NONEXISTENCE OF A KRUSKAL–KATONA TYPE THEOREM FOR SUBWORD ORDERS

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We consider the poset $SO(n)$ of all words over an n–element alphabet ordered by the subword relation. It is known that $SO(2)$ falls into the class of Macaulay posets, i.e. there is a theorem of Kruskal–Katona type for $SO(2)$. As the corresponding linear ordering of the elements of $SO(2)$ the *vip*–order can be chosen.

Daykin introduced the V–order which generalizes the *vip*–order to the $n \geq 2$ case. He conjectured that the V–order gives a Kruskal–Katona type theorem for $SO(n)$.

We show that this conjecture fails for all $n \geq 3$ by explicitely giving a counterexample. Based on this, we prove that for no $n \geq 3$ the subword order $SO(n)$ is a Macaulay poset.

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1. Introduction.

Macaulay posets. Let P be a ranked poset with the associated partial order $\leq$. Denote its i–th level (the set of all elements of rank i) by $N_i(P)$ resp. by N_i if there is no danger of ambiguity. For $x \in N_i$ the (down) shadow $\Delta(x)$ of x is the set of all $y \in N_{i-1}$ such that $y \leq x$. The shadow of a subset $X \subseteq N_i$ is the set $\Delta(X) = \bigcup_{x \in X} \Delta(x)$.

Let $\prec$ be some linear ordering of the elements of P. For $X \subseteq N_i$, the set $C(X)$ of the smallest $|X|$ elements of N_i with respect to $\prec$ is called the *compression* of X (w.r.t. $\prec$). We say that P is a *Macaulay poset* if there is a linear ordering $\prec$ such that the following two conditions are satisfied for all i and all nonempty $X \subseteq N_i$:

(1) $|\Delta(C(X))| \leq |\Delta(X)|$,

(2) $C(\Delta(C(X))) = \Delta(C(X))$.

If the conditions (1) and (2) are satisfied for some linear ordering $\prec$, then we will use a notion from [8] and say that $\prec$ is *suitable* (for P).

Since the fact that some ranked poset P is a Macaulay poset has many applications to extremal and optimization problems on P (see [12, 2, 13] for example), it is always interesting to know wheather or not some specific poset is a Macaulay poset. As the most popular examples of Macaulay posets let us mention Boolean lattices [18, 16], cartesian products of chains [21, 6], and cartesian products of stars [20, 3, 14, 19].

Subword orders. For an integer $n \geq 2$ the set $[n] := \{0, 1, \ldots, n-1\}$ is called the *alphabet* of the *subword order* $SO(n)$. The elements of $SO(n)$ are all strings (called *words*) that contain symbols (called *letters*) only from $[n]$. The partial order on $SO(n)$ is the subword relation, that is $\boldsymbol{x} \leq \boldsymbol{y}$ if $\boldsymbol{x}$ can be obtained from $\boldsymbol{y}$ by successively deleting letters. More precisely, $x_1x_2\ldots x_k \leq y_1y_2\ldots y_\ell$ holds if there is a subset $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, \ell\}$ such that $i_1 < i_2 < \cdots < i_k$ and $x_j = y_{i_j}$ for $j = 1, 2, \ldots, k$. The rank of an element of $SO(n)$ equals its length, that means $x_1x_2\ldots x_i \in N_i(SO(N))$. The only element of $N_0(SO(n))$ is the *empty word*, denoted by ϵ.

A linear ordering which is known to be suitable for $SO(2)$ is the *vip*–order. The i–th level of $SO(2)$ consists of all words of length i over the alphabet $\{0, 1\}$. Therefore, in an obvious way they can be considered as the vertices of the i–cube. The name *vip*–order stems from the fact that the ordering of the elements of $N_i(SO(2))$ induced by the *vip*–order on $SO(2)$ gives a solution of the vertex–isoperimetric problem for the i–cube [15].

