Interpolation on sparse grids and Nikol'skij–Besov spaces of dominating mixed smoothness

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Dedicated to Professor P.L. Butzer on the occasion of his 70th birthday

Running title:

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Abstract

We investigate the order of convergence of periodic interpolation on sparse grids (blending interpolation) in the framework of Nikol'skij–Besov spaces of dominating mixed smoothness. The main ingredients are a unified approach to error estimates in univariate Nikol'skij–Besov spaces and the tensor product characterization of the bivariate Nikol'skij–Besov spaces of dominating mixed smoothness.

Keywords: Periodic interpolation, Sparse grids, Nikol'skij–Besov spaces of dominating mixed smoothness.

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1 Introduction

In this paper, we deal with the error of approximation of periodic functions obtained by interpolation on sparse grids. The method itself is well–known, cf. e.g. Delvos and Schempp [4], Baszenski and Delvos [2], Pöplau and Sprengel [15] and Sprengel [22, 23]. What is new here is the choice of the underlying function spaces. Delvos and Schempp [4], Baszenski and Delvos [2] and Pöplau and Sprengel [15] dealt with tensor products of spaces defined by certain decay properties of the Fourier coefficients (Korobov spaces or potential spaces built on L_2, respectively). Based on [21], we are able to investigate the problem in more appropriate spaces, namely on Nikol'skij–Besov spaces and its tensor products, the classes $S^{s_1,s_2}_{p,\infty}B$.

Interpolation on sparse grids can be reduced to error estimates of corresponding interpolation processes in the one–dimensional situation. As a result we obtain estimates like

$$\| f - B_j f \,|L_p(\mathbb{T}^2)\| \leq C\, j\, 2^{-js}\,, \tag{1}$$

where B_j denotes the interpolation operator with respect to the sparse grid (having $j\,2^j$ knots approximately). Comparing this with interpolation on the full grid (having 2^{2j} knots approximately), we would end with an estimate

$$\| f - I_j f \,|L_p(\mathbb{T}^2)\| \leq C\; 2^{-js}\,. \tag{2}$$

So, except a logarithmic term the error is of the same order. The price we have to pay consists in the following: whereas (2) is true for all functions taken from the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T}^2)$ inequality (1) holds true for elements of the space of dominating mixed smoothness $S^{s,s}_{p,\infty}B(\mathbb{T}^2)$ only.

We restrict ourselves to the bivariate situation mostly for transparency. All can be done also in the higher dimensional situation.

Investigations of approximation procedures in spaces of dominating mixed smoothness attracted much attention in last time, cf. e.g. the monograph of Temlyakov [24] or the recent papers by DeVore, Konyagin and Temlyakov [5] and DeVore, Petrushev and Temlyakov [6].

The paper is organized as follows. In Section 2, we collect information about the underlying spaces. Further, to prepare those estimates as in (1), we investigate the approximation power of certain interpolation processes in the univariate situation. As a vehicle of (1), we have to deal also with error estimates in norm $\|\cdot|B^0_{p,1}(\mathbb{T})\|$. This will be done by the same methods (Fourier multipliers) as in case of $\|\cdot|L_p(\mathbb{T})\|$, presented before. All this can be found in Section 3. Finally, Section 4 deals with interpolation on sparse grids.

2 Periodic Besov spaces

2.1 Besov spaces, moduli of smoothness and best approximation

Recall some definitions. For our purpose, it will be sufficient to deal with the one–dimensional situation. Let $1 < p < \infty$. As usual, for a natural number m and $f \in L_p(\mathbb{T})$, we put

$$\omega_m(t,f)_p = \sup_{|h|<t} \|\, \Delta^m_h f \,|L_p(\mathbb{T})\|\,, \qquad t > 0\,,$$

where Δ_h^m denotes an m–th order difference with step–length h. Let $m > s$. Then the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T})$ is defined as the set of all 2π–periodic functions in $L_p(\mathbb{T})$ such that

$$\| f \,|B^s_{p,\infty}(\mathbb{T})\| = \| f \,|L_p(\mathbb{T})\| + \sup_{t<1} t^{-s}\, \omega_m(t,f)_p < \infty \,.$$

Different m lead to equivalent norms. We refer to Nikol'skij [12], Triebel [27] and DeVore and Lorentz [7] for the basics of theory of Besov spaces.

2.2 Besov spaces of dominating mixed smoothness

Here, we restrict ourselves to the two–dimensional situation. To introduce mixed differences we need the following modification. For natural numbers m_1, m_2, real numbers h_1, h_2, and $f \in L_p(\mathbb{T})$ let $\Delta^{m_1}_{h_1,1} f(x_1,x_2)$ be the difference of order m_1 taken in the first variable. Analoguously, $\Delta^{m_2}_{h_2,2} f(x_1,x_2)$ is defined. Finally, we put

$$\Delta^{(m_1,m_2)}_{h_1,h_2} f(x_1,x_2) = \Delta^{m_2}_{h_2,2}\,(\Delta^{m_1}_{h_1,1} f)(x_1,x_2)\,.$$

Definition 1 *Let $1 < p < \infty$ and s_1, $s_2 > 0$. Let m_1 and m_2 denote natural numbers such that $s_1 < m_1$ and $s_2 < m_2$. Then the Nikol'skij–Besov space of dominating mixed smoothness $S^{r_1,r_2}_{p,\infty}B(\mathbb{T}^2)$ is the set of all functions $f \in L_p(\mathbb{T}^2)$ such that*

$$\begin{aligned} \sup_{|h_1|>0} |h_1|^{-s_1} \| \Delta^{m_1}_{h_1,1} f \,|L_p(\mathbb{T}^2)\| \quad &+ \quad \sup_{|h_2|>0} |h_2|^{-s_2} \| \Delta^{m_2}_{h_2,2} f \,|L_p(\mathbb{T}^2)\| \\ &+ \quad \sup_{|h_1|>0} \sup_{|h_2|>0} |h_1|^{-s_1} |h_2|^{-s_2} \| \Delta^{(m_1,m_2)}_{h_1,h_2} f \,|L_p(\mathbb{T}^2)\| < \infty \,. \end{aligned}$$

For a detailed investigation of these classes of functions, we refer to Amanov [1] and Schmeisser and Triebel [20], cf. also Schmeisser [19]. In addition we refer to the recent monograph by Temlyakov [24], where similar spaces (but not the same) are investigated. In what follows, we need the following properties:

$$S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$$

if $\min(s_1,s_2) > \frac{1}{p}$, cf. [20, Chap. 2.4] and

$$B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_\lambda B^{s_2}_{p,\infty}(\mathbb{T}) = S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)\,, \tag{3}$$

where λ denotes a certain modification of the injective tensor norm, cf. [21] for details. The validity of (3) is not restricted to values of s_1, $s_2 > 0$ and $q = \infty$, it remains true for all $s_1, s_2 \in \mathbb{R}$ and all $0 < q \le \infty$. In particular, if $s_1 = s_2 = 0$ we have

$$B^0_{p,q}(\mathbb{T}) \otimes_\lambda B^0_{p,q}(\mathbb{T}) = S^{0,0}_{p,q}B(\mathbb{T}^2)\,.$$

For definitions and further characterizations of these more general Besov spaces, we refer to Schmeisser [19] or [21]. Also proved in [21], is the following (which is the main advantage of the tensor product characterization of $S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)$): Suppose $P, Q \in \mathcal{L}(C(\mathbb{T}))$ and define

$$(P \otimes Q)\,(f(x_1) \otimes g(x_2)) = P(f)(x_1) \otimes Q(g)(x_2)$$

for all $f, g \in C(\mathbb{T})$. Moreover, we assume $s_1, s_2 > \frac{1}{p}$, $P \in \mathcal{L}(B^{s_1}_{p,\infty}(\mathbb{T}), B^0_{p,1}(\mathbb{T}))$ and $Q \in \mathcal{L}(B^{s_2}_{p,\infty}(\mathbb{T}), B^0_{p,1}(\mathbb{T}))$. Then the restriction of $P \otimes Q$ to $S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)$ belongs to $\mathcal{L}(S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2), S^{0,0}_{p,1}B(\mathbb{T}^2))$ and

$$\begin{aligned} &\| (P \otimes Q)\, h \,|S^{0,0}_{p,1}B(\mathbb{T}^2)\| \\ &\quad \le C \,\| P \,|\mathcal{L}(B^{s_1}_{p,\infty}(\mathbb{T}), B^0_{p,1}(\mathbb{T}))\| \,\| Q \,|\mathcal{L}(B^{s_2}_{p,\infty}(\mathbb{T}), B^0_{p,1}(\mathbb{T}))\| \,\| h \,|S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)\| \end{aligned} \tag{4}$$

with some constant C independent of P, Q and $h \in S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)$.

3 Interpolation on equidistant grids

We are interested in chains of interpolation operators. For technical reasons, we will be restricted to those ones which are induced by interpolation processes on the real line. We shall start with recalling some well–known results of Ries and Stens [17] for approximation by sampling sums in the supremum–norm. Then we investigate the case $1 < p < \infty$. In a final step, we use complex interpolation of Besov spaces to weaken the restrictions under which we have derived our estimates for $1 < p < \infty$.

