Suborthogonal double covers of complete graphs

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Abstract. A family $\mathcal{G}$ of isomorphic copies of a given graph $\mathfrak{G}$ is said to be a suborthogonal double cover (SODC) of the complete graph $\mathfrak{K}_n$ by $\mathfrak{G}$, if every edge of $\mathfrak{K}_n$ belongs to exactly two members of $\mathcal{G}$ and any two different elements from $\mathcal{G}$ share at most one edge. We introduce the notion of an SODC as a generalization of the well-known concept of orthogonal double covers. Our objective is to investigate the sets $\mathcal{S}(\mathfrak{G})$ of integers n allowing an SODC of $\mathfrak{K}_n$ by a given graph $\mathfrak{G}$. We shall prove that $\mathcal{S}(\mathfrak{G})$ contains almost all integers n satisfying certain necessary conditions.

1 Introduction

All graphs considered in this paper are assumed to be finite, simple and with non-empty edge set.

An orthogonal double cover (ODC) of the complete graph $\mathfrak{K}_n$ on n vertices is a family $\mathcal{G} = \{\mathfrak{G}_1, \ldots, \mathfrak{G}_s\}$ of subgraphs of $\mathfrak{K}_n$ (called pages) such that:
(i) every edge of $\mathfrak{K}_n$ occurs in exactly two pages of $\mathcal{G}$ and
(ii) any two different pages of $\mathcal{G}$ have exactly one edge in common.

In addition, if all pages are isomorphic copies of a given graph $\mathfrak{G}$, the family $\mathcal{G}$ is said to be an ODC by $\mathfrak{G}$. The investigation of ODCs goes back to a paper of Demetrovics, Füredi and Katona [4] on minimum databases satisfying a certain closure condition. In the context of relational database theory, they asked for ODCs by graphs $\mathfrak{G}$, where $\mathfrak{G}$ consists of disjoint triangles and a singleton. Their question gave rise to a number of papers concerning ODCs by special graphs. The results focus on clique graphs [1, 5], graphs of maximum degree two [2, 3, 9] and graphs consisting of disjoint cycles [6, 7]. The existence of ODCs by graphs with 9 vertices or less was decided in [13].

By a simple counting argument, one can show that every ODC of $\mathfrak{K}_n$ consists of n pages each of them containing exactly $n-1$ edges. This gives a special motivation for investigating ODCs by trees [8, 14]. However, to construct an ODC by a given tree $\mathfrak{T}$ turns out to be difficult already for paths, although there are some results in this direction [11, 12]. According to [14], there exists an ODC for every tree $\mathfrak{T} \neq \mathfrak{P}_4$ with at most 14 vertices. For the path $\mathfrak{P}_4$ on 4 vertices, it is easy to check that there is no ODC by $\mathfrak{P}_4$. Generally, the following is conjectured in [8]:

Conjecture. *Let $\mathfrak{T}$ be a tree with n vertices and $\mathfrak{T} \neq \mathfrak{P}_4$. Then there exists an ODC of $\mathfrak{K}_n$ by $\mathfrak{T}$.*

The suspected exceptional case $\mathfrak{P}_4$ inspired us to consider the following question: Does there exist a family $\mathcal{G} = \{\mathfrak{G}_1, \ldots, \mathfrak{G}_s\}$ of subgraphs of the complete graph $\mathfrak{K}_n$ satisfying condition (i) and the relaxed condition
(ii') any two different pages of $\mathcal{G}$ have at most one edge in common ?

We call such a family $\mathcal{G}$ a suborthogonal double cover (SODC) of $\mathfrak{K}_n$. Note, that this time the cardinality of $\mathcal{G}$ does not necessarily coincide with the vertex number of $\mathfrak{K}_n$. However, if $\mathcal{G}$ consists of exactly n pages then it is just an ODC. In Section 2 we shall prove

Theorem 1. *There exists an SODC of the complete graph $\mathfrak{K}_n$ by $\mathfrak{P}_4$ iff $n \geq 6$ and $n \equiv 0$ or $1 \mod 3$.*

Hence, the non-existence of an ODC by $\mathfrak{P}_4$, i.e. of an SODC of the complete graph $\mathfrak{K}_4$ by $\mathfrak{P}_4$ is indeed an exception when dealing with suborthogonal double covers.

