AN INTERSECTION THEOREM FOR WEIGHTED SETS

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Abstract. A weight function $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ from the set of all subsets of $[n] = \{1, \dots, n\}$ to the nonnegative real numbers is called shift–monotone in $\{m+1, \dots, n\}$ if $\omega(\{i_1, \dots, i_m\}) \geq \omega(\{j_1, \dots, j_m\})$ holds for all $\{i_1, \dots, i_m\}, \{j_1, \dots, j_m\} \subseteq [n]$ with $i_\ell \leq j_\ell, \ell = 1, \dots, m$, and if $\omega(I) \geq \omega(J)$ holds for all $I, J \subseteq [n]$ with $I \subseteq J$ and $J \setminus I \subseteq \{m+1, \dots, n\}$. A family $\mathcal{F} \subseteq 2^{[n]}$ is called intersecting in $[m]$ if $F \cap G \cap [m] \neq \emptyset$ for all $F, G \in \mathcal{F}$. Let $\omega(\mathcal{F}) = \sum_{F \in \mathcal{F}} \omega(F)$. We show that $\max\{\omega(\mathcal{F}) : \mathcal{F} \subseteq 2^{[n]}, \mathcal{F}$ is intersecting in $[n]\} = \max\{\omega(\mathcal{F}) : \mathcal{F} \subseteq 2^{[n]}, \mathcal{F}$ is intersecting in $[m]\}$ provided that ω is shift–monotone in $\{m+1, \dots, n\}$. An application to the poset of colored subsets of a finite set is given.

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1. Introduction

Let $[n] = \{1, 2, \dots, n\}$, $2^{[n]} = \{A \subseteq [n]\}$ and consider a weight function $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the set of all nonnegative real numbers. Define the weight of a family $\mathcal{F} \subseteq 2^{[n]}$ as the sum of weights of its members,

$$\omega(\mathcal{F}) = \sum_{A \in \mathcal{F}} \omega(A).$$

A subset $\mathcal{F} \subseteq 2^{[n]}$ is called an intersecting family if no two sets in $\mathcal{F}$ are disjoint. Let us define

$$M(n, \omega) = \max\{\omega(\mathcal{F}) : \mathcal{F} \subseteq 2^{[n]} \text{ is intersecting}\}.$$

Theorem 1 (Erdős, Schönheim [8]). *Let $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be increasing, i.e. $\omega(A) \leq \omega(B)$ whenever $A \subseteq B$. Then*

$$M(n, \omega) = \frac{1}{2} \sum_{A \subseteq [n]} \max\{\omega(A), \omega([n] \setminus A)\}.$$

Note that the RHS is always an upper bound for $M(n, \omega)$, but equality need not occur for arbitrary weights.

We are concerned with decreasing weights satisfying an additional property. For positive numbers i, j let $[i, j] = \{i, i+1, \dots, j\}$. We call a family $\mathcal{F} \subseteq 2^{[n]}$ intersecting in $[i, j] \subseteq [n]$ if $A \cap B \cap [i, j] \neq \emptyset$ holds for all $A, B \in \mathcal{F}$.

Definition. *The weight function $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is called shift–monotone in $[m+1, n]$, $0 \leq m \leq n$, if*

$$\omega(a_1, \dots, a_j) \geq \omega(b_1, \dots, b_j) \text{ for all } \{a_1, \dots, a_j\}, \{b_1, \dots, b_j\} \subseteq [n] \\ \text{with } a_1 \leq b_1, a_2 \leq b_2, \dots, a_j \leq b_j,$$

and

$$\omega(A) \geq \omega(B) \text{ for all } A, B \subseteq [n] \text{ with } A \subseteq B \text{ and } B \setminus A \subseteq [m+1, n].$$

Theorem 2. *Let $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be shift–monotone in $[m+1, n]$, $1 \leq m \leq n$. Then*

$$M(n, \omega) = \max\{\omega(\mathcal{F}) : \mathcal{F} \subseteq 2^{[n]} \text{ is intersecting in } [1, m]\}.$$

In particular, if ω is shift–monotone in $[2, n]$, then

$$M(n, \omega) = \omega(\{A \subseteq [n] : 1 \in A\}).$$

The proof of Theorem 2 which is based on the concept of generating sets introduced by Ahlswede, Khachatrian [2] is given in the next section.

