AN ASYMPTOTIC COMPLETE INTERSECTION THEOREM FOR CHAIN PRODUCTS

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Abstract. Let $N_l(n,k)$ be the set of all n–tuples over the alphabet {0,1, ... ,k} whose component sum equals l. A subset $\mathcal{F} \subseteq N_l(n,k)$ is called a t–intersecting family if every two tuples in $\mathcal{F}$ have nonzero entries in at least t common coordinates. We determine the maximum size of a t–intersecting family in $N_{\lfloor \lambda n \rfloor}(n,k)$ asymptotically for all fixed λ ($0 < \lambda < k$) and $n \to \infty$.

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1. Introduction

For positive integers k, l and n let $N_l(n,k) = \{\boldsymbol{a} = (a_1, \dots, a_n) : a_i \in \{0, 1, \dots, k\},\ i = 1, \dots, n, \sum a_i = l\}$. A family $\mathcal{F} \subseteq N_l(n,k)$ is called t-intersecting if for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{F}$ there exist t coordinates $i_1, \dots, i_t$ such that $a_{i_j}, b_{i_j} \geq 1$ holds for $j = 1, \dots, t$. Define

$$M_l(n,k,t) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq N_l(n,k), \mathcal{F} \text{ is t-intersecting}\}.$$

$N_l(n,k)$ can be viewed as the l-th level of the direct product of n chains $0 \lessdot 1 \lessdot \cdots \lessdot k$ (see [3] for terminology not explained here). We identify $N_l(n,1)$ with $\binom{[n]}{l}$, the family of all l-subsets of $\{1, \dots, n\}$. Set $W_l(n,k) = |N_l(n,k)|$.

Define for $\boldsymbol{a} \in N_l(n,k)$ resp. $\mathcal{F} \subseteq N_l(n,k)$ the support of $\boldsymbol{a}$ resp. of $\mathcal{F}$ by $\operatorname{supp}(\boldsymbol{a}) = \{i : a_i > 0\}$ resp. $\operatorname{supp}(\mathcal{F}) = \{\operatorname{supp}(\boldsymbol{a}) : \boldsymbol{a} \in \mathcal{F}\}$. Obviously, $\mathcal{F} \subseteq N_l(n,k)$ is t-intersecting iff $\operatorname{supp}(\mathcal{F})$ is t-intersecting.

Let $\mathcal{S}_{r,l} = \{S \in \binom{[n]}{l} : |S \cap [1, t+2r]| \geq t + r\}$, where $r \in \{0\} \cup \mathbb{N}$ and $[i,j]$ is defined as $\{i, i+1, \dots, j\}$.

Theorem 1.1 (Ahlswede, Khachatrian [1])**.** *Let $n > 2l - t$ and $r \in \{0\} \cup \mathbb{N}$ such that $(l-t+1)(2 + \frac{t-1}{r+1}) \leq n < (l-t+1)(2 + \frac{t-1}{r})$. Then*

$$M_l(n,1,t) = |\mathcal{S}_{r,l}|.$$

For fixed $\lambda > 0$ define the families $\mathcal{F}_0, \dots, \mathcal{F}_{\lfloor \frac{n-t}{2} \rfloor} \subseteq N_{\lfloor \lambda n \rfloor}(n,k)$:

$$\mathcal{F}_r = \{\boldsymbol{a} \in N_{\lfloor \lambda n \rfloor}(n,k) : \operatorname{supp}(\boldsymbol{a}) \in \mathcal{S}_{r,j} \text{ for some } j\}.$$

Let $\alpha_{t,r}$ be the unique positive solution of the equation $x + x^2 + \cdots + x^k = \frac{1+r}{t+r}$ and define $\lambda_{t,-1} = 0$ and

$$\lambda_{t,r} = \frac{\sum_{i=1}^k i\alpha_{t,r}^i}{1 + \alpha_{t,r} + \cdots + \alpha_{t,r}^k} = \frac{t+r}{t+2r+1} \sum_{i=1}^k i\alpha_{t,r}^i.$$

Note that $\lambda_{t,0} < \lambda_{t,1} < \cdots$ and $\lim_{r \to \infty} \lambda_{t,r} = \lambda_{1,0}$.