3.1 Preliminaries

Let N be a natural number and denote by $J_N = \{k \in \mathbb{Z} :\ -N/2 \le k < N/2\}$ a related set of indices. Further

$$T_N = \Big\{ \sum_{k \in J_N} \eta_k\, e^{ikx} \ :\ \eta_k \in \mathbb{C} \Big\}$$

denotes a corresponding set of trigonometric polynomials. The discrete Fourier coefficients of a continuous function f are given by

$$c_k^N(f) = \frac{1}{N} \sum_{\ell \in J_N} f\Big(\frac{2\pi\ell}{N}\Big)\, e^{i2\pi k\ell/N}\,, \qquad k \in J_N\,.$$

Discrete Fourier coefficients $c_k^N(f)$ and Fourier coefficients $c_k(f)$ of a function f are connected by aliasing

$$c_k^N(f) = \sum_{\ell \in \mathbb{Z}} c_{k+\ell N}(f)\,,$$

as long as f belongs to the Wiener algebra $\mathcal{A}(\mathbb{T})$ of functions having an absolutely summable Fourier series. We shall consider interpolation on equidistant grids of type $\mathcal{T}_N = \{\, 2\pi k/N \ :\ k \in J_N \}$. The continuous and 2π–periodic function Λ_N^π is called a fundamental interpolant for $\mathcal{T}_N$ if

$$\Lambda_N^\pi\Big(\frac{2\pi k}{N}\Big) = \begin{cases} 1 & \text{if} \quad k = 0\,, \\ 0 & \text{if} \quad k \ne 0\,, \quad k \in J_N\,. \end{cases}$$

The associated interpolation operator I_N is defined as

$$I_N f(x) = \sum_{k \in J_N} f\Big(\frac{2\pi k}{N}\Big)\, \Lambda_N^\pi\Big(x - \frac{2\pi k}{N}\Big)\,. \tag{5}$$

The Fourier coefficients of $I_N f$ can be easily computed as

$$c_k(I_N f) = N\, c_k^N(f)\, c_k(\Lambda_N^\pi) = N\, c_k(\Lambda_N^\pi) \sum_{\ell \in \mathbb{Z}} c_{k+\ell N}(f) \tag{6}$$

as long as $f \in \mathcal{A}(\mathbb{T})$. Finally, we denote the N–th Fourier partial sum by

$$S_N f(x) = \sum_{k \in J_N} c_k(f)\, e^{ikx}\,, \qquad N \in \mathbb{N}\,, \qquad f \in L_1(\mathbb{T})\,.$$

3.2 The error of approximation in the supremum–norm

On the real line, there is a well–developed theory of so–called quasi–interpolation. Given some basic function ϕ then the rate of convergence of

$$\left\| f(x) - \sum_{k \in \mathbb{Z}} \alpha_{h,k}(f)\, \phi(hx-k) \,\middle|\, L_p(\mathbb{R}) \right\|, \qquad h \to 0\,, \quad h > 0\,,$$

can be determined in dependence of the local reproduction of polynomials by $\sum_{k \in \mathbb{Z}} \alpha(1,k)\, \phi(x-k)$. Here $\alpha_{h,k}$ are appropriate functionals, cf. DeVore and Lorentz [7, Chap. 13.7] and Jia and Lei [10] for details. Up to now, there seems to be no complete periodic counterpart. However, if we consider approximation in $\| \cdot |C(\mathbb{T})\|$, then simple arguments allow a transformation of estimates obtained in the non–periodic situation to the periodic one.

Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\Lambda(2\pi k) \;=\; \delta_{0,k}\,, \qquad k \in \mathbb{Z}\,, \tag{7}$$

$$\sum_{k \in \mathbb{Z}} |\Lambda(x - 2\pi k)| \qquad \text{converges uniformly on} \quad [0, 2\pi]\,. \tag{8}$$

The second assumption guarantees $\Lambda \in L_1(\mathbb{R})$ and hence, the Fourier transform of Λ

$$(\mathcal{F}\Lambda)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi}\, \Lambda(x)\, dx$$

is a continuous function on $\mathbb{R}$. Hence, the following definition makes sense

$$\Lambda_N^{\pi}(x) = \sum_{\ell=-\infty}^{\infty} \Lambda(Nx - 2\ell\pi N)\,, \qquad N \in \mathbb{N}\,, \quad x \in \mathbb{T}\,. \tag{9}$$

Some simple properties are collected in the following lemma.

Lemma 1 *Let Λ be a continuous function satisfying (7) and (8).*
(i) *The function Λ_N^{π} defines a 2π-periodic continuous fundamental interpolant, i.e.*

$$\Lambda_N^{\pi}\Big(\frac{2k\pi}{N}\Big) = \delta_{0,k}\,, \qquad k \in J_N\,.$$

Its Fourier series is given by

$$\frac{1}{N\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \mathcal{F}\Lambda\Big(\frac{k}{N}\Big)\, e^{ikx}\,.$$

(ii) *We have the equivalence of*

$$\sum_{\ell=-\infty}^{\infty} \Lambda(x - 2\pi\ell) = 1\,, \qquad x \in \mathbb{T}\,, \tag{10}$$

and

$$\mathcal{F}\Lambda(k) = \sqrt{2\pi}\, \delta_{0,k}\,, \quad k \in \mathbb{Z}\,. \tag{11}$$

(iii) *If Λ satisfies a refinement equation*

$$\frac{1}{2}\Lambda\Big(\frac{x}{2}\Big) = \sum_{k\in\mathbb{Z}} h_k\, \Lambda(x+2\pi k)\,, \qquad x\in\mathbb{R}\,, \tag{12}$$

with the mask $\{h_k\}_{k\in\mathbb{Z}}\in \ell_1$, then

$$\Lambda_N^\pi(x) = 2\sum_{k=0}^{2N-1}\Big(\sum_{r=-\infty}^{\infty} h_{k+2rN}\Big)\,\Lambda_{2N}^\pi\Big(x+\frac{2\pi k}{2N}\Big)\,.$$

Proof Continuity of Λ_N^π becomes a consequence of (8). Also the interpolation conditions can be derived from this requirement taking into account (7). The calculation of the Fourier series is elementary. Part (ii) is taken from Ries and Stens [17]. Finally, (iii) is again elementary. ■

The absolute moment of order $\alpha > 0$ of Λ is given by

$$m_\alpha(\Lambda) = \sup_{0\le x\le 2\pi}\sum_{k\in\mathbb{Z}} |x-2\pi k|^\alpha\, |\Lambda(x-2\pi k)|\,.$$

Proposition 1 *Let $\Lambda:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (7), (8) and (11). Let Λ_N^π and I_N be the associated fundamental interpolant and interpolation operator, respectively.*
(i) *We assume $m_\alpha(\Lambda)<\infty$ for some $0<\alpha<1$. Let $0<s\le\alpha$. Then there exists a constant C (independent of N) such that*

$$\|\, f - I_N f\,|C(\mathbb{T})\| \le C\, N^{-s}\, \|\, f\,|B^s_{\infty,\infty}(\mathbb{T})\| \tag{13}$$

holds for all $f\in B^s_{\infty,\infty}(\mathbb{T})$.
(ii) *We assume $m_r(\Lambda)<\infty$ and $(\mathcal{F}\Lambda)^{(j)}(k)=0\,,\ j=1,2,\ldots,r-1\,,\ k\in\mathbb{Z}\,,$ for some $r\in\mathbb{N}$. Let $0<s<r$. Then (13) holds.*

Proof Employing periodicity of the interpolated function and Lemma 1, we obtain

$$\begin{aligned} f(x) - I_N f(x) &= \sum_{k=0}^{N-1}\Big(f(x) - f\Big(\frac{2\pi k}{N}\Big)\Big)\,\Lambda_N\Big(x-\frac{2\pi k}{N}\Big) \\ &= \sum_{k=0}^{N-1}\Big(f(x) - f\Big(\frac{2\pi k}{N}\Big)\Big)\sum_{\ell=-\infty}^{\infty}\Lambda\Big(Nx-2\pi k-2\pi\ell N\Big) \\ &= \sum_{m=-\infty}^{\infty}\Big(f(x) - f\Big(\frac{2\pi m}{N}\Big)\Big)\,\Lambda(Nx-2\pi m)\,. \end{aligned}$$

Hence, the periodic situation can be traced back to the non–periodic one; for this see Ries and Stens [17]. ■

Remark 1 As we stated before Proposition 1 has been proved by Ries and Stens [17] in the non–periodic case.

3.3 The error of approximation in the L_p-norm

We consider functions $f\in B^s_{p,\infty}(\mathbb{T})\cap\mathcal{A}(\mathbb{T})$. Our approach is based on Fourier multiplier assertions and this requires additional restrictions compared with the preceding subsection.

In what follows, we shall need from time to time a smooth cut–off function. To this end, let ψ be a C^∞–function defined on the real line such that $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge 2$.

Proposition 2 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (7) and (8). Moreover, we assume*

$$\int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi(\xi)\, |\xi|^{-\alpha} \Big(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}} \Big) \Big\}(w) \right| dw < \infty\,, \tag{14}$$

$$\sum_{\ell \neq 0} \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi(\xi)\, |\xi|^{-\alpha}\, \mathcal{F}\Lambda(\xi+\ell) \Big\}(w) \right| dw < \infty\,, \tag{15}$$

$$\sum_{\ell \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ (1-\psi(\xi))\, |\xi|^{-\beta}\, \mathcal{F}\Lambda(\xi-\ell) \Big\}(w) \right| dw < \infty \tag{16}$$

for some $\alpha > 0$ and $\beta \ge 0$.
Let Λ_N^π (cf. (9)) and I_N (cf. (5)) be the associated fundamental interpolants and interpolation operator, respectively. Let $0 < s < \alpha$, $\beta < s$ and let $1 < p < \infty$. Then there exists a constant C (independent of N) such that

$$\| f - I_N f \,|L_p(\mathbb{T})\| \le C\, N^{-s}\, \| f \,|B^s_{p,\infty}(\mathbb{T})\| \tag{17}$$

holds for all $f \in B^s_{p,\infty}(\mathbb{T}) \cap \mathcal{A}(\mathbb{T})$.

Remark 2 In case $p = 2$ one can simplify the conditions on Λ very much. Even one can avoid to take the long way around the non–periodic case. All conditions can be formulated in terms of the Fourier coefficients of the periodic fundamental interpolant Λ_N^π, cf. Brumme [3] or Pöplau and Sprengel [15].