We give three applications of Theorem 2, two of them being known results.

Recall that $\mathcal{I} \subseteq 2^{[n]}$ is called an ideal (or downset) if $A \in \mathcal{I}, B \subseteq A$ implies $B \in \mathcal{I}$.

Theorem 3 (Chvátal [3])**.** *Let $\mathcal{I} \subseteq 2^{[n]}$ be an ideal with the following property: $\{a_1, \ldots, a_j\} \in \mathcal{I}$ and $b_i \leq a_i, i = 1, \ldots, j$, implies $\{b_1, \ldots, b_j\} \in \mathcal{I}$. Then the star*

$$\{A \in \mathcal{I} : 1 \in A\}$$

is a maximum intersecting subfamily of $\mathcal{I}$.

Theorem 3 holds under the weaker hypothesis that the ideal $\mathcal{I}$ is compressed with respect to 1, i.e. replacing $a \neq 1$ by 1 in an element of $\mathcal{I}$ gives another element of $\mathcal{I}$, see Snevily [11].

Theorem 4 (Ahlswede, Cai [1])**.** *Let $\omega_1, \omega_2, \ldots, \omega_n$ be positive real numbers and let $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be defined by*

$$\omega(A) = \prod_{a \in A} \omega_a \text{ for } A \subseteq [n].$$

If $\omega_1 \geq \ldots \geq \omega_m > 1 \geq \omega_{m+1} \geq \ldots \geq \omega_n$ then

$$M(n, \omega) = \begin{cases} \omega(\{A \subseteq [n] : 1 \in A\}) & \text{if } m \leq 1 \\ \frac{1}{2} \sum_{A \subseteq [m]} \max\{\omega'(A), \omega'([m] \setminus A)\} & \text{otherwise,} \end{cases}$$

where $\omega' : 2^{[m]} \to \mathbb{R}_{\geq 0}$ is given by

$$\omega'(A) = \omega(A)\omega(2^{[m+1,n]}) \text{ for } A \subseteq [m].$$

Theorem 4 follows from Theorem 1 and 2. The original proof in [1] is by induction on n and can be adopted to shift–monotone weights which then yields the special case of Theorem 2.

Our third application is an intersecting theorem for the poset of colored subsets and is given in Section 3. We will need the following theorem which solves the weighted intersection problem for weight functions depending only on the size of the sets.

Theorem 5 (Erdős, Frankl, Katona [9])**.** *Let $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a weight function satisfying $\omega(A) = \omega(B)$ for all $A, B \subseteq [n]$ with $|A| = |B|$. Then*

$$M(n, \omega) = \max_{1 \leq i \leq \frac{n+1}{2}} \{\omega(\mathcal{F}_i)\},$$

where the families $\mathcal{F}_i \subseteq 2^{[n]}$, $1 \leq i \leq \frac{n+1}{2}$, are given by

$$\mathcal{F}_i = \{A \subseteq [n] : i \leq |A| \leq n - i, 1 \in A\} \cup \{A \subseteq [n] : |A| \geq n - i + 1\}.$$

2. Proof of Theorem 2

For every family $\mathcal{F} \subseteq 2^{[n]}$ let $\mathcal{M} = \mathcal{M}(\mathcal{F})$ denote the set of all minimal elements of $\mathcal{F}$ (with respect to set inclusion). Define

$$s^+(A) = \max\{i : i \in A\} \text{ for } A \subseteq [n],$$
$$s^+(\mathcal{F}) = \max\{s^+(A) : A \in \mathcal{F}\} \text{ for } \mathcal{F} \subseteq 2^{[n]}.$$

Now choose under all optimal families $\mathcal{A} \subseteq 2^{[n]}$ (i.e. $\omega(\mathcal{A}) = M(n,\omega)$) one for which

$$s^+ = s^+(\mathcal{M}(\mathcal{A})) \tag{1}$$

is minimal, and under all such families choose one for which

$$|\{A \in \mathcal{M}(\mathcal{A}) : s^+ \in A\}| \tag{2}$$

is minimal.