Theorem 1.2. *Let k, t and λ be fixed and let n tend to infinity. Then*

a) $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim |\mathcal{F}_r|$ *if* $\lambda_{t,r-1} < \lambda \leq \lambda_{t,r}$,
b) $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim \frac{1}{2} W_{\lfloor \lambda n \rfloor}(n,k)$ *if* $\lambda = \lambda_{1,0}$,
c) $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim W_{\lfloor \lambda n \rfloor}(n,k)$ *if* $\lambda > \lambda_{1,0}$.

The case a) with $0 < \lambda < \lambda_{t,0}$, and the case c) were proven by Engel, Frankl [4]. The key ingredient in their proof is the exact bound of the Erdős–Ko–Rado theorem (see [6] and [8]).

The proof of case a) with strict inequality $\lambda_{t,r-1} < \lambda < \lambda_{t,r}$ is a generalization of [4] using the Ahlswede–Khachatrian theorem.

The proof of case a) with equality $\lambda = \lambda_{t,r}$ will follow from a somewhat more general version of the Ahlswede–Khachatrian theorem.

Define $\mathcal{S}_r = \cup_l \mathcal{S}_{r,l}$.

Lemma 1.3. *Let $\omega_0, \omega_1, \ldots, \omega_n$ be nonnegative real weights. For $\mathcal{S} \subseteq 2^{[n]}$ define $\omega(\mathcal{S}) = \sum_{i=0}^{n} |\mathcal{S} \cap \binom{[n]}{i}|\, \omega_i$. If there exist $r \in \{0\} \cup \mathbb{N}$ such that*

$$\omega_i = 0 \text{ unless } \frac{n}{2 + \frac{t-1}{r}} + t - 1 < i < \frac{n}{2 + \frac{t-1}{r+2}} + t - 1$$

then

$$\max\{\omega(\mathcal{S}) : \mathcal{S} \subseteq 2^{[n]}, \mathcal{S} \text{ is } t\text{-intersecting}\} \leq \max\{\omega(\mathcal{S}_r), \omega(\mathcal{S}_{r+1})\}.$$

Sketch of proof. It follows from the proof method of [1] resp. [2] that a left compressed optimal family $\mathcal{S}$ is generated in $[1, t + 2r + 2]$, i.e.

$$|S_1 \cap S_2 \cap [1, t + 2r + 2]| \geq t \text{ for all } S_1, S_2 \in \mathcal{S},$$

resp. is invariant in $[1, t + 2r]$, i.e.

$$S_{i,j} \in \mathcal{S} \text{ for all } S \in \mathcal{S},\ i, j \in [1, t + 2r],$$

where $S_{i,j}$ is obtained from S by exchanging the coordinates i, j.
It follows easily that $\mathcal{S} = \mathcal{S}_r$ or $\mathcal{S} = \mathcal{S}_{r+1}$. □

The proof of case b) will require some tedious calculations.

We remark that the case $t = 1$ is settled (in principle) by a profile polytope theorem for intersecting families due to Erdős, Frankl, Katona [5] (see also [3, p.114]). In particular, Wu [9] showed $M_l(n, k, 1) = |\mathcal{F}_{\lfloor \frac{n-1}{2} \rfloor}|$ if $l \geq n$.

2. Proof of case a)

First we recall a lemma from [4] on the average size of nonzeros in members of $N_{\lfloor \lambda n \rfloor}(n, k)$. Let λ be fixed and α be the unique positive solution of the equation

$$\sum_{i=1}^{k} i\alpha^i = \lambda(\sum_{i=0}^{k} \alpha^i).$$

Define $p = 1 - \frac{1}{\sum_{i=0}^{k} \alpha^i}$.