We subdivide the proof of (17) into two lemmata. In the first one, we investigate the error for trigonometric polynomials taken from T_N.
Fairly often we have to consider sums of the type $\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} a_m$. In such situation, we simply write $\sum_{m\neq 0} a_m$.

Lemma 2 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a function satisfying (7), (8), (14) and (15).*
Let $0 < s < \alpha$ and $1 < p < \infty$. Then there exists a constant C (independent of N) such that

$$\| f - I_N f \,|L_p(\mathbb{T})\| \le C\, N^{-s}\, \| f \,|B^s_{p,\infty}(\mathbb{T})\| \tag{18}$$

holds for all $f \in T_N$.

Proof We shall employ the following Fourier–analytical characterization of Nikol'skij–Besov spaces. We put

$$\| f \,|B^s_{p,\infty}(\mathbb{T})\|^* = |\, c_0(f)\, | + \sup_{j=0,1,\ldots} 2^{js} \left\| \sum_{2^j \le |k| < 2^{j+1}} c_k(f)\, e^{ikx} \,\Big|L_p(\mathbb{T}) \right\|. \tag{19}$$

Then $f \in B^s_{p,\infty}(\mathbb{T})$ if and only if $f \in L_p(\mathbb{T})$ and $\| f \,|B^s_{p,\infty}(\mathbb{T})\|^* < \infty$. Moreover, $\| \cdot \,|B^s_{p,\infty}(\mathbb{T})\|^*$ yields an equivalent norm, cf. e.g. [20, Chap. 3.7.2].
Our assumption $f \in T_N$ together with Lemma 1 and (6) yield

$$\begin{aligned} f(x) - I_N f(x) &= \sum_{m=-\infty}^{\infty} \Big[c_m(f) - N \sum_{\ell \in \mathbb{Z}} c_{m+\ell N}(f)\, c_m(\Lambda_N^\pi) \Big]\, e^{imx} \\ &= \sum_{m \neq 0} \Big[c_m(f) - N\, c_m(f)\, \frac{1}{N\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big) \Big]\, \psi\Big(\frac{m}{N}\Big)\, e^{imx} \\ &\quad - \sum_{m \in \mathbb{Z}} \sum_{\ell \neq 0} \frac{1}{\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\, c_{m+\ell N}(f)\, e^{imx}\,. \end{aligned}$$

The two parts of the sum we denote by

$$
\begin{aligned}
F_1(x) &= \sum_{m\in\mathbb{Z}} \Big[c_m(f) - N\, c_m(f)\, \frac{1}{N\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\Big]\, \psi\Big(\frac{m}{N}\Big)\, e^{imx} \\
F_2(x) &= \sum_{m\neq 0}\sum_{\ell\neq 0} \frac{1}{\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\, c_{m+\ell N}(f)\, e^{imx}\, .
\end{aligned}
$$

Next, we employ that the function

$$
M(\xi) = \psi(\xi/N)\, |\xi|^{-\alpha} \left(1 - \frac{\mathcal{F}\Lambda(\xi/N)}{\sqrt{2\pi}}\right)
$$

is continuous in $\mathbb{R}\setminus\{0\}$ and has an absolutely integrable Fourier transform. Hence, interpreting F_1 as a convolution (cf. [20, Chap. 3.3.4, formula (2)]), we find

$$
\begin{aligned}
\| F_1\, | L_p(\mathbb{T})\| &= \left\| \sum_{m\neq 0} c_m(f)\ \psi\Big(\frac{m}{N}\Big) \left(1 - N\, \frac{1}{N\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\right) e^{imx}\, \Big| L_p(\mathbb{T}) \right\| \\
&= \frac{1}{\sqrt{2\pi}} \left\| \int_{-\infty}^{\infty} \Big(\mathcal{F}^{-1} M(y)\Big) \Big(\sum_{m\neq 0} c_m(f)\, |m|^{\alpha}\, e^{im(x-y)}\Big)\, dy\ \Big| L_p(\mathbb{T}) \right\| \\
&\le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} M(y)|\, dy \qquad \left\| \sum_{m\neq 0} c_m(f)\, |m|^{\alpha}\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| . \qquad (20)
\end{aligned}
$$

Observe, that the homogeneity of the Fourier transform implies

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\mathcal{F}^{-1} M(y)|\, dy &= \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi\Big(\frac{\xi}{N}\Big)\, |\xi|^{-\alpha} \Big(1 - \frac{\mathcal{F}\Lambda(\xi/N)}{\sqrt{2\pi}}\Big)\Big\}(y) \right| dy \\
&= N^{-\alpha} \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi(\xi)\, |\xi|^{-\alpha} \Big(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\Big)\Big\}(y) \right| dy \\
&= N^{-\alpha}\, C\, , \qquad (21)
\end{aligned}
$$

where C is a positive number independent of N. It remains to estimate $\| \sum_{m\neq 0} c_m(f)\, |m|^{\alpha}\, e^{imx}\, | L_p(\mathbb{T})\|$. Suppose $2^r \le N < 2^{r+1}$. We find

$$
\begin{aligned}
N^{-\alpha} \left\| \sum_{m\neq 0} c_m(f)\, |m|^{\alpha}\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| &\le N^{-\alpha} \sum_{\ell=0}^{r} \left\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, |m|^{\alpha}\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| \\
&\le C_1\, N^{-\alpha} \sum_{\ell=0}^{r} 2^{\ell\alpha} \left\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| \\
&\le C_1\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|\, N^{-\alpha} \sum_{\ell=0}^{r} 2^{\ell(\alpha - s)} \\
&\le C_2\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|\, N^{-s}\, , \qquad (22)
\end{aligned}
$$

where we used the Fourier multiplier assertion

$$
\left\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, 2^{-\ell\alpha} |m|^{\alpha}\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| \le C_1 \left\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, e^{imx}\, \Big| L_p(\mathbb{T}) \right\| .
$$

Here C_1 can be estimated as follows

$$
C_1 \le c\, \|\, |\xi|^{\alpha}\, \chi(\xi)\, | W^1_2(\mathbb{R})\|\, ,
$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $\chi(\xi) = 1$ if $1 \le \xi \le 2$ and $\chi(\xi) = 0$ if $|\xi| < 1/2$, cf. Schmeisser and Triebel [20, Chap. 3.3.4]. Hence, C_1 is bounded independent of ℓ. Consequently, C_2

becomes a constant independent of N. Plugging this into (20), taking into account (21), then (18) follows with F_1 instead of $f - I_N f$.

We shall derive an estimate of F_2. We put

$$M_\ell(\xi) = \psi\Big(\ell + \frac{\xi}{N}\Big)\, |\xi + \ell N|^{-\alpha}\, \mathcal{F}\Lambda\Big(\frac{\xi}{N}\Big)\, .$$

Similar as above we obtain

$$\begin{aligned} \| F_2 \,| L_p(\mathbb{T})\| &\le \Big\| \sum_{m\in\mathbb{Z}} \sum_{\ell\neq 0} c_{m+\ell N}(f)\, \mathcal{F}\Lambda(\frac{m}{N})\, e^{imx} \Big| L_p(\mathbb{T})\Big\| \\ &\le \sum_{\ell\neq 0} \Big\| \int_{-\infty}^{\infty} \Big(\mathcal{F}^{-1} M_\ell(y)\Big) \Big(\sum_{m\in\mathbb{Z}} c_{m+\ell N}(f)\, |m+\ell N|^\alpha\, e^{i(m+\ell N)(x-y)}\Big)\, dy\, \Big| L_p(\mathbb{T})\Big\| \\ &\le \sum_{\ell\neq 0} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} M_\ell(y)|\, dy \qquad \Big\| \sum_{m\in\mathbb{Z}} c_m(f)\, |m|^\alpha\, e^{imx} \Big| L_p(\mathbb{T})\Big\| . \end{aligned}$$

The same arguments as used in case of F_1 can be applied to finish the estimate of F_2 (here one has to apply (15)) ending up with (18) but with F_2 instead of $f - I_N f$. ■

Remark 3 With some additional effort (one has to use a different equivalent norm in $B^s_{p,\infty}(\mathbb{T})$), one can extend the validity of above lemma to $p = 1$ and $p = \infty$.

Lemma 3 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ satisfy (7), (8) and (16).*
Let $\beta < s$ and let $1 < p < \infty$. Then there exists a constant C (independent of N) such that

$$\| I_N(f - S_N f)\, | L_p(\mathbb{T})\| \le C\, N^{-s}\, \| f\, | B^s_{p,\infty}(\mathbb{T})\| \tag{23}$$

holds for all $f \in B^s_{p,\infty}(\mathbb{T}) \cap \mathcal{A}(\mathbb{T})$.

