On Orthogonal Double Covers by Hamilton Paths

Volker Leck
Universität Rostock, Fachbereich Mathematik
18051 Rostock
Germany

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Abstract

An Orthogonal Double Cover (ODC) of the complete graph K_n is a set of isomorphic graphs such that two of them share exactly one edge and all together cover the complete graph twice. In the case of paths the problem is also known as self-orthogonal Hamilton path decomposition of $2K_n$. Solutions generated by a group correspond to self-orthogonal 2-sequencings of this group. We present small solutions which can be applied to a well-known recursive construction. This gives new classes of such ODCs.

1 Introduction

Let K_n denote the complete graph with n vertices ($n \geq 2$). A collection $\mathcal{G}$ of n spanning subgraphs $\mathcal{G} = \{G_1, \ldots, G_n\}$ of K_n is an *orthogonal double cover* (ODC) of K_n if

1. every edge of K_n belongs to exactly two subgraphs
2. any two distinct subgraphs have exactly one common edge.

The elements of $\mathcal{G}$ are called *pages*. If all of them are isomorphic to a graph G we call $\mathcal{G}$ an ODC by G. Clearly, G must have $n-1$ edges. ODCs are also known as self-orthogonal decompositions of $2K_n$. An example is given in figure 1.

ODCs have been considered for several classes of graphs (cf.[1, 11]), e.g. triangle factors [7], short cycles [8, 17], graphs of maximum degree 2 [10], trees of small diameter [9, 19, 18, 16]. Also the directed analogue was examined. But the problem considered first is that of an ODC by an "almost-hamiltonian" cycle, that is a cycle of length $n-1$ [14, 13, 2]. It turns out that such an ODC (of K_{4n}) can be constructed from an ODC (of K_n) by a Hamilton path [12]. Hence ODCs by paths become of interest.

In [12] an ODC of K_n by a Hamilton path P_n is given for $2 \leq n \leq 20$ and $n \neq 4$. Most of these ODCs are *cyclic*, that means generated by the cyclic group; all pages can be derived from a given labeled path P by adding k with respect to

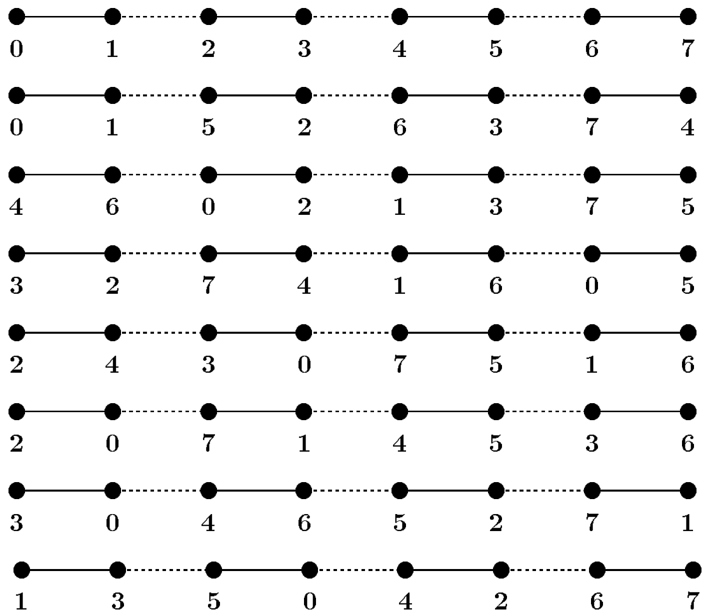

Figure 1: an ODC of K_8 by the path P_8

the cyclic group to each vertex label, $0 \leq k \leq n-1$. A path P with this property is called *ODC–starter* with respect to the cyclic group. This idea extends to arbitrary groups.

Let the vertex set of K_n be the elements of an abelian group $\mathcal{A}$. Define the *length* ℓ of edge xy by the set of differences $\ell(xy) := \{x-y, y-x\}$. Two edges x_1y_1 and x_2y_2 of the same length have *distance* k ($k>0$), if either adding k to x_1 and to y_1 or subtracting k from both gives the vertices x_2 and y_2 (in some order). Clearly, if two edges have distance k they also have distance $-k$. If $x-y = y-x$ (i.e. $x-y$ is an element of order 2) then xy has distance $k=-k=x-y$ to itself.

The following lemma gives necessary and sufficient conditions for a graph to be an ODC–starter with respect to $\mathcal{A}$.

Lemma 1 (cf. [15]) *A graph G is an ODC-starter with respect to $\mathcal{A}$ iff the following conditions hold:*

1. *For every $a \in \mathcal{A}$ with order 2 there is exactly one edge e with $a \in \ell(e)$.*
2. *For every $a \in \mathcal{A}$ with order not 2 there are exactly two distinct edges e_1 and e_2 with $a \in \ell(e_1)$ and $a \in \ell(e_2)$.*
3. *The set of distances is $\mathcal{A} \setminus \{0\}$.*

An ODC by a Hamilton path P_n is said to be 2*–colorable* if the edges of each path of the ODC can be colored alternately such that every edge of the complete graph is covered twice by the same color (see [15]). Such an ODC cannot exist for $n \equiv 3 \bmod 4$. The additional condition for an ODC-starter to generate a 2–colorable ODC is that edges of the same length have the same color, i.e. there is an odd number of edges inbetween any such pair. The following theorem gives a construction that makes use of 2-colorable ODCs.