Lemma 2.1. *Let $\epsilon > 0$ be given. The number of elements $\boldsymbol{a} \in N_{\lfloor \lambda n \rfloor}(n, k)$ with $|supp(\boldsymbol{a})| \notin [(p - \epsilon)n, (p + \epsilon)n]$ is exponentially small (in n) with respect to $W_{\lfloor \lambda n \rfloor}(n, k)$.*

See [4] or [3, p.329] for a proof. □

Lemma 2.2. $\frac{W_l(n,k)}{|\mathcal{F}_r|} \leq \binom{n}{t+2r}$ *for $l \geq k(t + r)$.*

Proof. If $l \geq k(t + r)$ then $|\text{supp}(\boldsymbol{a})| \geq t + r$ for all $\boldsymbol{a} \in N_l(n, k)$. Thus $N_l(n, k) = \cup_{T \subseteq [1,n] : |T| = t+2r} \{\boldsymbol{a} \in N_l(n, k) : |\text{supp}(\boldsymbol{a}) \cap T| \geq t + r\}$. □

Now let $\mathcal{F} \subseteq N_{\lfloor \lambda n \rfloor}(n,k)$ be a maximum t-intersecting family. Consider

$$\begin{aligned}
\mathcal{F}' &= \{\boldsymbol{a} \in \mathcal{F} : \operatorname{supp}(\boldsymbol{a}) \in [(p-\epsilon)n, (p+\epsilon)n]\},\\
\mathcal{F}_r' &= \{\boldsymbol{a} \in \mathcal{F}_r : \operatorname{supp}(\boldsymbol{a}) \in [(p-\epsilon)n, (p+\epsilon)n]\},\\
W_{\lfloor \lambda n \rfloor}'(n,k) &= |\{\boldsymbol{a} \in N_{\lfloor \lambda n \rfloor}(n,k) : \operatorname{supp}(\boldsymbol{a}) \in [(p-\epsilon)n, (p+\epsilon)n]\}|.
\end{aligned}$$

From the preceding two lemmata we conclude that

$$\begin{aligned}
\frac{W_{\lfloor \lambda n \rfloor}(n,k) - W_{\lfloor \lambda n \rfloor}'(n,k)}{|\mathcal{F}|} &\leq \frac{W_{\lfloor \lambda n \rfloor}(n,k) - W_{\lfloor \lambda n \rfloor}'(n,k)}{|\mathcal{F}_r|}\\
&= \frac{W_{\lfloor \lambda n \rfloor}(n,k) - W_{\lfloor \lambda n \rfloor}'(n,k)}{W_{\lfloor \lambda n \rfloor}(n,k)} \frac{W_{\lfloor \lambda n \rfloor}(n,k)}{|\mathcal{F}_r|}
\end{aligned}$$

tends to zero for $n \to \infty$, which implies

$$|\mathcal{F}'| \sim |\mathcal{F}| \text{ and } |\mathcal{F}_r'| \sim |\mathcal{F}_r| \text{ for } n \to \infty.$$

If $\lambda_{t,r-1} < \lambda < \lambda_{t,r}$ then $\frac{1}{2+\frac{t-1}{r}} < p < \frac{1}{2+\frac{t-1}{r+1}}$. Choose $\epsilon, \delta > 0$ such that

$$[p-\epsilon, p+\epsilon] \subseteq [\frac{1}{2+\frac{t-1}{r}} + \delta, \frac{1}{2+\frac{t-1}{r+1}} - \delta].$$

Then, using the Ahlswede–Khachatrian theorem, we have for large n

$$\begin{aligned}
|\mathcal{F}'| &= \sum_j |\operatorname{supp}(\mathcal{F}') \cap \binom{[n]}{j}|\, W_{\lfloor \lambda n \rfloor - j}(j, k-1)\\
&\leq \sum_j |\mathcal{S}_{r,j}|\, W_{\lfloor \lambda n \rfloor - j}(j, k-1) = |\mathcal{F}_r'|,
\end{aligned}$$

where the summation is extended over all $j \in [(p-\epsilon)n, (p+\epsilon)n]$. It follows $|\mathcal{F}| \sim |\mathcal{F}_r|$ for $n \to \infty$.