In the proof of this lemma, we will also need Sobolev spaces $H^s_p(\mathbb{T})$ of fractional order on the torus equipped with the norm

$$\| f\, | H^s_p(\mathbb{T})\| = \Big\| \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s/2}\, c_k(f)\, e^{ikx} \Big| L_p(\mathbb{T})\Big\| .$$

Proof By assumption, f is continuous and has absolutely summing Fourier coefficients. So we may apply (6), too. Similar as in proof of the preceding lemma, we derive

$$I_N(f - S_N f)(x) = \sum_{\ell\in\mathbb{Z}} \sum_{m=-\infty}^{\infty} \Big[\Big(1 - \psi\Big(2\,\frac{m+\ell N}{N}\Big)\Big) N\, c_{m+\ell N}(f - S_N f)\, \frac{1}{N\sqrt{2\pi}}\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\Big]\, e^{imx}\, .$$

This yields

$$|\, I_N(f - S_N f)(x)\, | \le \sum_{\ell\in\mathbb{Z}} \Big| \sum_{m=-\infty}^{\infty} \Big[\Big(1 - \psi\Big(2\,\frac{m}{N} + 2\,\ell\Big)\Big)\, c_{m+\ell N}(f - S_N f)\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\Big]\, e^{i(m+\ell N)x} \Big| .$$

Employing that the functions

$$M_\ell(\xi) = \Big(1 - \psi\Big(2\,\frac{\xi}{N} + 2\,\ell\Big)\Big)\, |\xi + \ell N|^{-\beta}\, \mathcal{F}\Lambda\Big(\frac{\xi}{N}\Big)$$

are continuous in $\mathbb{R}$ and have integrable Fourier transforms, we obtain

$$
\begin{aligned}
&\| I_N(f - S_N f) | L_p(\mathbb{T})\| \\
&\leq \sum_{\ell\in\mathbb{Z}} \Big\| \sum_{m\in\mathbb{Z}} c_{m+\ell N}(f - S_N f) \Big(1 - \psi\Big(2\,\frac{m}{N} + 2\,\ell\Big)\Big)\, \mathcal{F}\Lambda\Big(\frac{m}{N}\Big)\, e^{i(m+\ell N)x} \Big| L_p(\mathbb{T})\Big\| \\
&\leq \sum_{\ell\in\mathbb{Z}} \Big\| \int_{-\infty}^{\infty} \mathcal{F}^{-1} M_\ell(y) \Big(\sum_{m\in\mathbb{Z}} c_{m+\ell N}(f - S_N f)\, |m + \ell N|^\beta\, e^{i(m+\ell N)(x-y)} \Big)\, dy \Big| L_p(\mathbb{T})\Big\| \\
&\leq \sum_{\ell\in\mathbb{Z}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} M_\ell(y)|\, dy\, \Big\| \sum_{m\in\mathbb{Z}} c_{m+\ell N}(f - S_N f)\, |m + \ell N|^\beta\, e^{i(m+\ell N)x} \Big| L_p(\mathbb{T})\Big\| \\
&\leq \sum_{\ell\in\mathbb{Z}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} M_\ell(y)|\, dy\ \| f - S_N f\, | H_p^\beta(\mathbb{T})\| . \qquad (24)
\end{aligned}
$$

The homogeneity of the Fourier transform leads to

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\mathcal{F}^{-1} M_\ell(y)|\, dy &= \int_{-\infty}^{\infty} \Big| \mathcal{F}^{-1}\Big\{ \Big(1 - \psi\Big(2\,\frac{\xi}{N} + 2\,\ell\Big)\Big)\, |\xi + \ell N|^{-\beta}\, \mathcal{F}\Lambda\Big(\frac{\xi}{N}\Big)\Big\}(y) \Big|\, dy \\
&= N^{-\beta} \int_{-\infty}^{\infty} \big| \mathcal{F}^{-1}\big\{ \big(1 - \psi(2\,\xi + 2\,\ell)\big)\, |\xi + \ell|^{-\alpha}\, \mathcal{F}\Lambda(\xi)\big\}(y) \big|\, dy \\
&= N^{-\beta} \int_{-\infty}^{\infty} \big| \mathcal{F}^{-1}\big\{ \big(1 - \psi(2\,\xi)\big)\, |\xi|^{-\alpha}\, \mathcal{F}\Lambda(\xi - \ell)\big\}(y) \big|\, dy\, . \qquad (25)
\end{aligned}
$$

In case $s > \beta$ and q arbitrary, one knows

$$\| f - S_N f\, | B^\beta_{p,q}(\mathbb{T})\| \leq C\, N^{\beta - s}\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|\, ,$$

where $B^\beta_{p,q}(\mathbb{T})$ denotes a Besov space, cf. e.g. Pietsch [14]. Because of $B^\beta_{p,q}(\mathbb{T}) \hookrightarrow H^\beta_p(\mathbb{T})$, $q \leq \min(p, 2)$, cf. e.g. [20, Chap. 3.5.4], this yields

$$\| f - S_N f\, | H^\beta_p(\mathbb{T})\| \leq C\, N^{\beta - s}\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|\, , \qquad (26)$$

where again the constant C does not depend on N and f. Inserting this inequality into (24), taking into account (25), then (23) follows. ∎

Remark 4 Again, one could try to extend the lemma to values $p = 1$ and $p = \infty$. But then the approximation properties of the partial sums $S_N f$ are not longer sufficient. Replacing $S_N f$ by the following de la Vallée Poussin type means

$$V_N f(x) = \sum_{k\in\mathbb{Z}} \psi(k/N) c_k(f)\, e^{ikx}\, , \qquad N \in \mathbb{N}, \qquad f \in D'(\mathbb{T}),$$

one can prove

$$\| I_N(f - V_N f)\, | L_p(\mathbb{T})\| \leq C\, N^{-s}\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|$$

also if $p = 1$ or $p = \infty$.

Proof of Proposition 2 This can be done in one line now. Lemma 2, Lemma 3 and (26) (applied with $\beta = 0$) yield

$$
\begin{aligned}
\| f - I_N f\, | L_p(\mathbb{T})\| &\leq \| f - S_N f\, | L_p(\mathbb{T})\| + \| S_N f - I_N(S_N f)\, | L_p(\mathbb{T})\| \\
&\quad + \| I_N(f - S_N f)\, | L_p(\mathbb{T})\| \\
&\leq C\, N^{-s}\, \| f\, | B^s_{p,\infty}(\mathbb{T})\|\, .
\end{aligned}
$$

The proof is complete. ∎

Remark 5 We compare the conditions of Proposition 1 and Proposition 2. Because of

$$\begin{aligned} |\,\psi(\xi)\,|\xi|^{-\alpha}\Big(1-\frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\Big)| &\le \int_{-\infty}^{\infty}\Big|\mathcal{F}^{-1}\Big\{\psi(\xi)\,|\xi|^{-\alpha}\Big(1-\frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\Big)\Big\}(w)\Big|\,dw\,, \\ |\psi(\xi)\,|\xi|^{-\alpha}\,\mathcal{F}\Lambda(\xi+\ell)\,| &\le \int_{-\infty}^{\infty}\Big|\mathcal{F}^{-1}\Big\{\psi(\xi)\,|\xi|^{-\alpha}\,\mathcal{F}\Lambda(\xi+\ell)\Big\}(w)\Big|\,dw\,, \end{aligned}$$

we see that in case $\alpha > 1$ the conditions in Proposition 1 are satisfied if in addition $m_\alpha(\Lambda) < \infty$. In particular, if (14) and (15) are satisfied for some $\alpha > 0$, then (11) holds. Further, if $\mathcal{F}\Lambda$ is $(r-1)$–times differentiable in a neighborhood of the integers, then we may apply Proposition 1(ii) as long as $\alpha > r-1$ and $m_r(\Lambda) < \infty$.

Remark 6 We add a second observation. The inequality

$$|\,(1-\psi(\xi+\ell))\,|\xi+\ell|^{-\beta}\,\mathcal{F}\Lambda(\xi)\,| \le \int_{-\infty}^{\infty}|\,\mathcal{F}^{-1}\{\,(1-\psi(\xi+\ell)\,|\xi+\ell|^{-\beta}\,\mathcal{F}\Lambda(\xi))\,\}(w)\,|\,dw\,,$$

evaluated in point $\xi = 0$ and summed up over ℓ, shows that β must be strictly larger than 1 in any case.

Remark 7 Recall, if X is a Banach space and W a subspace of X, then the linear N–width is defined as

$$\lambda_N(W,X) = \inf_{\substack{L_N\in Lin_N(X)\\ \Lambda\in\mathcal{L}(X,L_N)}}\ \sup_{f\in W}\,\|\,f-\Lambda f\,|X\|\,,$$

where the infimum is taken over all subspaces L_N of X of finite dimension $\le N$ and all linear operators Λ from X to L_N. Here we are interested in $X = L_p(\mathbb{T})$ and W the unit ball in the Nikol'skij–Besov space $B^s_{p,\infty}(\mathbb{T})$, denoted by $B^s_p(\mathbb{T})$. If $s > 0$, then

$$\lambda_N(B^s_p(\mathbb{T}), L_p(\mathbb{T})) \sim N^{-s} \tag{27}$$

cf. Lorentz, von Golitschek and Makovoz [11, Theorem 14.3.8]. In this sense, approximation with those interpolation operators I_N is nearly optimal (nearly optimal means the order of approximation is correct but may be not the constants). More details about widths may be found in Tichomirov [25] and Lorentz, von Golitschek and Makovoz [11].

Remark 8 The conditions of Proposition 2 can be simplified by means of Szasz' theorem, cf. e.g. Jawerth [8] or [20, Proposition 1.7.5]. It holds that

$$\int_{-\infty}^{\infty}|\mathcal{F}^{-1}f(w)|\,dw \le C\,\|\,(1+|\xi|^2)^{1/4}\,\mathcal{F}f(\xi)\,|L_2(\mathbb{R})\|\,.$$

Here the right–hand side represents the norm of f in the fractional order Sobolev space $H^{1/2}_2(\mathbb{R})$, which coincides with the Besov space $B^{1/2}_{2,1}(\mathbb{R})$ (equivalent norms).

As it turns out if $s > 1$, then Proposition 2 will be sufficient for our purposes. In case $s < 1$, we add the following observation. In the proof of Proposition 2, we applied the most simple convolution inequality. So, our philosophy is as follows. If p is approaching 1 or ∞, then the quality of the estimates stated in Proposition 2 becomes better. Hence, interpolating between spaces with p close to 1 (here we apply Proposition 2) and the spaces $C(\mathbb{T})$ and $B^s_{\infty,\infty}(\mathbb{T})$, respectively (here we apply Proposition 1), we can improve the restrictions on s. We do not formulate a general result, we shall describe it in detail during the investigations of some examples.

3.4 Examples

The conditions in Proposition 2 look rather technical. We shall show by some examples that they are not very restrictive.