Theorem 1 (**[15]**) *Let $m \geq 5$ be a prime power and let $n \geq 3$. Suppose that there exist ODCs by P_m and by P_n. Furthermore, suppose that the ODC by P_m is 2–colorable. Then there exists an ODC by P_{mn}.*

In addition, 2–colorable ODC-starters are known for $n = 5, 9, 10, 12, 13, 17, 29$ [15]. Applying theorem 1 this gives infinite classes of ODCs by paths. Apart from this, such ODC-starters are known to exit asymptotically for certain classes of prime powers [4, 3].

2 Solutions from starters

If an ODC-starter is a path, then it corresponds to a *self–orthogonal 2–sequencing* of $\mathcal{A}$ (see [4]). A special class of such 2–sequencings can be constructed using starters. A *starter* (cf. [5]) in the additive abelian group $\mathcal{A}$, where $|\mathcal{A}| = 2n+1$, is a set of unordered pairs $S := \{\{s_i, t_i\} : 1 \leq i \leq n\}$ which satisfies:

1. $\{s_i : 1 \leq i \leq n\} \cup \{t_i : 1 \leq i \leq n\} = \mathcal{A} \setminus \{0\}$
2. $\{\pm(s_i - t_i) : 1 \leq i \leq n\} = \mathcal{A} \setminus \{0\}$

A *translate* $S + a$ of a starter S is the set $\{\{s_i + a, t_i + a\} : 1 \leq i \leq n\}$ for some fixed $a \in \mathcal{A}$.

Consider the union of a starter S_1 and a translate $S_2 + a$ and let the pairs of this union be edges. By the starter definition, condition 1 and 2 of Lemma 1 hold. If also condition 3 holds, then S_1 and $S_2 + a$ are said to be *skew–orthogonal*, i.e. they form an ODC–starter with respect to $\mathcal{A}$. If this ODC-starter is a path we speak of a *starter–translate* self–orthogonal 2–sequencing.

In [4] the authors conjecture that if $2n+1 \geq 11$ then $\mathbb{Z}_{2n+1}$ has a starter-translate self–orthogonal 2–sequencing.

There is a very efficient Hill Climbing approach to construct orthogonal starters [6]. This can easily be modified to find a starter and a starter-translate which are skew-orthogonal. If in addition the union of them forms a path, we have found the desired ODC. We implemented this construction. The results are summarized in the following lemma.

Lemma 2 *Let $11 \leq 2n+1 \leq 101$. Then $\mathbb{Z}_{2n+1}$ has a starter-translate self-orthogonal 2–sequencing.*

Unfortunately, a starter-translate self-orthogonal 2–sequencing is never 2–colorable since edges of the same length lie in different starters and thus receive different colors. In order to construct 2–colorable solutions we change from starters to half–starters.

3 Solutions from half–starters

A *half–starter* [4] in the additive abelian group $\mathcal{A}$, where $|\mathcal{A}| = 4n+1$, is a set of unordered pairs $H := \{\{s_i, t_i\} : 1 \leq i \leq 2n\}$ which satisfies:

1. $\{s_i : 1 \leq i \leq 2n\} \cup \{t_i : 1 \leq i \leq 2n\} = \mathcal{A} \setminus \{0\}$

2. $\{\pm(s_i - t_i) : 1 \leq i \leq 2n\}$ contains exactly half the nonzero elements of $\mathcal{A}$ and each such element occurs exactly twice.

Again a *translate* $H + a$ of a half–starter H is the set $\{\{s_i + a, t_i + a\} : 1 \leq i \leq 2n\}$ for some fixed $a \in \mathcal{A}$.

Consider the union of a half–starter H_1 and a translate $H_2 + a$ and let the pairs in this union be edges. By the half–starter definition, condition 1 and 2 of Lemma 1 hold. If also condition 3 holds, then H_1 and $H_2 + a$ are said to be *supplementary*, i.e. they form an ODC–starter with respect to $\mathcal{A}$. If this ODC–starter is a path, then this path is 2–colorable since edges of the same half–starter receive the same color.

It is straight forward to modify the Hill Climbing algorithm [6] to construct supplementary half–starters. Also the case when n is even and there is exactly on element of order 2 in the group can be included by fixing the edge with length of order 2 before starting the algorithm. We implemented this modified algorithm. The results are summarized in the following lemma.

Lemma 3 *Let* $10 \leq n \leq 101$, $n \not\equiv 3 \bmod 4$. *Then* $\mathbb{Z}_n$ *has a self-orthogonal* 2*–sequencing consisting of a half–starter and a translate of a half–starter which are supplementary.*

It is known [15] that there are no group–generated 2–colorable ODCs by P_6 and P_8. A complete computer search showed that there is in fact no 2–colorable ODC by P_6. But we found one for P_8 (see figure 1). Applying theorem 1 we can summarize the results in the following theorem.

Theorem 2 *Let* $n = 2^a q$, *where* q *is a product of (not necessarily distinct) prime powers* p_i *with* $p_i \equiv 1 \bmod 4$, $5 \leq p_i \leq 101$ *and* $a \geq 0, a \neq 1, 2$. *Further suppose that there exists an ODC of* K_m *by a Hamilton path. Then there exists an ODC of* K_{nm} *by a Hamilton path.*

Remark

The 2–sequencings in Lemma 2 and 3 are available at the author.

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