We may assume that $\mathcal{A}$ is a filter (upset), i.e. $A \in \mathcal{A}, A \subseteq B$ implies $B \in \mathcal{A}$.

Recall the well–known Shifting–operation $S_{ij} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ (see [7]) defined for $i, j \in [n]$ by

$$S_{ij}(\mathcal{F}) = \{S_{ij}(A) : A \in \mathcal{F}\} \text{ for } \mathcal{F} \subseteq 2^{[n]},$$

where

$$S_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\}, & \text{if this is not an element of } \mathcal{F} \text{ and} \\ & A \cap \{i,j\} = \{j\} \\ A & \text{otherwise.} \end{cases}$$

Note that if $\mathcal{F}$ is intersecting then $S_{ij}(\mathcal{F})$ is intersecting too. Furthermore, applying S_{ij} for $i < j$ to $\mathcal{F}$ does not increase the parameters defined in (1) and (2).

Hence, since ω is shift–monotone, we can assume that our optimal family $\mathcal{A}$ is left–shifted (= stable with respect to left–shifting), i.e.

$$S_{ij}(\mathcal{A}) = \mathcal{A} \text{ for all } i < j.$$

We will show that $s^+(\mathcal{M}(\mathcal{A})) \leq m$ which finishes the proof. Assume the contrary, i.e.

$$s^+ = s^+(\mathcal{M}(\mathcal{A})) > m.$$

Consider a set $A \in \mathcal{M} = \mathcal{M}(\mathcal{A})$ with $s^+ \in A$. Let $A' = A \setminus \{s^+\}$, $B = [s^+] \setminus A'$ and $B' = B \setminus \{s^+\}$.

First note that no set $C' \cup \{s^+\}$ with $C' \subsetneq B'$ is contained in $\mathcal{A}$. This is true because otherwise $(A' \cup \{i\}) \cap (C' \cup \{s^+\}) = \emptyset$ for some $i \in [s^+] \setminus (A \cup C')$ but $A' \cup \{i\} \in \mathcal{A}$ and $\mathcal{A}$ is intersecting, a contradiction.

Next we claim that $B \in \mathcal{M}$. Otherwise, the previous remark shows $B \notin \mathcal{A}$. But then $\mathcal{A} \cup \{A'\}$ is intersecting which contradicts our choice of $\mathcal{A}$.

Now consider the filter $\mathcal{A}_1$ resp. $\mathcal{A}_2$ generated by $\mathcal{M}_1 = (\mathcal{M} \setminus \{A, B\}) \cup \{A'\}$ resp. $\mathcal{M}_2 = (\mathcal{M} \setminus \{A, B\}) \cup \{B'\}$, i.e.

$$\mathcal{A}_i = \{A \subseteq [n] : A \supseteq B \text{ for some } B \in \mathcal{M}_i\}, \quad i = 1, 2.$$

Note that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are intersecting. We claim that

$$\max\{\omega(\mathcal{A}_1), \omega(\mathcal{A}_2)\} \geq \omega(\mathcal{A}) \tag{3}$$

which contradicts again our choice of $\mathcal{A}$.

Obviously, $\mathcal{A}\backslash\mathcal{A}_1$ consists exactly of those sets of $\mathcal{A}$ which are generated only by B. Furthermore, these sets are exactly the sets $B \cup C$ with $C \subseteq [s^+ + 1, n]$. It follows

$$\omega(\mathcal{A} \setminus \mathcal{A}_1) = \sum_{C \subseteq [s^+ + 1, n]} \omega(B \cup C).$$

Similary, it is easy to see that $\mathcal{A}_1 \setminus \mathcal{A}$ consists exactly of the sets $A' \cup C$ with $C \in [s^+ + 1, n]$. It follows

$$\omega(\mathcal{A}_1 \setminus \mathcal{A}) = \sum_{C \subseteq [s^+ + 1, n]} \omega(A' \cup C).$$