Periodized B–splines

The cardinal centralized B–spline $\mathcal{M}_r$ of order r is defined as

$$\mathcal{M}_r(x) \quad = \quad (\underbrace{\mathcal{M}_1 * \ldots * \mathcal{M}_1}_{\text{m-fold}})(x)\,, \qquad x \in \mathbb{R}\,, r \in \mathbb{N}\,.$$

Here $\mathcal{M}_1$ denotes the characteristic function of the interval $[-1/2, 1/2]$. Elementary calculations give

$$\mathcal{F}\mathcal{M}_r(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin \frac{\xi}{2}}{\frac{\xi}{2}} \right)^r .$$

To construct a fundamental interpolant on $\mathbb{R}$ we follow a standard procedure, cf. e.g. Jetter [9]. The symbol corresponding to $\mathcal{M}_r$ is given by

$$\mathcal{M}_r{}^{\sim}(\xi) = \sum_{m \in \mathbb{Z}} \mathcal{F}\mathcal{M}_r(2\pi\xi + 2\pi m)\,.$$

We define

$$\mathcal{F}\Lambda_{2r}(\xi) = \sqrt{2\pi}\, \frac{\mathcal{F}\mathcal{M}_{2r}(2\pi\xi)}{\mathcal{M}_{2r}{}^{\sim}(\xi)}\,, \qquad \xi \in \mathbb{R}\,, \quad r \in \mathbb{N}\,.$$

The decay properties of $\mathcal{M}_{2r}$ and the well–known positivity of the symbol imply that the function Λ_{2r} is a fundamental interpolant on $\mathbb{R}$.

Thanks to the well–known exponential decay of Λ_{2r}, periodization makes sense and we may investigate the periodic fundamental interpolants $\Lambda^{\pi}_{2r,N}$, cf. (9). The function $\Lambda^{\pi}_{2r,N}$ is a periodic spline of order $2r$ and with nodes $2\pi k/N$.

Lemma 4 *Let* $r \in \mathbb{N}$.
(i) *The function* Λ_{2r} *satisfies (14) with* $\alpha < 2r$.
(ii) *The function* Λ_{2r} *satisfies (15) with* $\alpha < 2r$.
(iii) *The function* Λ_{2r} *satisfies (16) if* $\beta > 1$.

Proof *Step 1.* We have

$$1 - \frac{\mathcal{F}\Lambda_{2r}\xi}{\sqrt{2\pi}} = \frac{\sum\limits_{m\neq 0} \mathcal{F}\mathcal{M}_{2r}(2\pi\xi + 2\pi m)}{\mathcal{M}_{2r}{}^{\sim}(\xi)} = \pi^{-2r}\, \sin^{2r}\, \pi\xi\, \frac{\sum_{m\neq 0} \frac{1}{|\xi+m|^{2r}}}{\mathcal{M}_{2r}{}^{\sim}(\xi)}\,.$$

The functions

$$F_1(\xi) = \psi(4\xi)\, \frac{\sum_{m\neq 0} \frac{1}{|\xi+m|^{2r}}}{\mathcal{M}_{2r}{}^{\sim}(\xi)}$$

and

$$F_2(\xi) = (1 - \psi(4\xi))\, \psi(\xi)\, \frac{\sum_{m\neq 0} \mathcal{F}\mathcal{M}_{2r}(2\pi\xi + 2\pi m)}{\mathcal{M}_{2r}{}^{\sim}(\xi)}$$

belong to $C^1(\mathbb{R})$. Making use of Szazs' Theorem, cf. Remark 8, and of assertions on pointwise multipliers in Besov spaces, cf. [27, Chap. 2.8] or [18, Chap. 4.7], it follows

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi(\xi)\, |\xi|^{-\alpha} \Big(1 - \frac{\mathcal{F}\Lambda_{2r}(\xi)}{\sqrt{2\pi}}\Big)\Big\}(w) \right| dw \\
&\le\ C_1 \left\| \psi(\xi)\, |\xi|^{-\alpha} \Big(1 - \frac{\mathcal{F}\Lambda_{2r}(\xi)}{\sqrt{2\pi}}\Big) \Big| B^{1/2}_{2,1}(\mathbb{R}) \right\| \\
&\le\ C_2 \Big(\|\, F_1(\xi)\, \sin^{2r}\pi\xi\ |\xi|^{-\alpha}\, |B^{1/2}_{2,1}(\mathbb{R})\| \ +\ \|\, F_2(\xi)\, |\xi|^{-\alpha}\, |B^{1/2}_{2,1}(\mathbb{R})\| \Big) \\
&\le\ C_3 \Big(\|\, F_1\, |C^1(\mathbb{R})\|\ \|\, \psi^2(\xi/2)\, |\xi|^{-\alpha}\, \sin^{2r}\pi\xi\, |B^{1/2}_{2,1}(\mathbb{R})\| \\
&\qquad\qquad + \|\, F_2\, |C^1(\mathbb{R})\|\ \|\, (1-\psi(8\xi))\, \psi(\xi/2)\, |\xi|^{-\alpha}\, |B^{1/2}_{2,1}(\mathbb{R})\| \Big) \\
&\le\ C_4 \bigg(\Big\| \psi(\xi/2)\, |\xi|^{2r-\alpha} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\|\ \Big\| \psi(\xi/2) \Big(\frac{\sin \pi\xi}{\pi\xi}\Big)^{2r} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\| \\
&\qquad\qquad + \Big\| (1-\psi(8\xi))\, \psi(\xi/2)\, |\xi|^{-\alpha} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\| \bigg) . \qquad (28)
\end{aligned}$$

The membership of power–type functions in Besov spaces is well–known, cf. e.g. [18, Chap. 2.3.1]. As a consequence, the right–hand side in (28) is finite as long as $\alpha < 2r$.

Step 2. To prove (ii) we shall employ

$$\left\| \psi(\xi/2)\, \frac{1}{|\xi+\ell|^{2r}} \right| B^{1/2}_{2,1}(\mathbb{R}) \bigg\| \le C_1\, |\ell|^{-2r}$$

with C_1 independent of ℓ, $|\ell| \ge 5$. Hence, as above we conclude

$$\begin{aligned}
&\sum_{|\ell|\ge 5} \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ \psi(\xi))\, |\xi|^{-\alpha}\, \mathcal{F}\Lambda_{2r}(\xi+\ell) \Big\}(w) \right| dw \\
&\le\ C_2 \sum_{|\ell|\ge 5} \Big\| \frac{\psi(\xi/2)}{|\xi+\ell|^{2r}} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\|\ \Big\| \psi(\xi)\, |\xi|^{2r-\alpha} \Big(\frac{\sin \pi\xi}{\pi\xi}\Big)^{2r} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\| \\
&\le\ C_3 \Big\| \psi(\xi/2)\, |\xi|^{2r-\alpha} \Big| B^{1/2}_{2,1}(\mathbb{R}) \Big\|\ \Big\| \Big(\frac{\sin \pi\xi}{\pi\xi}\Big)^{2r} \Big| C^1(\mathbb{R}) \Big\| . \qquad (29)
\end{aligned}$$

It remains to observe the finiteness of the summands in (15) with $|\ell| < 5$. That can be done as in Step 1 by isolating the singularities of $|\xi|^{-\alpha}$ and $|\xi+\ell|^{-2r}$, respectively.

Step 3. Similar as above we derive

$$\begin{aligned}
&\sum_{\ell\in\mathbb{Z}} \int_{-\infty}^{\infty} \left| \mathcal{F}^{-1}\Big\{ (1-\psi(2\xi))\, |\xi|^{-\beta}\, \mathcal{F}\Lambda_{2r}(\xi-\ell) \Big\}(w) \right| dw \\
&\le\ C_1 \sum_{\ell\in\mathbb{Z}} \|\, (1-\psi(2\xi))\, |\xi|^{-\beta}\, \mathcal{F}\Lambda_{2r}(\xi-\ell) \big| B^{1/2}_{2,1}(\mathbb{R}) \| \\
&\le\ C_2 \|\, (\mathcal{M}^{\sim}_{2r}(\xi))^{-1}\, |C^1(\mathbb{R})\| \sum_{\ell\in\mathbb{Z}} \|\, (1-\psi(2\xi))\, |\xi|^{-\beta}\, \mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell)) \big| B^{1/2}_{2,1}(\mathbb{R}) \| \\
&\le\ C_3 \sum_{\ell\in\mathbb{Z}} \|\, (1-\psi(2\xi))\, |\xi|^{-\beta}\, \mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell)) \big| W^1_2(\mathbb{R}) \| ,
\end{aligned}$$

where we have used the embedding $W^1_2(\mathbb{R}) \hookrightarrow B^{1/2}_{2,1}(\mathbb{R})$. First, notice the norm in case $\ell = 0$ is finite. Let $\ell > 0$. Observe

$$\int_{1/2}^{\infty} |\xi|^{-2\beta}\, |\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell))|^2\, d\xi = \int_{1/2}^{\ell/2} |\xi|^{-2\beta}\, |\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell))|^2\, d\xi$$

$$
\begin{aligned}
&+\int_{\ell/2}^{3\ell/2} |\xi|^{-2\beta}\, |\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell))|^2\, d\xi + \int_{3\ell/2}^{\infty} |\xi|^{-2\beta}\, |\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell))|^2\, d\xi \\
\le\ & C_1 \left(\ell^{-4r} \int_{1/2}^{\infty} |\xi|^{-2\beta}\, d\xi + \ell^{-2\beta} \int_0^{\ell/2} |\mathcal{F}\mathcal{M}_{2r}(2\pi\xi)|^2\, d\xi \right) \\
\le\ & C_2\, (\ell^{-2\beta} + \ell^{-4r})\, .
\end{aligned} \tag{30}
$$