If $\lambda = \lambda_{t,r}$ then choose $\epsilon, \delta > 0$ such that

$$[p-\epsilon, p+\epsilon] \subseteq [\frac{1}{2+\frac{t-1}{r}} + \delta, \frac{1}{2+\frac{t-1}{r+2}} - \delta].$$

Lemma 1.3 yields in this case

$$|\mathcal{F}'| \leq \max\{|\mathcal{F}_r'|, |\mathcal{F}_{r+1}'|\}.$$

Thus, $|\mathcal{F}| \sim |\mathcal{F}_r|$ will follow from $|\mathcal{F}_r| \sim |\mathcal{F}_{r+1}|$.

Lemma 2.3. *Let $\lambda = \lambda_{t,r}$. Then $|\mathcal{F}_r| \sim |\mathcal{F}_{r+1}|$ for $n \to \infty$.*

Proof. The proof is a routine application of a local limit theorem (see [3, ch.7]). Let $\lambda = \lambda_{t,r}, \alpha = \alpha_{t,r}$. Define

$$W_l(n,k;m,j) = |\{(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \in [0,k]^n \times [0,j]^m : \sum a_i = l\}|.$$

Note that

$$|\mathcal{F}_r \setminus \mathcal{F}_{r+1}| = \binom{t+2r}{t+r} W_{\lfloor \lambda n \rfloor - (t+r)}(t+r, k-1; n-(t+2r+2), k),$$

$$|\mathcal{F}_{r+1} \setminus \mathcal{F}_r| = \binom{t+2r}{t+r-1} W_{\lfloor \lambda n \rfloor - (t+r+1)}(t+r+1, k-1; n-(t+2r+2), k).$$

Consider the discrete random variables ξ_1, ξ_2 defined by $\mathrm{Prob}(\xi_1 = i) = \alpha^i/(1+\alpha+\cdots+\alpha^k), i = 0, \ldots, k$ and $\mathrm{Prob}(\xi_2 = i) = \alpha^i/(1+\alpha+\cdots+\alpha^{k-1}), i = 0, \ldots, k-1$. The random variable ξ_1 has expected value λ, let σ^2 be the variance of ξ_1.
Define a third random variable ζ_n as the sum of a independent copies of ξ_2 and $n-a$ independent copies of ξ_1, $a \in \mathbb{N}$. Note that

$$\mathrm{Prob}(\zeta_n = l) = \frac{\alpha^l}{(1+\alpha+\cdots+\alpha^{k-1})^a(1+\alpha+\cdots+\alpha^k)^{n-a}} W_l(a, k-1; n-a, k).$$

If a is constant then it follows from a central limit theorem ([7, ch.8.2.]) that the sequence $\{\zeta_n\}$ is asymptotically normal with mean $n\lambda$ and variance $n\sigma^2$. Moreover, since the sequence $\mathrm{Prob}(\zeta_n = j)$ is properly log concave in j (see [3, p.307]) we have local asymptotic normality, i.e.

$$\lim_{n\to\infty} \sigma\sqrt{n}\mathrm{Prob}(\zeta_n = \lfloor \sigma\sqrt{n}x + \mu n \rfloor) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1}$$

uniformly for all $x \in \mathbb{R}$. It follows (with $x = c/\sigma\sqrt{n}$) that

$$W_{\lfloor \lambda n \rfloor + c}(a, k-1; n-a, k) \sim \frac{1}{\sqrt{2\pi n}\sigma} \frac{(1+\alpha+\cdots+\alpha^{k-1})^a(1+\alpha+\cdots+\alpha^k)^{n-a}}{\alpha^{\lfloor \lambda n \rfloor + c}} \tag{2}$$

for $n \to \infty$, where c is an arbitrary constant.
Applying this asymptotic formula to $|\mathcal{F}_r \setminus \mathcal{F}_{r+1}|$ and $|\mathcal{F}_{r+1} \setminus \mathcal{F}_r|$ yields the claim. $\square$