To define the *vip*–order on $SO(2)$, for $\boldsymbol{x} = x_1x_2\ldots x_k \in SO(2)$ we introduce the notation $w(\boldsymbol{x}) := |\{j : x_j = 1, 1 \leq j \leq k\}|$, and compare any two elements $\boldsymbol{x} = x_1x_2\ldots x_k$ and $\boldsymbol{y} = y_1y_2\ldots y_\ell$ of $SO(2)$ by:

(a) $\boldsymbol{x} \prec_{vip} \boldsymbol{y}$ if $w(\boldsymbol{x}) < w(\boldsymbol{y})$,

(b) $\boldsymbol{x} \prec_{vip} \boldsymbol{y}$ if $w(\boldsymbol{x}) = w(\boldsymbol{y})$ and there is some $j \leq \min\{k, \ell\}$ such that $x_j > y_j$ and $x_h = y_h$ for $h = 1, 2, \ldots, j-1$,

(c) $\boldsymbol{x} \prec_{vip} \boldsymbol{y}$ if $w(\boldsymbol{x}) = w(\boldsymbol{y})$, $k < \ell$ and $x_j = y_j$ for $j = 1, 2, \ldots, k$.

Theorem 1.1. [1, 5, 7] *The vip–order is suitable for $SO(2)$.* ∎

Let us remark that there are also other suitable orderings for $SO(2)$, all these orderings were determined by Daykin [10].

2. Daykin's conjecture.

Daykin [9] (see also [8]) introduced the *V–order*, a linear ordering of the elements of $SO(n)$. He conjectured that this ordering is suitable for $SO(n)$ (for arbitrary n).

Definition of the $\boldsymbol{V}$–order. The following definition of the V–order on $SO(n)$ is essentially the one given in [8].

Consider an element $\boldsymbol{x} = x_1x_2\ldots x_k \neq \epsilon$ of $SO(n)$. Let $i \in \{1, 2, \ldots, k\}$ be the least index such that $x_i \leq x_{i+1} \leq \cdots \leq x_k$. Denote by $\boldsymbol{x}^*$ the word obtained from $\boldsymbol{x}$ by deleting the letter x_i, and for $s = 1, 2, \ldots, k$ denote by $\boldsymbol{x}^{s*}$ the word obtained from $\boldsymbol{x}$ by successively applying this operation s times. Finally define $\boldsymbol{x}^{0*} := \boldsymbol{x}$ for all $\boldsymbol{x} \in SO(n)$. Note that, by this definition, $\boldsymbol{x}^{k*} = \epsilon$ holds for all $k = 0, 1, 2, \ldots$ and $\boldsymbol{x} \in N_k$.

Therefore, for all $\boldsymbol{x}, \boldsymbol{y} \in SO(n)$ there are non–negative integers s, t such that $\boldsymbol{x}^{s*} = \boldsymbol{y}^{t*}$, and among all $\boldsymbol{z}$ with $\boldsymbol{z} = \boldsymbol{x}^{s*} = \boldsymbol{y}^{t*}$ for some s, t there is a unique element of maximum rank. Let us denote this element by $\inf^*(\boldsymbol{x}, \boldsymbol{y})$. Now the following two conditions establish the V–order on $SO(n)$:

(i) $\boldsymbol{x} \prec_V \boldsymbol{y}$ if $\inf^*(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$,

(ii) $\boldsymbol{x} \prec_V \boldsymbol{y}$ if $\inf^*(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^{s*} = \boldsymbol{y}^{t*}$ with $s, t \geq 1$ and for $\boldsymbol{x}^{(s-1)*} = u_1u_2\ldots u_m$, $\boldsymbol{y}^{(t-1)*} = v_1v_2\ldots v_m$ there is an $i \in \{1, 2, \ldots, m\}$ such that $u_i < v_i$, $u_j = v_j$ $(j = i+1, i+2, \ldots, m)$.

Properties of the $\boldsymbol{V}$–order. The following properties of the V–order were shown in [7] and [8], respectively. Properties 3.,4.,5. provide a possibility to construct the V–order inductively over n.