By obvious modifications, we obtain the same estimate for the integral $\int_{-\infty}^{-1/2} \ldots d\xi$ and of course, also if $\ell < 0$. Next, we consider

$$
\begin{aligned}
&\frac{d}{d\xi}\, (1-\psi(2\xi))\, |\xi|^{-\beta}\, \mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell)) \\
=\ & -2\, \psi'(2\xi)\, |\xi|^{-\beta}\, \mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell)) + (-\beta)\, \mathrm{sign}\, \xi\, (1-\psi(2\xi))\, |\xi|^{-\beta-1}\, \mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell)) \\
&+ 2r\, (1-\psi(2\xi))\, |\xi|^{-\beta} \left(\frac{\sin(\pi(\xi-\ell))}{\pi(\xi-\ell)} \right)^{2r-1} \frac{\pi\, (\xi-\ell)\, \cos(\pi(\xi-\ell)) - \sin(\pi(\xi-\ell))}{\pi(\xi-\ell)^2} \\
=\ & T_1 + T_2 + T_3\, .
\end{aligned}
$$

In order to estimate the L_2-norm of this expression, one observes that in case of the term T_1 one may proceed exactly as in (30) because of ψ' is vanishing in a neigborhood of the origin. The estimate of T_2 can be done also as there (only β has to be replaced by $\beta+1$). What concerns T_3, then the result follows from the boundedness of

$$
\left(\frac{\sin(\pi(\xi-\ell))}{\pi(\xi-\ell)} \right)^{2r-1} \frac{\pi\, (\xi-\ell)\, \cos(\pi(\xi-\ell)) - \sin(\pi(\xi-\ell))}{\pi(\xi-\ell)^2}
$$

in the neigborhood of ℓ and the decay properties which are the same as of $\mathcal{F}\mathcal{M}_{2r}(2\pi(\xi-\ell))$. If $\min(\beta, 2r) > 1$, then this yields the finiteness of (16). ■

Corollary 1 *Let $r \in \mathbb{N}$, $1 < p < \infty$, and suppose*

$$
\frac{1}{p} < s < 2r\, . \tag{31}
$$

Let I_N^{2r} be the interpolation operator induced by Λ_{2r}. Then periodic spline interpolation has the following properties: for all $f \in B^s_{p,\infty}(\mathbb{T})$, it holds that

$$
\| f - I_N^{2r} f\, |L_p(\mathbb{T})\| \le C\, N^{-s}\, \| f\, |B^s_{p,\infty}(\mathbb{T})\|\, , \tag{32}
$$

where C denotes a constant independent of N and f; vice versa, if $f \in C(\mathbb{T})$ satisfies

$$
\sup_{N \in \mathbb{N}} N^s\, \| f - I_N^{2r} f\, |L_p(\mathbb{T})\| < \infty\, ,
$$

then f belongs to $B^s_{p,\infty}$.

Proof *Step 1.* We shall prove (32). By the above lemma, Proposition 2 yields the result as long as $1 < s < 2r$, because of $B^s_{p,\infty}(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T})$ under this condition. It remains to investigate $1/p < s \le 1$. Let $1 < p_0 < 2$. Then (31) implies $B^{s_0}_{p_0,\infty}(\mathbb{T}) \hookrightarrow B^{1/2}_{2,1}(\mathbb{T}) \hookrightarrow \mathcal{A}(\mathbb{T})$. Hence, if p_0 is close to 1, then Proposition 2 and Lemma 4 prove (32) as long as $1/p_0 < 1 < s_0 < 2r$. If $p_1 = \infty$, Proposition 1(ii) can be applied as long as $0 < s_1 < 2r-1$, $r > 1$: If $r = 1$, then Proposition 1(i) can be applied for

$0 < s_1 < 1$. Now we proceed by complex interpolation. To this end, let $1 < p < \infty$ and let s be as in (31). Then there exists some Θ, $0 < \Theta < 1$, $p_0 > 1$, $1 < s_0 < 2r$ and $0 < s_1 < 2r$ such that

$$\frac{1}{p} = \frac{1-\Theta}{p_0} \qquad \text{and} \qquad s = (1-\Theta)s_0 + s_1 \, .$$

Complex interpolation yields

$$[b^{s_0}_{p_0,\infty}(\mathbb{T}), b^{s_1}_{\infty,\infty}(\mathbb{T})]_\Theta = b^s_{p,\infty}(\mathbb{T}) \qquad \text{and} \qquad [L_{p_0}(\mathbb{T}), L_\infty(\mathbb{T})]_\Theta = L_p(\mathbb{T}) \, ,$$

cf. e.g. Triebel [26, Chap. 1.18.1,1.18.4,2.4.1]. Here $b^s_{p,\infty}(\mathbb{T})$ denotes the closure of the set of trigonometric polynomials in $B^s_{p,\infty}(\mathbb{T})$. This leads to

$$\begin{aligned} & \| \, (E - I_N^{2r}) \, |\mathcal{L}(b^s_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\| \\ \leq \quad & \| \, (E - I_N^{2r}) \, |\mathcal{L}(b^{s_0}_{p_0,\infty}(\mathbb{T}), L_{p_0}(\mathbb{T}))\|^{1-\Theta} \, \| \, (E - I_N^{2r}) \, |\mathcal{L}(b^{s_1}_{\infty,\infty}(\mathbb{T}), L_\infty(\mathbb{T}))\|^{\Theta} \, . \end{aligned}$$

Consequently

$$\| \, f - I_N^{2r} f \, | L_p(\mathbb{T})\| \leq C \, N^{-s} \, \| \, f \, | B^s_{p,\infty}(\mathbb{T})\| \tag{33}$$

for all $f \in b^s_{p,\infty}(\mathbb{T})$. Hence, the above inequality is true at least for all polynomials f. Replacing f in (33) by its Fourier partial sum $S_M f$, we find

$$\| \, S_M f - I_N^{2r} S_M f \, | L_p(\mathbb{T})\| \leq C_1 \, N^{-s} \, \| \, S_M f \, | B^s_{p,\infty}(\mathbb{T})\| \leq C_2 \, N^{-s} \, \| \, f \, | B^s_{p,\infty}(\mathbb{T})\| \, .$$

On the left–hand side, we have a convergent sequence (with respect to M). This may be seen as follows. Because of $f \in B^{s-\varepsilon}_{p,1}(\mathbb{T})$ we may apply (33) with $s - \delta > 1/p$ (if δ is sufficiently small) and end up with

$$\| \, S_{M_1} f - S_{M_2} f - I_N^{2r} (S_{M_1} f - S_{M_2} f) \, | L_p(\mathbb{T})\| \leq C \, N^{-s+\delta} \, \| \, S_{M_1} f - S_{M_2} f \, | B^{s-\delta}_{p,1}(\mathbb{T})\| < \varepsilon$$

if M_1, M_2 are large enough. This proves (33) for all $f \in B^s_{p,\infty}(\mathbb{T})$.
Step 2. The second statement of the corollary is a consequence of (27). ■

Remark 9 Results as stated in the Corollary 1 are not new, cf. e.g. Oswald [13].

De la Vallée Poussin means

Let $0 < \lambda < 1/2$ be fixed. We put

$$\varphi_\lambda(x) = 2 \, \frac{\sin \frac{x}{2} \, \sin \lambda x}{\lambda \, x^2} \, .$$

This implies

$$\mathcal{F}\varphi_\lambda(\xi) = \sqrt{2\pi} \begin{cases} 1 & \text{if} \quad |\xi| \leq \frac{1}{2} - \lambda \, , \\ \frac{1}{2\lambda}(\frac{1}{2} + \lambda - \xi) & \text{if} \quad \frac{1}{2} - \lambda < |\xi| < \frac{1}{2} + \lambda \, , \\ 0 & \text{if} \quad \frac{1}{2} + \lambda \leq |\xi| \, , \end{cases}$$

Some calculations yield

$$\Lambda^\pi_{\lambda,N}(x) = \frac{2}{N} \, \frac{1}{K_{\lambda,N}} \sum_{(1/2-\lambda)N \leq \ell < (1/2+\lambda)N} D_\ell(x) \, ,$$

where D_ℓ denotes the Dirichlet kernel of order ℓ and $K_{\lambda,N}$ the number of summands in $\sum_{(1/2-\lambda)N\le\ell<(1/2+\lambda)N}$. Proposition 1 is applicable for all s. Moreover, because of the fact that $\mathcal{F}\Phi_r$ is identical $\sqrt{2\pi}$ in a neigborhood of the origin the singularity of $|\xi|^{-\alpha}$ does not enter the picture, so (14) is finite for all $\alpha > 0$. Finally, the compactness of the support of $\mathcal{F}\Phi_r$ and its weak singularities in $-1/2-\lambda$, $-1/2+\lambda$, $1/2-\lambda$, $1/2+\lambda$ yield the finiteness of (16) for all $\beta > 1$. Complementing Propositions 1 and 2 by complex interpolation (as in proof of Corollary 1), we end up with the following.

Corollary 2 *Let $0 < \lambda < 1/2$. Let $1 < p < \infty$ and suppose $s > 1/p$. Let I_N^λ be the periodic interpolation operator induced by $\Lambda^\pi_{\lambda,N}$. Then: for all $f \in B^s_{p,\infty}(\mathbb{T})$ it holds that*

$$\| f - I_N^\lambda f \,|L_p(\mathbb{T})\| \le C\, N^{-s}\, \| f \,|B^s_{p,\infty}(\mathbb{T})\|\,,$$

where C denotes a constant independent of N and f; vice versa, if $f \in C(\mathbb{T})$ satisfies

$$\sup_{N\in\mathbb{N}} N^s \,\| f - I_N^\lambda f \,|L_p(\mathbb{T})\| < \infty\,,$$

then f belongs to $B^s_{p,\infty}(\mathbb{T})$.

3.5 Approximation in the norm of a Besov space

For the applications we have in mind, we need a modification of the results obtained so far. Because of $L_p(\mathbb{T}) \otimes_\lambda L_p(\mathbb{T}) \neq L_p(\mathbb{T}^2)$ (here λ denotes again the injective tensor norm) we have difficulties to study operators $P \otimes Q$ mapping a tensor product like $B^{r_1}_{p,q}(\mathbb{T}) \otimes_\lambda B^{r_2}_{p,q}(\mathbb{T})$ into $L_p(\mathbb{T}^2)$. These troubles disappear when replacing $L_p(\mathbb{T}^2)$ by a Besov space of dominating mixed smoothness. For this reason, we study $\| f - I_N f \,|B^0_{p,q}(\mathbb{T})\|$. It will be sufficient to deal with $q = 1$.