2 SODCs by $\mathfrak{P}_4$

It can easily be checked that whenever there exists an SODC of $\mathfrak{K}_n$ by a given graph $\mathfrak{G}$ on at most n vertices then $n(n-1)$ is a multiple of the edge number of $\mathfrak{G}$. This observation implies $n(n-1) \equiv 0 \mod 3$ as a necessary condition for the existence of an SODC by $\mathfrak{P}_4$. Hence, our objective is to show that this condition is also sufficient for $n \geq 6$.

Throughout, let the vertex set of $\mathfrak{K}_n$ be $V = \{0, 1, \ldots, n-1\}$. All addition of vertex labels shall be done modulo n. For a subgraph $\mathfrak{G}$ of $\mathfrak{K}_n$, let $\mathfrak{G} + i$ denote the isomorphic image of $\mathfrak{G}$ under the mapping $v \to v + i$. The set of graphs $\mathfrak{G} + i$, $i \in V$ is called the orbit of $\mathfrak{G}$. The path with edges ab, bc and cd shall be denoted by (a, b, c, d), for short. The following lemma provides some direct constructions for small n which shall be used for recursion, later on.

Lemma 2. *There exist SODCs of $\mathfrak{K}_n$ by $\mathfrak{P}_4$ for $n = 6, 7, 9$ and 10.*

Proof: For $n = 6$, the SODC consists of the orbit of $(0, 1, 5, 2)$ and the pages $(1, 3, 4, 0)$, $(2, 4, 5, 1)$, $(3, 5, 0, 2)$ as well as $(0, 1, 2, 3)$.

For $n = 7$, the SODC is the union of the orbits of $(0, 1, 2, 4)$ and $(0, 2, 5, 1)$.

For $n = 9$, the SODC consists of the orbits of $(0, 1, 3, 7)$ and $(0, 3, 6, 1)$ together with the pages $(0, 1, 2, 3)$, $(3, 4, 5, 6)$ and $(6, 7, 8, 0)$ as well as the pages $(0, 2, 4, 6)$, $(6, 8, 1, 3)$ and $(3, 5, 7, 0)$.

Finally, for $n = 10$, the SODC is the union of the orbits of $(0, 1, 2, 4)$, $(0, 2, 5, 8)$ and $(0, 4, 8, 3)$. □

Although we defined only suborthogonal double covers of complete graphs $\mathfrak{K}_n$, the definition can easily be generalized to arbitrary graphs $\mathfrak{H}$. Hence, we are looking for subgraphs of $\mathfrak{H}$ satisfying conditions (i) and (ii'). We will use such SODCs of certain complete bipartite graphs to establish Theorem 1.

Lemma 3. *There is an SODC of the complete bipartite graph $\mathfrak{K}_{3,3}$ by $\mathfrak{P}_4$.*

Proof: Denote the two partition classes of $\mathfrak{K}_{3,3}$ by $W = \{0, 1, 2\}$ and $W' = \{0', 1', 2'\}$. Let the SODC consist of the pages $(0, 1', 1, 2')$, $(1, 2', 2, 0')$ and $(2, 0', 0, 1')$ as well as $(0', 1, 1', 2)$, $(1', 2, 2', 0)$ and $(2', 0, 0', 1)$. □

In the sequel, we shall describe a recursive construction which makes use of the following simple observation.

Lemma 4. *Assume there exist SODCs of certain graphs $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$ by a given graph $\mathfrak{G}$. If a graph $\mathfrak{H}$ can be decomposed into edge-disjoint copies of the graphs $\mathfrak{H}_1, \ldots, \mathfrak{H}_m$, then there exists an SODC of $\mathfrak{H}$ by $\mathfrak{G}$, too.*

This gives us the following corollary of Lemma 3.