1. The V–order on $SO(2)$ reduces to the *vip*–order.
2. $C_V(\Delta(C_V(X))) = \Delta(C_V(X))$ for all i and $\emptyset \subset X \subseteq N_i$, where $C_V(X)$ denotes the compression of X w.r.t. the V–order.
3. For $\boldsymbol{x} \in SO(n)$, let $\mu(\boldsymbol{x})$ the maximum element of $[n]$ occuring in $\boldsymbol{x}$ as a letter, where $\mu(\epsilon) := 0$. If $\mu(\boldsymbol{x}) < \mu(\boldsymbol{y})$, then $\boldsymbol{x} \prec_V \boldsymbol{y}$.
4. Let $\nu(\boldsymbol{x})$ be the number of letters of $\boldsymbol{x}$ equal to $\mu(\boldsymbol{x})$. If $\mu(\boldsymbol{x}) = \mu(\boldsymbol{y})$ and $\nu(\boldsymbol{x}) < \nu(\boldsymbol{y})$, then $\boldsymbol{x} \prec_V \boldsymbol{y}$.
5. Let $\mu(\boldsymbol{x}) = \mu(\boldsymbol{y}) = g$ and $\nu(\boldsymbol{x}) = \nu(\boldsymbol{y}) = k$. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are of the forms $\boldsymbol{x} = \boldsymbol{x_0}g\boldsymbol{x_1}g\boldsymbol{x_2}g\ldots g\boldsymbol{x_k}$ and $\boldsymbol{y} = \boldsymbol{y_0}g\boldsymbol{y_1}g\boldsymbol{y_2}g\ldots g\boldsymbol{y_k}$, where $\mu(\boldsymbol{x_j}) < g$, $\mu(\boldsymbol{y_j}) < g$ for $j = 0, 1, \ldots, k$. (Note that some $\boldsymbol{x_j}$'s and $\boldsymbol{y_j}$'s can be the empty word.) If $\boldsymbol{x_j} \prec_V \boldsymbol{y_j}$ for the least j with $\boldsymbol{x_j} \neq \boldsymbol{y_j}$, then $\boldsymbol{x} \prec_V \boldsymbol{y}$.

3. The counterexample.

The example. Let $n \geq 3$, and let X be a subset of $N_5(SO(n))$ with $|X| = 68$. By the definition of the V–order and its properties given in the previous section, it is easy to see that $C_V(X) = R \cup S \cup T$, where

$$\begin{aligned}
R &= \{x_1x_2x_3x_4x_5 : \ x_i \in \{0,1\} \text{ for } i = 1,2,3,4,5\},\\
S &= \{x_1x_2x_3x_4x_5 : \ \exists i \in \{1,2,3,4,5\} \text{ such that } x_i = 2, x_j = 0 \text{ for } 1 \leq j < i \text{ and}\\
&\qquad x_j \in \{0,1\} \text{ for } i < j \leq 5\},\\
T &= \{12000, 12100, 12010, 12001, 12110\}.
\end{aligned}$$

Let us now specify X saying that X is the subset of N_5 which is obtained from $C_V(X)$ replacing the elements $00020, 00021, 00002$ by $12101, 12011, 12111$. Obviously, the shadow of $\{12101, 12011, 12111\}$ is contained in $\Delta(C_V(X))$ which yields $\Delta(X) \subseteq \Delta(C_V(X))$. On the other hand, 0002 is in $\Delta(C_V(X))$, but not in $\Delta(X)$. Therefore, $|\Delta(X)| < |\Delta(C_V(X))|$ holds.

Consequently, the V–order is not suitable for $SO(n)$ with $n \geq 3$.

Discussion. Since it will be needed in the next section let us briefly recall some basic facts and (roughly) describe how the above example was found.