Analogously, it holds

$$\omega(\mathcal{A} \setminus \mathcal{A}_2) = \sum_{C \subseteq [s^+ + 1, n]} \omega(A \cup C),$$

$$\omega(\mathcal{A}_2 \setminus \mathcal{A}) = \sum_{C \subseteq [s^+ + 1, n]} \omega(B' \cup C).$$

Since ω is shift–monotone in $[m + 1, n]$, the last four equalities imply

$$\omega(\mathcal{A} \setminus \mathcal{A}_1) + \omega(\mathcal{A} \setminus \mathcal{A}_2) \leq \omega(\mathcal{A}_1 \setminus \mathcal{A}) + \omega(\mathcal{A}_2 \setminus \mathcal{A})$$

which shows (3).

3. An application

Consider a finite set C of $k_1 + k_2 + \ldots + k_n$ elements that is partitioned into n sets (color classes) C_i, $i = 1, \ldots, n$, each set C_i having size k_i. The family of all subsets of C which intersect each color class C_i in at most one element is denoted by $Col(k_1, \ldots, k_n)$. With respect to set inclusion, $Col(k_1, \ldots, k_n)$ is a ranked poset called poset of colored subsets (see [6, p.344]). Let $W_k(k_1, \ldots, k_n)$ denote the k-th Whitney number of $Col(k_1, \ldots, k_n)$, i.e. $W_k(k_1, \ldots, k_n) = |Col(k_1, \ldots, k_n) \cap \binom{C}{k}|$. We assume that $k_1 \leq \ldots \leq k_n$.

It is not difficult to see (and follows also from the proof below) that a maximum intersecting subfamily of $Col(k_1, \ldots, k_n)$ has cardinality $\frac{1}{k_1+1}|Col(k_1, \ldots, k_n)|$. We study the uniform case.

Theorem 6. *Let $1 = k_1 = \ldots = k_m < k_{m+1} \leq \ldots \leq k_n$. Then one of the families $\mathcal{C}_i \cap \binom{C}{k}$, $0 \leq i \leq \frac{m+1}{2}$, is a maximum k-uniform intersecting subfamily of $Col(k_1, \ldots, k_n)$, where*

$$\begin{aligned}\mathcal{C}_i = \{A \in Col(k_1, \ldots, k_n) : i \leq |j \in [m] : A \cap C_j \neq \emptyset| \leq m - i \text{ and} \\ A \cap C_1 \neq \emptyset\} \\ \cup \{A \in Col(k_1, \ldots, k_n) : |j \in [m] : A \cap C_j \neq \emptyset| \geq m - i + 1\}.\end{aligned}$$

The case $k_1 = k_2 = \ldots = k_n > 1$ was treated by Meyer [10], Frankl [4] and Engel [5].

Proof. W.l.o.g. let $C_i = [k_1 + \ldots + k_{i-1} + 1, k_1 + \ldots + k_i]$. We can assume that an optimal family $\mathcal{A} \subseteq Col(k_1, \ldots, k_n)$ is left–shifted inside each color class C_i, i.e. $S_{jk}(\mathcal{A}) = \mathcal{A}$ for all $j, k \in C_i$, $j < k$. Then, it is easy to see that

$$supp(\mathcal{A}) = \big\{\{i : A \text{ intersects } C_i \text{ in its first element}\} : A \in \mathcal{A}\big\}$$

is an intersecting family in $2^{[n]}$. Thus, we have a weighted intersection problem with weight function $\omega : 2^{[n]} \to \mathbb{R}_{\geq 0}$ given by

$$\omega(\{i_1, \ldots, i_r\}) = W_{k-r}\big(\pi_{\{i_1,\ldots,i_r\}}(k_1 - 1, \ldots, k_n - 1)\big),$$

where π_I denotes the projection map onto the coordinates that are not contained in I. This weight function is easily seen to be shift–monotone in $[m+1, n]$. Moreover,

$$\omega(\{A \subseteq [n] : A \cap [m] = B\})$$

depends only on $|B|$. This allows us to apply Theorem 2 and 5. □

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