3. Proof of case b)

Let $\lambda = \lambda_{1,0}$, $\alpha = \alpha_{1,0}$. We consider again the discrete random variables ξ_1, ξ_2 defined in the proof of case a). Let μ_j resp. σ_j^2 denote the expected value resp. the variance of $\xi_j, j = 1, 2$, and define $\sigma^2 := \sigma_1^2$. Note that $\mu_1 = \lambda$, $\mu_2 = 2\lambda - 1$ and $\sigma_2^2 = 2(\sigma^2 + \lambda^2)$. Let further ξ_3 be the random variable defined by $\mathrm{Prob}(\xi_3 = i) = \frac{1}{2}$, $i = 0, 1$, with expected value $\mu_3 = 1/2$ and variance ${\sigma_3}^2 = 1/4$. Again, the sum of n independent copies of ξ_j is locally asymptotically normal (in n) with mean $n\mu_j$ and variance $n\sigma_j^2$.

Lemma 3.1. $\lim_{r\to\infty} \lim_{n\to\infty} \frac{|\mathcal{F}_r|}{W_{\lfloor \lambda n \rfloor}(n,k)} = \frac{1}{2}$.

Proof. First let r be constant. We have

$$|\mathcal{F}_r| = \sum_{i=0}^{r} \binom{t+2r}{t+r+i} W_{\lfloor \lambda n \rfloor - (t+r+i)}(t+r+i, k-1; n-(t+2r), k).$$

From (2) we know

$$W_{\lfloor \lambda n\rfloor-(t+r+i)}(t+r+i,k-1;n-(t+2r),k)\sim\frac{1}{\sqrt{2\pi n}\sigma}\frac{2^{n-(t+2r)}}{\alpha^{\lfloor \lambda n\rfloor}}.$$

Applying (1) (with $x=0$) to a sum of n independent copies of ξ_1 yields

$$W_{\lfloor \lambda n\rfloor}(n,k)\sim\frac{1}{\sqrt{2\pi n}\sigma}\frac{2^n}{\alpha^{\lfloor \lambda n\rfloor}}. \tag{3}$$

Thus,

$$\lim_{n\to\infty}\frac{|\mathcal{F}_r|}{W_{\lfloor \lambda n\rfloor}(n,k)}=\frac{1}{2^{t+2r}}\sum_{i=0}^{r}\binom{t+2r}{t+r+i}=\frac{1}{2}-\frac{1}{2^{t+2r}}\sum_{0\le i<\frac{t}{2}}{}'\binom{t+2r}{t/2+r+i},$$

where $i\in\mathbb{N}$ if $2\mid t$, $i\in\mathbb{N}+\frac{1}{2}$ if $2\nmid t$ and $\sum_i'$ means taking half the (possible) summand for $i=0$.

Now let r vary. Applying again (1) (with $x=i/\sigma_3\sqrt{t+2r}$) to a sum of $t+2r$ independent copies of ξ_3 yields

$$\binom{t+2r}{t/2+r+i}\sim\frac{1}{\sqrt{2\pi(t+2r)}\sigma_3}2^{t+2r}\quad\text{for } r\to\infty.$$

The claim follows. □

Lemma 3.1 shows $M_{\lfloor \lambda n\rfloor}(n,k,t)\gtrsim\frac{1}{2}W_{\lfloor \lambda n\rfloor}(n,k)$. Case b) follows already from the converse if $t=1$.

Lemma 3.2. $M_{\lfloor \lambda n\rfloor}(n,k,1)\lesssim\frac{1}{2}W_{\lfloor \lambda n\rfloor}(n,k)$.

Proof. Let $\epsilon>0$ be given. From (3) we know

$$(1-\epsilon)W_{\lfloor \lambda n\rfloor}(n,k)<\frac{1}{\sqrt{2\pi n}\sigma}\frac{2^n}{\alpha^{\lfloor \lambda n\rfloor}}<(1+\epsilon)W_{\lfloor \lambda n\rfloor}(n,k) \tag{4}$$

for sufficiently large n.