It is well–known (cf. [4, 12]) that some ordering $\prec$ is suitable for some poset P iff the reverse of $\prec$ is suitable for the dual of P. In other words, the conditions (1) and (2) hold iff (1') and (2') below are satisfied for all i and $\emptyset \subset X \subseteq N_i$:

(1') $|\nabla(L(X))| \leq |\nabla(X)|$,

(2') $L(\nabla(L(X))) = \nabla(L(X))$,

where $L(X)$ is the set of the greatest $|X|$ elements of N_i w.r.t. $\prec$. For $x \in N_i$, the *upper shadow* $\nabla(x)$ is the set of all $y \in N_{i+1}$ with $x \leq y$, and the upper shadow of $X \subseteq N_i$ is $\nabla(X) = \bigcup_{x \in X} \nabla(x)$.

Let $P = SO(n)$, and assume that there is a suitable ordering $\prec$ for P. Without loss of generality, we may assume that $0 \prec 1 \prec \cdots \prec n-1$ holds for the elements of N_1. Now $\mu(\boldsymbol{x}) < \mu(\boldsymbol{y}) \Longrightarrow \boldsymbol{x} \prec \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in N_i$ follows by (2'). Consequently, for $m < n$ the ordering of the elements of $SO(m)$ induced by $\prec$ is suitable for $SO(m)$. In particular, if $SO(n)$ was a Macaulay poset for some $n > 3$, then $SO(3)$ would be a Macaulay poset, too. We will use this simple observation in the next section.

Now let $P = SO(3)$ and consider a subset $X \subseteq N_i$. For $j = 0,1,2$ let X_j denote the set of all words $x_2x_3\dots x_i$ such that $jx_2\dots x_i \in X$. Counting seperately the numbers of elements of $\nabla(X)$ with first letter equal to 0,1, and 2, respectively, gives the inequality

$$|\nabla(X)| \geq \max\{|X|, |\nabla(X_0)|\} + \max\{|X|, |\nabla(X_1)|\} + \max\{|X|, |\nabla(X_2)|\}$$

(which is related to an approach of Eckhoff and Wegner [11] and Katona [17] for Boolean lattices).

For $X = L_V(X)$, using the properties of the V–order given in the previous section, it is not hard to verify that we have equality in the above inequality. Moreover, we have $|X| \leq |\nabla(X_j)|$ for $j = 0,1,2$, and therefore $|\nabla(X)| = |\nabla(X_0)| + |\nabla(X_1)| + |\nabla(X_2)|$. Furthermore, if $|X| < |\nabla(X_1)|$ and $\boldsymbol{x_0}, \boldsymbol{x_1} \in N_{i-1}$ are the largest element not contained in X_0 and the smallest element of X_1, respectively, then

$$|\nabla(X \setminus \{1\boldsymbol{x_1}\} \cup \{0\boldsymbol{x_0}\})| = |\nabla(X_0 \cup \{\boldsymbol{x_0}\})| + |\nabla(X_1 \setminus \{\boldsymbol{x_1}\})| + |\nabla(X_2)|.$$

Therefore, in this case $|\nabla_{new}(\boldsymbol{x_1})| \leq |\nabla_{new}(\boldsymbol{x_0})|$ is a necessary condition for the V–order to satisfy (1'), where the (upper) *new–shadow* $\nabla_{new}(\boldsymbol{x})$ of $\boldsymbol{x} \in N_{i-1}$ is defined to be the set of all elements of $\nabla(\boldsymbol{x})$ not contained $\nabla(\{\boldsymbol{y} : \boldsymbol{x} \prec_V \boldsymbol{y}\})$.

Consider the set $X = L_V(X) \subset N_4$ whose smallest element is 1211. The set Y in the proof of the proposition below is obtained from X replacing 1211 by 0002, where 002 is the greatest element of N_3 which is not in X_0. The proposition provides a counterexample to the validity of (1') for the V–order since $\nabla_{new}(211) = \{2111, 2011, 2101, 2110\}$ and $\nabla_{new}(002) = \{0002, 0021, 0020\}$.