The investigations will be made parallel to our previous ones. So we will be very rough here. The Besov space $B^0_{p,1}(\mathbb{T})$ can be characterized with the help of the following norm, cf. (19):

$$\| f \,|B^0_{p,1}(\mathbb{T})\|^* = |\, c_0(f)\,| + \sum_{j=0}^\infty \Big\| \sum_{2^j\le|k|<2^{j+1}} c_k(f)\, e^{ikx} \,\Big| L_p(\mathbb{T})\Big\|,$$

cf. [20, Chap. 3.7.2].

Proposition 3 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (7) and (8). Moreover, we assume that (14) and (16) hold for some $\alpha > 0$ and $\beta \ge 0$.*

Let Λ^π_N (cf. (9)) and I_N (cf. (5)) be the associated fundamental interpolants and interpolation operator, respectively. Let $0 < s < \alpha$, $\beta < s$ and let $1 < p < \infty$. Then there exists a constant C (independent of N) such that

$$\| f - I_N f \,|B^0_{p,1}(\mathbb{T})\| \le C\, \| f \,|B^s_{p,\infty}(\mathbb{T})\|$$

holds for all $f \in B^s_{p,\infty}(\mathbb{T})$.

Proof *Step 1.* We employ that the set of Fourier multipliers of $B^0_{p,1}(\mathbb{T})$ is larger than the set of all Fourier multipliers of $L_p(\mathbb{T})$. More exactly, the following convolution inequality holds true

$$\Big\| \sum_{k\in\mathbb{Z}} c_k(f)\, M(k)\, e^{ikx} \,\Big| B^0_{p,1}(\mathbb{T})\Big\| \le \frac{1}{\sqrt{2\pi}} \int |\mathcal{F}^{-1} M(y)|\, dy \quad \| f \,|B^0_{p,1}(\mathbb{T})\|\,.$$

This allows us to follow the same path as described above. The modification in case of inequality (20) reads as follows

$$\| f - I_N f \,|B^0_{p,1}(\mathbb{T})\| \le \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\mathcal{F}^{-1} M(y)| \, dy \quad \Big\| \sum_{m\neq 0} c_m(f)\, |m|^\alpha \, e^{imx} \Big| L_p(\mathbb{T}) \Big\| .$$

Also (21) and (22) remain true: if $2^r \le N < 2^{r+1}$, then

$$\begin{aligned} \Big\| \sum_{m\neq 0} c_m(f)\, |m|^\alpha \, e^{imx} \Big| B^0_{p,1}(\mathbb{T}) \Big\| &\le \sum_{\ell=0}^{r} \Big\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, |m|^\alpha \, e^{imx} \Big| L_p(\mathbb{T}) \Big\| \\ &\le C_1 \sum_{\ell=0}^{r} 2^{\ell\alpha} \Big\| \sum_{2^\ell \le |m| < 2^{\ell+1}} c_m(f)\, e^{imx} \Big| L_p(\mathbb{T}) \Big\| \\ &\le C_1 \, \| f \,|B^s_{p,\infty}(\mathbb{T})\| \sum_{\ell=0}^{r} 2^{\ell(\alpha - s)} \\ &\le C_2 \, \| f \,|B^s_{p,\infty}(\mathbb{T})\| \, N^{\alpha - s} . \end{aligned}$$

Here C_2 denotes a constant independent of N. This establishes the counterpart of Lemma 2.

Concerning the counterpart of Lemma 3, observe that (as in (24))

$$\begin{aligned} \| I_N(f - S_N f) | B^0_{p,1}(\mathbb{T})\| &\le \sum_{\ell\in\mathbb{Z}} \Big(\int_{-\infty}^{\infty} |\mathcal{F}^{-1} M_\ell(y)| \, dy \Big) \; \| f - S_N f \,| B^\beta_{p,1}(\mathbb{T})\| \\ &\le C\, N^\beta \, \| f - S_N f \,| B^\beta_{p,1}(\mathbb{T})\| . \end{aligned}$$

The remainder part is the same as in proof of Lemma 3. ■

We do not have estimates of the interpolation error in norm $B^0_{\infty,1}(\mathbb{T})$, at least not in case $0 < s < 1$. So we have to add a few remarks about the interpolation argument we used several times before. Instead of $[L_{p_0}(\mathbb{T}), L_\infty(\mathbb{T})]_\Theta = L_p(\mathbb{T})$ we shall use

$$\begin{aligned} (B^0_{p_0,1}(\mathbb{T}), L_\infty(\mathbb{T}))_\Theta &\hookrightarrow (B^0_{p_0,1}(\mathbb{T}), B^0_{\infty,\infty}(\mathbb{T}))_\Theta = B^0_{p,q}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T}) , \\ \frac{1}{q} &= \frac{1-\Theta}{q_0} , \qquad \frac{1}{p} = \frac{1-\Theta}{p_0} , \qquad q \le \min(2,p) , \end{aligned} \tag{34}$$

cf. [20, 3.5.4,3.6.1]. Here $(\,\cdot\,,\,\cdot\,)_\Theta$ denotes Triebel's modified complex interpolation method.

4 Interpolation on sparse grids

4.1 Preliminaries

We need some preparations. By E we denote always the identity (even on different spaces without indicating this). First we introduce the parameter dependent extension of our interpolation operators. We put

$$\begin{aligned} I^x_K f(x,y) &= \sum_{\ell\in J_K} f\Big(\frac{2\pi\ell}{K}, y\Big) \, \Lambda_K\Big(x - \frac{2\pi\ell}{K}\Big) , \\ I^y_N f(x,y) &= \sum_{\ell\in J_N} f\Big(x, \frac{2\pi\ell}{N}\Big) \, \Lambda_N\Big(y - \frac{2\pi\ell}{N}\Big) . \end{aligned}$$

What is important for us are the following identities

$$I^x_K f = (I_K \otimes E) f , \qquad \text{and} \qquad I^y_N f = (E \otimes I_N) f , \qquad f \in C(\mathbb{T}) \otimes C(\mathbb{T}) .$$

4.2 Interpolation on sparse grids

We follow Delvos and Schempp [4]. A sequence of projections $\{P_j\}_{j=1}^{\infty}$ is called a chain if P_j commutates with P_{j+1} and $P_{j+1}\, P_j = P_j\, P_{j+1} = P_j$ holds for all j. The interpolation operators I_N, considered on $C(\mathbb{T})$, form a chain if the range spaces satisfy the inclusion relation

$$\mathcal{R}(I_N) \subset \mathcal{R}(I_M)\,, \qquad N \le M\,, \tag{35}$$

and if $\mathcal{T}_N \subset \mathcal{T}_M$, $N \le M$. The second condition will be always satisfied if we concentrate on a sequence $N_j = d\, 2^j$ for some natural number d. The first condition has to be checked in each example separately.

Now, having a chain I_{N_j}, $j \in \mathbb{N}$, of interpolation operators, then also the parameter dependent extensions $I^x_{N_j}$ and $I^y_{N_j}$ form chains. The Boolean sum of order j is defined as

$$B_j = \bigoplus_{r=0}^{j} I^x_{N_r}\, I^y_{N_{j-r}} = \sum_{r=0}^{j} I^x_{N_r}\, I^y_{N_{j-r}} - \sum_{r=0}^{j-1} I^x_{N_r}\, I^y_{N_{j-r-1}}\,.$$

Then B_j form a chain, cf. Baszenski and Delvos [2] or Delvos and Schempp [4, Prop.1.3.1]. They have range space

$$\mathcal{R}(B_j) = \sum_{r=0}^{j} \mathcal{R}(I_{N_r}) \otimes \mathcal{R}(I_{N_{j-r}})$$

and satisfy the interpolation conditions with respect to the sparse grid

$$\mathcal{T}(B_j) = \bigcup_{r=0}^{j} \mathcal{T}_{N_r} \times \mathcal{T}_{N_{j-r}}\,.$$

The total number of knots in this grid is equal to $d^2\,(j2^{j-1} + 2^j)$ and is much less then the number of knots $d^2\, 2^{2j}$ in the full grid $\mathcal{T}_{N_j} \times \mathcal{T}_{N_j}$. Finally, the remainder operator $E - B_j$ has the representation

$$E - B_j = (E - I^x_{N_j}) + (E - I^y_{N_j}) - \sum_{r=0}^{j} (E - I^x_{N_r})\,(E - I^y_{N_{j-r}}) + \sum_{r=0}^{j-1} (E - I^x_{N_r})\,(E - I^y_{N_{j-r-1}})\,, \tag{36}$$

cf. [4, Prop.1.4.2].

Now we are ready to formulate the main result of this paper.

Theorem 1 *Suppose that I_{N_j}, $j \in \mathbb{N}$, form a chain and satisfy*

$$\sup_{j\in\mathbb{N}} N_j^{s_k}\, \|\, f - I_{N_j} f\, | B^0_{p,q}(\mathbb{T})\| \le C_k\, \|\, f\, | B^{s_k}_{p,\infty}(\mathbb{T})\| \tag{37}$$

with some constants C_k independent of f and for some fixed s_1, s_2, p and q, $1 < p < \infty$, $\min(s_1, s_2) > 1/p$, and $q \le \min(2, p)$.