In addition to ϵ let $M>0$ be given. (M will be determined by ϵ later.) We will apply (1) to ξ_2 and ξ_3 with specified error estimates. Let $u\in\mathbb{N}$ if $2\mid n$ and $u\in\mathbb{N}+1/2$ if $2\nmid n$. Applying (1) (with $x=\frac{u}{\sigma_2\sqrt{n}}$) to a sum of n independent copies of ξ_3 yields

$$\left|\sigma_3\sqrt{n}\,\mathrm{Prob}\left(\sum_{i=1}^{n}\xi_3=\frac{n}{2}+u\right)-\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2\sigma_3{}^2n}}\right|<\epsilon\frac{1}{\sqrt{2\pi}}e^{-\frac{M^2}{2\sigma_3{}^2}}$$

for sufficiently large n. It follows for $|u|\le M\sqrt{n}$ and large n

$$(1-\epsilon)\frac{1}{\sqrt{2\pi n}\sigma_3}2^ne^{-\frac{u^2}{2\sigma_3{}^2n}}<\binom{n}{\frac{n}{2}+u}<(1+\epsilon)\frac{1}{\sqrt{2\pi n}\sigma_3}2^ne^{-\frac{u^2}{2\sigma_3{}^2n}}. \tag{5}$$

Applying (1) (with $x = -2\lambda u/\sigma_2\sqrt{\frac{n}{2}+u}$) to a sum of $\frac{n}{2}+u$ independent copies of ξ_2 yields

$$\left|\sigma_2\sqrt{n/2+u}\,\mathrm{Prob}\left(\sum_{i=1}^{n/2+u}\xi_2 = \lfloor\lambda n\rfloor - (\frac{n}{2}+u)\right) - \frac{1}{\sqrt{2\pi}}e^{-\frac{2\lambda^2u^2}{\sigma_2{}^2(n/2+u)}}\right| < \epsilon\frac{1}{\sqrt{2\pi}}e^{-\frac{8\lambda^2M^2}{\sigma_2{}^2}}$$

for sufficiently large n. Note that for $|u| \le M\sqrt{n}$

$$n/2+u > n/4, \quad (1-\epsilon)\frac{1}{\sqrt{n/2}} < \frac{1}{\sqrt{n/2+u}} < (1+\epsilon)\frac{1}{\sqrt{n/2}}$$

and

$$(1-\epsilon)e^{-\frac{4\lambda^2u^2}{\sigma_2{}^2n}} < e^{-\frac{2\lambda^2u^2}{\sigma_2{}^2(n/2+u)}} < (1+\epsilon)e^{-\frac{4\lambda^2u^2}{\sigma_2{}^2n}}$$

if n is large enough. It follows for $|u| \le M\sqrt{n}$ and large n

$$(6)\quad (1-\epsilon)^3\frac{1}{\sqrt{\pi n}\sigma_2}\frac{1}{\alpha^{\lfloor\lambda n\rfloor}}e^{-\frac{4\lambda^2u^2}{\sigma_2{}^2n}} < W_{\lfloor\lambda n\rfloor-(n/2+u)}(n/2+u,k-1) < (1+\epsilon)^3\frac{1}{\sqrt{\pi n}\sigma_2}\frac{1}{\alpha^{\lfloor\lambda n\rfloor}}e^{-\frac{4\lambda^2u^2}{\sigma_2{}^2n}}$$

and consequently

$$(7)\quad \left(\frac{1-\epsilon}{1+\epsilon}\right)^3 W_{\lfloor\lambda n\rfloor-(n/2-u)}(n/2-u,k-1) < W_{\lfloor\lambda n\rfloor-(n/2+u)}(n/2+u,k-1) < \left(\frac{1+\epsilon}{1-\epsilon}\right)^3 W_{\lfloor\lambda n\rfloor-(n/2-u)}(n/2-u,k-1).$$