Proposition 3.1. *There is a subset Y of $N_4(SO(3))$ with $|Y| = 47$ and $|\nabla(Y)| = 175$.*

Proof. Choose $Y = U \cup V \cup W$, where

$U = \{x_1x_2x_3x_4 : \exists 1 \leq i < j \leq 4 \text{ such that } x_i = x_j = 2\}$,

$V = \{x_1x_2x_3x_4 : \exists i \in \{3,4\} \text{ such that } x_i = 2, x_j \in \{0,1\} \text{ for } j \neq i \text{ and } \sum_{j=1}^{i-1} x_j \geq 1\}$,

$W = \{0002\}$.

Then, clearly, $|Y| = 33 + 13 + 1 = 47$ and $|\nabla(Y)| = 131 + 41 + 3 = 175$ holds. ∎

Using the (simple) proof of the equivalence [(1) and (2)]$\Longleftrightarrow$[(1') and (2')] (the one given in [12] for instance), it is easy to figure out that Y corresponds to the counterexample to the validity of (1) for V–order given above.

By the help of the inequality $|\nabla(X)| \geq \sum_{j=0,1,2} \max\{|X|, |\nabla(X_j)|\}$ (and with some effort of time), one can observe that the V–order is suitable for the subposet of $SO(3)$

consisting just of the words with at most 4 letters. This illustrates that Proposition 3.1 yields a strong condition which a suitable ordering for $SO(3)$ must satifsy. In fact, we will show that this condition does not allow the existence of a suitable ordering for $SO(3)$.

4. The main result.

Theorem 4.1. *There is no suitable ordering for $SO(n)$ with $n > 2$.*

Proof. By the arguments of the previous section, it suffices to show that there is no ordering of the elements of $SO(3)$ satisfying (1') and (2'). Assume there is a suitable ordering $\prec$ for $SO(3)$.

The proof follows a very simple strategy: Using (1') and (2'), we will obtain necessary consitions on the structure of $\prec$ on the levels N_1, N_2, N_3, N_4 and N_5. Finally, Proposition 3.1 will imply a contradiction to (1').

Without loss of generality, we may assume $0 \prec 1 \prec 2$. Now the (w.r.t. $\prec$) greatest 5 elements of N_2 are those containing 2 as a letter. Also w.l.o.g. we can suppose that the greatest element of N_2 is $a2$, where $a \in \{0,1,2\}$. Now the greatest 7 elements of N_3 are those from $\nabla(a2)$, the greatest 33 elements of N_4 are those from $\nabla(\nabla(a2))$, and the greatest 131 elements of N_5 are those from $\nabla(\nabla(\nabla(a2)))$.

Let $\boldsymbol{x}$ be the element of N_2 preceding $a2$ in the ordering $\prec$. Then $|\nabla(a2) \cap \nabla(\boldsymbol{x})| \leq 2$ with equality iff the second letter of $\boldsymbol{x}$ is 2. Consequently, $\boldsymbol{x} = b2$ with $b \in \{0,1,2\}, b \neq a$. Furthermore, $\nabla_{new}(b2) = \{bb2, bc2, cb2, b20, b21\}$, where c is the only element of $\{0,1,2\} \setminus \{a,b\}$.

Let Z be the set of the greatest 3 elements of $\nabla_{new}(b2)$. Then $|\nabla_{new}(Z)| \geq 13$ with equality iff $Z = \{bb2, bc2, cb2\}$. Therefore, $Z = \{bb2, bc2, cb2\}$ must hold. Obviously, $|\nabla_{new}(\nabla_{new}(Z))| = 41$.

Altogether, we have the following information about $\prec$: The greatest 46 elements of N_4 are those containing $a2, bb2, bc2$, or $cb2$ as a subword, and the greatest 172 elements of N_5 are those satisfying the same condition. The element of N_3 preceding the first element of Z must be in $\nabla_{new}(b2)$, so it is either $b20$ or $b21$. Consequently, in N_4 the predecessor of the greatest 46 elements is of the form $b2de$ with $d, e \in \{0,1\}$, and its upper new–shadow is of cardinality 4. This implies that the upper shadow of the greatest 47 elements of N_4 has cardinality 176, and we obtained the desired contradiction. ∎

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