Corollary 5. *There is an SODC of the complete bipartite graph $\mathfrak{K}_{3p,3q}$ by $\mathfrak{P}_4$ for all integers $p \geq 1$ and $q \geq 1$.*

Proof: $\mathfrak{K}_{3p,3q}$ can be decomposed into pq edge-disjoint copies of $\mathfrak{K}_{3,3}$. $\square$

Finally, we obtain the proof of our theorem.

Proof: (Theorem 1.) It remains to prove the sufficiency of the condition $n \equiv 0, 1 \mod 3$ for $n \geq 6$. We consider the two possible cases separately. First assume, $n = 3t$ for some positive integer t. We proceed by induction on t. For $t = 2$ and 3, Lemma 2 ensures the existence of an SODC of $\mathfrak{K}_6$ and $\mathfrak{K}_9$, respectively. Let $t \geq 4$ and suppose, the claim is true for all positive integers smaller than t. Obviously, $\mathfrak{K}_n$ can be decomposed into complete graphs $\mathfrak{K}_6$ with vertex set $U = \{0, 1, \ldots, 5\}$ and $\mathfrak{K}_{n-6}$ with vertex set $W = \{6, 7, \ldots, n-1\}$ and a complete bipartite graph $\mathfrak{K}_{6,n-6}$ with partition classes U and W. However, $n - 6 = 3(t-2)$ with $t - 2 \geq 2$. By Lemma 2, the induction hypothesis and Corollary 5, we have SODCs of $\mathfrak{K}_6$, $\mathfrak{K}_{n-6}$ and $\mathfrak{K}_{6,n-6}$. Now, Lemma 4 implies the statement.

On the other hand, assume $n = 3t + 1$ for some positive integer t. Again, we proceed by induction on t. For $t = 2$ and 3, the existence of an SODC of $\mathfrak{K}_7$ and $\mathfrak{K}_{10}$, respectively, follows by Lemma 2. Let $t \geq 4$ and suppose the correctness of the claim for all smaller positive integers. This time, $\mathfrak{K}_n$ can be decomposed into complete graphs $\mathfrak{K}_7$ with vertex set $U = \{0, 1, \ldots, 6\}$ and $\mathfrak{K}_{n-6}$ with vertex set $W = \{0\} \cup \{7, 8, \ldots, n-1\}$ together with a complete bipartite graph $\mathfrak{K}_{6,n-7}$ with partition classes $U \backslash \{0\}$ and $W \backslash \{0\}$. Note, that $n - 7 = 3(t-2)$ with $t - 2 \geq 2$. For each of the graphs $\mathfrak{K}_7$, $\mathfrak{K}_{n-6}$ and $\mathfrak{K}_{6,n-7}$, there exists an SODC by $\mathfrak{P}_4$ according to Lemma 2, the induction hypothesis and Corollary 5. Hence, Lemma 4 concludes the proof. $\square$

3 Asymptotic existence of SODCs

A natural question to be asked at this point is, for which pairs $(n, \mathfrak{G})$ there exists an SODC of $\mathfrak{K}_n$ by $\mathfrak{G}$. Of course, this question is hopeless in general, since the answer would solve the existence problem of ODCs, too. Nevertheless, we are able to decide the existence of an SODC of $\mathfrak{K}_n$ by a given graph $\mathfrak{G}$ for almost all n.

