}_3 \times \mathbb{Z}_3 - generated$	

Table 2: Solutions for $n = 4, 5, 7, 9$

Applying Theorem 8 and Theorem 7 we have to present solutions for $n = 12, 15, 18, 19, 23, 26, 27, 30$ in order to complete the proof of Theorem 4 only. The solutions will be given in Table 3.

It remains to show that there is a solution for $n = 51$. Take a $TD(5, 11)$, delete 5 points of the last group. Adjoin to every of the 5 groups a new point ∞. By adding the new groups as blocks we obtain a $PBD[51, \{4, 5, 7, 12\}]$. Since there is an idempotent solution for $n = 12$ we obtain the desired result by applying Theorem 7.

5 Proof of Theorem 4

Solutions for $n = 10$ and 11 are given in Table 4.

It remains to show that there is a solution for $n = 54$. Take a $TD(5, 11)$, delete 1 point of the last group. By adding the new groups as blocks we obtain a $PBD[54, \{4, 5, 10, 11\}]$. Since there are corresponding idempotent solutions for $n = 10$ and 11 we obtain the desired result by applying Theorem 7.

n	solution
12	$\vec{G} = \{((0,0)),((0,1),(1,3),(1,1),(0,4)),((0,2),(1,0),(1,2),(0,3)),((0,5),(1,4),(1,5))\}$ $\mathbb{Z}_2 \times \mathbb{Z}_6 - generated$
15	$\vec{G} = \{(0),(1,14,4,12),(2,11,13,5),(3,9,8),(6,7,10)\} \mathbb{Z}_{15} - generated$
18	$\vec{G}_1 = \{((0,0)),((0,2),(0,1),(1,1),(1,5)),((0,8),(0,3),(1,2),(1,4)),((0,7),(0,5),(1,8)),$ $((0,4),(1,0),(1,6)),((0,6),(1,7),(1,3))\}$ $(-,\mathbb{Z}_9) - generated$ $\vec{G}_2 = \{((1,0)),((0,3),(0,0),(1,6),(1,7)),((0,4),(0,6),(1,4),(3)),((0,1),(0,2),(0,5)),$ $((0,7),(1,2),(1,5)),((0,8),(1,1),(1,8))\}$ $(-,\mathbb{Z}_9) - generated$
19	$\vec{G} = \{(0),(1,7,11),(2,14,3),(4,9,6),(5,16,17),(8,18,12),(10,13,15)\}$ $\mathbb{Z}_{19} - generated$
23	$\vec{G} = \{(0),(1,19,8),(2,17,11),(3,4,6),(5,15,13,20),(7,18,21),(9,14,10),(12,16,22)\}$ $\mathbb{Z}_{23} - generated$
26	$\vec{G}_1 = \{((0,0)),((0,1),(1,12),(0,3)),((0,2),(0,8),(0,7)),((0,4),(1,3),(0,11),(1,5)),$ $((0,5),(1,11),(1,4)),((0,6),(0,9),(1,1)),((0,10),(1,6),(1,10)),((0,12),(1,0),(1,9)),$ $((1,2),(1,7),(1,8))\}$ $(-,\mathbb{Z}_{13}) - generated$ $\vec{G}_2 = \{((1,0)),((0,0),(1,3),(1,2)),((0,2),(1,4),(0,11)),((0,1),(1,1),(1,12),(0,9)),$ $((0,3),(1,11),(0,7)),((0,4),(1,8),(0,10)),((0,5),(0,6),(0,8)),((0,12),(1,9),(1,6)),$ $((1,5),(1,7),(1,10))\},$ $(-,\mathbb{Z}_{13}) - generated$
27	$\vec{G} = \{(0),(1,17,5),(2,12,20,23),(3,24,11),(4,22,6),(7,19,14),(8,21,26),$ $(9,13,10),(15,16,18,25)\}$ $\mathbb{Z}_{27} - generated$
30	$\vec{G}_1 = \{((0,0)),((0,1),(1,8),(0,5)),((0,2),(1,5),(1,2)),((0,3),(0,7),(1,12)),$ $((0,4),(0,11),(0,8),(0,10)),((0,6),(1,4),(1,14)),((0,9),(1,0),(1,7),(1,13)),$ $((0,12),(1,6),(1,9)),((0,13),(1,10),(1,11)),((0,14),(1,3),(1,1))\}$ $(-,\mathbb{Z}_{15}) - generated$ $\vec{G}_2 = \{((1,0)),((0,0),(0,14),(0,7),(0,5)),((0,1),(0,4),(0,9),(0,10)),$ $((0,2),(1,13),(1,7)),((0,3),(1,3),(1,14)),((0,6),(1,1),(1,5)),((0,8),(1,10),(1,9)),$ $((0,11),(1,12),(1,2)),((0,12),(1,11),(1,4)),((0,13),(1,6),(1,8))\},$ $(-,\mathbb{Z}_{15}) - generated$

Table 3: Solutions for $n = 12, 15, 18, 19, 23, 26, 27, 30$

n	solution
10	$\vec{G}_1 = \{(0),(1,2,3),(4,5,6),(7,8,9)\}$ $\vec{G}_2 = \{(1),(0,3,2,4,9),(5,7,6,8)\}$ $\vec{G}_3 = \{(2),(0,4,1,9,6),(3,5,8,7)\}$ $\vec{G}_4 = \{(3),(0,1,8,4),(2,6,7,9,5)\}$ $\vec{G}_5 = \{(4),(0,8,6,3,7),(1,5,9,2)\}$ $\vec{G}_6 = \{(5),(0,6,2,8),(1,3,9,4,7)\}$ $\vec{G}_7 = \{(6),(0,5,4,8,3),(1,7,2,9)\}$ $\vec{G}_8 = \{(7),(0,9,8,2,5),(1,4,3,6)\}$ $\vec{G}_9 = \{(8),(0,2,7,5,1),(3,4,6,9)\}$ $\vec{G}_{10} = \{(9),(0,7,4,2),(1,6,5,3,8)\}$
11	$\vec{G} = \{(0),(1,3,9,5,4),(2,6,7,10,8)\}$ $\mathbb{Z}_{11} - generated$

Table 4: Solutions for $n = 10$ and $n = 11$

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