Then in case $s_1 = s_2 = s$, we find

$$\|\, f - B_j f\, | L_p(\mathbb{T}^2)\| \le C\, (j+1)\, N_j^{-s}\, \|\, f\, | S^{s,s}_{p,\infty} B(\mathbb{T}^2)\|\,, \tag{38}$$

whereas in case $s_1 \ne s_2$, it holds that

$$\|\, f - B_j f\, | L_p(\mathbb{T}^2)\| \le C\, N_j^{-\min(s_1,s_2)}\, \|\, f\, | S^{s_1,s_2}_{p,\infty} B(\mathbb{T}^2)\|\,. \tag{39}$$

In both situations C denotes a constant independent on j and f.

Proof We shall prove (38) and (39) with $L_p(\mathbb{T}^2)$ replaced by $S^{0,0}_{p,q}B(\mathbb{T}^2)$. In view of $S^{0,0}_{p,q}B(\mathbb{T}^2) \hookrightarrow L_p(\mathbb{T}^2)$ for $q \le \min(2,p)$, cf. [20, Chap. 2.2.3] this will be sufficient.

We employ (36), $E - I^x_K = (E - I_K) \otimes E$ and $E - I^y_N = E \otimes (E - I_N)$. Hence, (4) yields

$$\begin{aligned} \| f - B_j f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| \;\le\; & \| f - I^x_{N_j} f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| + \| f - I^y_{N_j} f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| \\ & + \sum_{r=0}^{j} \| (E - I^x_{N_r})\,(E - I^y_{N_{j-r}}) f \,|S^{0}_{p,q}B(\mathbb{T}^2)\| \\ & + \sum_{r=0}^{j-1} \| (E - I^x_{N_r})\,(E - I^y_{N_{j-r-1}}) f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| \,. \end{aligned} \tag{40}$$

By means of $E - I^x_K = (E - I_K) \otimes E$, $S^{0,0}_{p,q}B(\mathbb{T}^2) = B^0_{p,q}(\mathbb{T}) \otimes_\lambda B^0_{p,q}(\mathbb{T})$ and $S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2) = B^{s_1}_{p,\infty}(\mathbb{T}) \otimes_\lambda B^{s_2}_{p,\infty}(\mathbb{T})$, cf. (4), we estimate

$$\begin{aligned} & \| f - I^x_{N_j} f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| \\ \le\; & \| E - I_{N_j} \,|\mathcal{L}(B^{s_1}_{p,\infty}(\mathbb{T}),\, B^0_{p,q}(\mathbb{T}))\| \, \| E \,|\mathcal{L}(B^{s_2}_{p,\infty}(\mathbb{T}),\, B^0_{p,q}(\mathbb{T}))\| \, \| f \,|S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)\| \,. \end{aligned} \tag{41}$$

Next we use

$$(E - I^x_{N_r})\,(E - I^y_{N_{j-r}}) = \big((E - I_{N_r}) \otimes E\big)\,\big(E \otimes (E - I_{N_{j-r}})\big) = (E - I_{N_r}) \otimes (E - I_{N_{j-r}})\,.$$

Analoguously to (41), we derive

$$\begin{aligned} & \| (E - I^x_{N_r})\,(E - I^y_{N_{j-r}}) f \,|S^{0,0}_{p,q}B(\mathbb{T}^2)\| \\ \le\; & \| E - I_{N_j} \,|\mathcal{L}(B^{s_1}_{p,\infty}(\mathbb{T}),\, B^0_{p,q}(\mathbb{T}))\| \, \| E - I_{N_{j-r}} \,|\mathcal{L}(B^{s_2}_{p,\infty}(\mathbb{T}),\, B^0_{p,q}(\mathbb{T}))\| \, \| f \,|S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)\| \,. \end{aligned} \tag{42}$$

Inserting (37) into (41) and into (42), making use of the obtained estimates in (40), then the desired result follows. ■

4.3 Examples

Periodized B-splines

First we have to check whether the corresponding sequences I_{N_j} form a chain. But this is clear from the nestedness of the underlying sets of the spline nodes (which equal the interpolation knots $\mathcal{T}_{N_j}$ here).

Theorem 2 *Let $r \in \mathbb{N}$, $1 < p < \infty$, and suppose*

$$\max\left(\frac{1}{2}, \frac{1}{p}\right) < s_1, s_2 < 2r\,.$$

Let I^{2r}_N be the interpolation operator induced by Λ_{2r}. The corresponding Boolean sum of order j we denote by B^{2r}_j. Then periodic spline interpolation on the sparse grids $\mathcal{T}(B^{2r}_j)$ satisfies

$$\| f - B^{2r}_j f \,|L_p(\mathbb{T}^2)\| \le C\, \| f \,|S^{s_1,s_2}_{p,\infty}B(\mathbb{T}^2)\| \begin{cases} N_j^{-\min(s_1,s_2)} & \text{if} \quad s_1 \ne s_2\,, \\ (j+1)\, N_j^{-s} & \text{if} \quad s_1 = s_2 = s\,, \end{cases}$$

where C denotes a constant independent of j and f.

Proof A direct application of Lemma 4, Proposition 3 and Theorem 1 yield (37) in case $1 < s < 2r$. It remains to investigate $1/p < s \leq 1$. To indicate how we interpolate this with the result from Proposition 1 we consider the following figure:

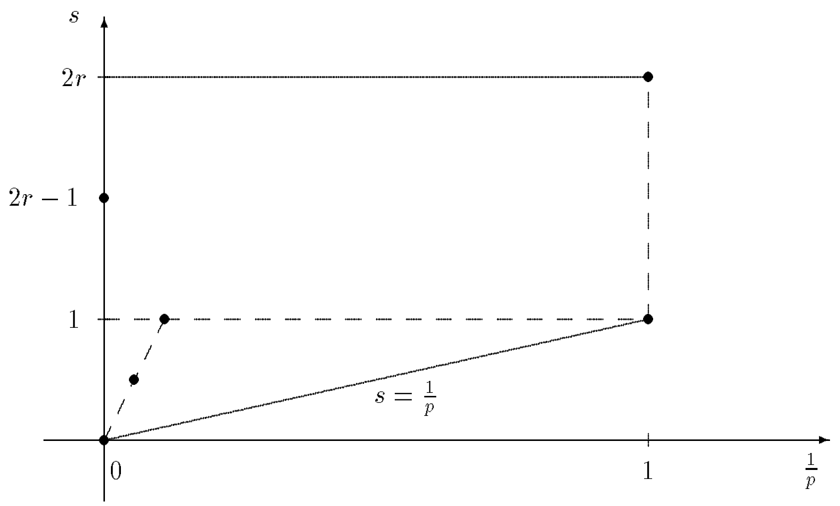

Fig. 1

The parameter p_1 will be always ∞ and s_1 will be chosen near zero. We choose p_0 as close as possible to p. This means starting with $s = \lambda/p$ for some $\lambda > 1$ we end up with s_0 is nearly 1 and $p_0 \sim \lambda$. Now, as indicated in (34), we have

$$(B^0_{p_0,1}(\mathbb{T}), L_\infty(\mathbb{T}))_\Theta \hookrightarrow (B^0_{p_0,1}(\mathbb{T}), B^0_{\infty,\infty}(\mathbb{T}))_\Theta = B^0_{p,q}(\mathbb{T}) \,,$$

where

$$\frac{1}{p} = \frac{1-\Theta}{p_0} \sim \frac{1-\Theta}{\lambda} \qquad \text{and} \qquad \frac{1}{q} = 1 - \Theta \sim \frac{\lambda}{p} \,.$$

This space turns out to be a subspace of $L_p(\mathbb{T})$ if

$$q = \frac{1}{1-\Theta} \sim \frac{p}{\lambda} = \frac{1}{s} < \min(2,p)$$

Hence, if $s > 1/2$ this is always possible. As a consequence, we obtain that the conditions in Theorem 1 are satisfied under the above restrictions. This proves the claim. ■

De la Vallée Poussin means

In case of the de la Vallée Poussin means, the condition (35) is violated in general. Lemma 1 offers a simple possibility to obtain a sufficient condition to guarantee (35). To this end, we investigate the refinement equation (12) in the Fourier image:

$$\mathcal{F}\Lambda(2\xi) = \left(\sum_{k\in\mathbb{Z}} h_k \, e^{-i2\pi k\xi}\right) \mathcal{F}\Lambda(\xi) \,, \qquad \xi \in \mathbb{R} \,.$$

In our concrete situation, it is obvious that if $\lambda \leq 1/6$, then

$$\mathcal{F}\varphi_\lambda(2\xi) = \mathcal{F}\varphi_\lambda(2\xi)\mathcal{F}\varphi_\lambda(\xi) = \left(\sum_{k\in\mathbb{Z}} h_k \, e^{-i2\pi k\xi}\right) \mathcal{F}\varphi_\lambda(\xi)$$

holds, where

$$h_k = \int_0^1 \mathcal{F}\varphi_\lambda(2\xi) e^{i2\pi k\xi}\, d\xi \,.$$

The 1–periodic extension of $\mathcal{F}\varphi_\lambda(2\xi)$ is a Lipschitz function. Hence, its Fourier coefficients are absolutely summable. So we may apply Lemma 1. Propositon 1 and Proposition 3 together with complex interpolation guarantee the following.

Theorem 3 *Let $1 < p < \infty$ and suppose $1/p < s_1, s_2$. Let $0 < \lambda \le 1/6$. Let I_N^λ be interpolation operator associated to φ_λ and denote by B_j^λ its Boolean sum of order j. Then we have*

$$\| f - B_j^\lambda f \,| L_p(\mathbb{T}^2)\| \le C \,\| f \,| S^{s_1,s_2}_{p,\infty} B(\mathbb{T}^2)\| \begin{cases} N_j^{-\min(s_1,s_2)} & \text{if} \quad s_1 \neq s_2 \,, \\ (j+1)\, N_j^{-s} & \text{if} \quad s_1 = s_2 = s \,, \end{cases}$$

where C denotes a constant independent of j and f.

Remark 10 Investigations of periodic fundamental interpolants of de la Vallée Poussin type have been made also in Prestin and Selig [16] in a multiresolution setting.

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