Let $e(\mathfrak{G})$ denote the edge number of $\mathfrak{G}$ and $d(\mathfrak{G})$ the greatest common divisor of the vertex degrees in $\mathfrak{G}$. The existence of an SODC of the complete graph $\mathfrak{K}_n$ by $\mathfrak{G}$ immediately implies the following two necessary conditions which we record as

Lemma 6. *If there exists an SODC of $\mathfrak{K}_n$ by $\mathfrak{G}$ then*

$$n(n-1) \equiv 0 \mod e(\mathfrak{G}), \tag{1}$$

$$2(n-1) \equiv 0 \mod d(\mathfrak{G}). \tag{2}$$

The difficult thing is to prove the sufficiency of these conditions, at least asymptotically. To do this, we use a recent Wilson-type result on decompositions of complete digraphs [10]. Let $\vec{\mathfrak{D}}_n$ denote the complete (symmetric) digraph on n vertices, and let $\vec{\mathfrak{G}}$ be a subdigraph of $\vec{\mathfrak{D}}_n$. A suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ is a decomposition of $\vec{\mathfrak{D}}_n$ into isomorphic copies of $\vec{\mathfrak{G}}$ (again called pages) such that every pair of different pages contains at most one pair of oppositely directed arcs. Obviously, the existence of a suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$ implies the existence of an SODC of $\mathfrak{K}_n$ by the underlying graph $\mathfrak{G}$ of $\vec{\mathfrak{G}}$.

Necessary conditions for the existence of arbitrary $\vec{\mathfrak{G}}$-decompositions are well-known [16]. For any vertex w of $\vec{\mathfrak{G}}$, let $deg^-(w)$ and $deg^+(w)$ denote the indegree and the outdegree of w in $\vec{\mathfrak{G}}$, respectively. Further, let $d^{\pm}(\vec{\mathfrak{G}})$ be the smallest positive integer in the set

$$\{y \in \mathbb{Z} \ :\ \forall w \in \vec{\mathfrak{G}}\ \exists x_w \in \mathbb{Z} \text{ s.t. } \sum_{w\in\vec{\mathfrak{G}}} x_w deg^-(w) = \sum_{w\in\vec{\mathfrak{G}}} x_w deg^+(w) = y\}.$$

Whenever $\vec{\mathfrak{D}}_n$ has a $\vec{\mathfrak{G}}$-decomposition, then we have

$$n(n-1) \equiv 0 \mod e(\vec{\mathfrak{G}}), \tag{3}$$
$$n-1 \equiv 0 \mod d^{\pm}(\vec{\mathfrak{G}}). \tag{4}$$

In [10], we proved that these conditions are sufficient even for the existence of a suborthogonal $\vec{\mathfrak{G}}$-decomposition whenever n is large enough. This implies the following

Corollary 7. *Let $\mathfrak{G}$ be a graph with $d(\mathfrak{G})$ even, and let n be a sufficiently large positive integer satisfying the necessary conditions from Lemma 6. Then there is an SODC of $\mathfrak{K}_n$ by $\mathfrak{G}$.*

Proof: If $d(\mathfrak{G})$ is even, then all vertex degrees in $\mathfrak{G}$ are even, too. Hence, we are able to orientate $\mathfrak{G}$ such that in the obtained digraph $\vec{\mathfrak{G}}$ every vertex w satisfies $deg^-(w) = deg^+(w)$. This implies $d^{\pm}(\vec{\mathfrak{G}}) = \frac{1}{2}d(\mathfrak{G})$. Thus, the necessary conditions (1, 2) and (3, 4) are equivalent. The result on suborthogonal $\vec{\mathfrak{G}}$-decompositions [10] implies the claim. □

Unfortunately, in general the value $d^{\pm}(\vec{\mathfrak{G}})$ might be much larger than the value $d(\mathfrak{G})$ of the underlying graph $\mathfrak{G}$ of $\vec{\mathfrak{G}}$. Nevertheless, we shall use the result on suborthogonal decompositions to prove the asymptotic existence of SODCs. Therefore, we need the notion of a pairwise balanced design (PBD).