Combining (5) and (6) yields

$$(1-\epsilon)^4\frac{2^n}{\sqrt{2\pi n}\sigma}\frac{1}{\alpha^{\lfloor\lambda n\rfloor}}\frac{\sigma}{\sqrt{\pi}\sigma_2\sigma_3}\sum_{|u|\le M\sqrt{n}}\frac{1}{\sqrt{n}}e^{-\frac{u^2}{n}(\frac{4\lambda^2}{\sigma_2{}^2}+\frac{1}{2\sigma_3{}^2})} < \sum_{|u|\le M\sqrt{n}}\binom{n}{n/2+u}W_{\lfloor\lambda n\rfloor-(n/2+u)}(n/2+u,k-1) < (1+\epsilon)^4\frac{2^n}{\sqrt{2\pi n}\sigma}\frac{1}{\alpha^{\lfloor\lambda n\rfloor}}\frac{\sigma}{\sqrt{\pi}\sigma_2\sigma_3}\sum_{|u|\le M\sqrt{n}}\frac{1}{\sqrt{n}}e^{-\frac{u^2}{n}(\frac{4\lambda^2}{\sigma_2{}^2}+\frac{1}{2\sigma_3{}^2})}.$$

We have

$$\int_{-\infty}^{\infty}e^{-t^2(\frac{4\lambda^2}{\sigma_2{}^2}+\frac{1}{2\sigma_3{}^2})}\,dt = \frac{\sqrt{\pi}}{\sqrt{\frac{4\lambda^2}{\sigma_2{}^2}+\frac{1}{2\sigma_3{}^2}}} = \frac{\sqrt{\pi}\sigma_2\sigma_3}{\sigma}$$

and hence

$$(1-\epsilon) < \frac{\sigma}{\sqrt{\pi}\sigma_2\sigma_3} \sum_{|u|\le M\sqrt{n}} \frac{1}{\sqrt{n}} e^{-\frac{u^2}{n}\left(\frac{4\lambda^2}{\sigma_2{}^2}+\frac{1}{2\sigma_3{}^2}\right)} < (1+\epsilon)$$

provided n and M are large enough. Note that here M is determined by ϵ only. In view of (4) we conclude for sufficiently large n

$$(8) \quad (1-\epsilon)^6 W_{\lfloor \lambda n \rfloor}(n,k) < \sum_{|u|\le M\sqrt{n}} \binom{n}{n/2+u} W_{\lfloor \lambda n \rfloor-(n/2+u)}(n/2+u, k-1)$$
$$< (1+\epsilon)^6 W_{\lfloor \lambda n \rfloor}(n,k).$$

Finally, combining (7) and (8) yields

$$(9) \quad \frac{1}{2}\frac{(1-\epsilon)^9}{(1+\epsilon)^3} W_{\lfloor \lambda n \rfloor}(n,k)$$
$$< \sideset{}{'}\sum_{0\le u\le M\sqrt{n}} \binom{n}{n/2+u} W_{\lfloor \lambda n \rfloor-(n/2+u)}(n/2+u, k-1) <$$
$$\frac{1}{2}\frac{(1+\epsilon)^9}{(1-\epsilon)^3} W_{\lfloor \lambda n \rfloor}(n,k).$$

Now let $\mathcal{F} \subseteq N_{\lfloor \lambda n \rfloor}(n,k)$ be an arbitrary intersecting family. Let

$$\mathcal{F}_1 = \left\{ \boldsymbol{a} \in \mathcal{F} : |\mathrm{supp}(\boldsymbol{a})| \in [n/2 - M\sqrt{n}, n/2 + M\sqrt{n}] \right\},$$
$$\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1,$$

where M is chosen such that (8) and (9) are satisfied. From (8) we know

$$|\mathcal{F}_2| < 6\epsilon W_{\lfloor \lambda n \rfloor}(n,k)$$

for sufficiently large n. Since $\mathcal{F}$ is intersecting we have for all u

$$\left| \mathrm{supp}(\mathcal{F}) \cap \binom{n}{n/2+u} \right| + \left| \mathrm{supp}(\mathcal{F}) \cap \binom{n}{n/2-u} \right| \le \binom{n}{n/2+u}.$$

It follows with (7) and (9) for sufficiently large n

$$|\mathcal{F}_1| < \frac{1}{2}\frac{(1+\epsilon)^{12}}{(1-\epsilon)^6} W_{\lfloor \lambda n \rfloor}(n,k).$$

Hence, $|\mathcal{F}| \lesssim \frac{1}{2} W_{\lfloor \lambda n \rfloor}(n,k)$. □

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