Let n be a positive integer and K a subset of $\mathbb{N}$. A pairwise balanced design $PBD(n,K)$ is a pair $(V,\mathcal{F})$, where V is a set of size n and $\mathcal{F}$ is a family of subsets of V (called blocks), such that every pair of distinct elements of V belongs to exactly one block and all block sizes are in K. Hence, a $PBD(n,K)$ is equivalent to a decomposition of the complete graph $\mathfrak{K}_n$ into edge-disjoint copies of complete graphs $\mathfrak{K}_b$, $b \in K$.

A set K of positive integers is said to be PBD-closed if the existence of a $PBD(n,K)$ implies that n belongs to K. By Lemma 4, the following claim is self-evident.

Lemma 8. *Let $\mathcal{S}(\mathfrak{G})$ denote the set of all integers n allowing an SODC of $\mathfrak{K}_n$ by $\mathfrak{G}$. Then, the set $\mathcal{S}(\mathfrak{G})$ is PBD-closed.*

A set K of positive integers is said to be eventually periodic with period π if, for any $n \in K$ and all sufficiently large integers $m \equiv n \mod \pi$, we obtain $m \in K$. A deep theorem of Wilson [15] states that a PBD-closed set K is eventually periodic with period $\beta(K) = g.c.d.\{n(n-1) : n \in K\}$, and hence, with period $\pi = \lambda\beta(K)$ for any positive integer λ.

Lemma 9. *The set $\mathcal{S}(\mathfrak{G})$ is eventually periodic with period $2e(\mathfrak{G})$.*

Proof: Let $\vec{\mathfrak{G}}$ be any orientation of $\mathfrak{G}$ and let $\mathcal{S}(\vec{\mathfrak{G}})$ denote the set of all positive integers n allowing a suborthogonal $\vec{\mathfrak{G}}$-decomposition of $\vec{\mathfrak{D}}_n$. From [10], we know that $2e(\vec{\mathfrak{G}})$ is a multiple of $\beta(\mathcal{S}(\vec{\mathfrak{G}}))$, and thus, a multiple of $\beta(\mathcal{S}(\mathfrak{G}))$ since $\mathcal{S}(\vec{\mathfrak{G}})$ is a subset of $\mathcal{S}(\mathfrak{G})$. Hence, $\mathcal{S}(\mathfrak{G})$ is eventually periodic with period $2e(\vec{\mathfrak{G}}) = 2e(\mathfrak{G})$. $\square$

Theorem 10. *Let $\mathfrak{G}$ be a given graph, and let n be a sufficiently large positive integer satisfying the necessary conditions from Lemma 6. Then there is an SODC of $\mathfrak{K}_n$ by $\mathfrak{G}$.*

Proof: Let n be any positive integer satisfying (1, 2). According to Lemma 9 it remains to show that there exists an integer m allowing an SODC of $\mathfrak{K}_m$ by $\mathfrak{G}$ such that $m \equiv n \mod 2e(\mathfrak{G})$. Actually, we shall determine such an integer m satisfying $m \in \mathcal{S}(\vec{\mathfrak{H}}) \subseteq \mathcal{S}(\mathfrak{G})$ for a suitable digraph $\vec{\mathfrak{H}}$.

Put $p = 2e(\mathfrak{G}) + 1 \geq 2$, and let $\vec{\mathfrak{G}}$ be any orientation of $\mathfrak{G}$. By reversing every arc from $\vec{\mathfrak{G}}$ we obtain a second orientation $\vec{\mathfrak{G}}'$ of $\mathfrak{G}$. Let $\vec{\mathfrak{H}}$ be the vertex-disjoint union of $\vec{\mathfrak{G}}$ and $p-1$ copies of $\vec{\mathfrak{G}}'$. As usual, let $d(\mathfrak{G})$ denote the $g.c.d.$ of the vertex degrees in $\mathfrak{G}$, i.e. the smallest positive in the set

$$\{y \in \mathbb{Z} \ : \ \forall w \in \mathfrak{G} \ \exists x_w \in \mathbb{Z} \text{ s.t. } \sum_{w \in \mathfrak{G}} x_w deg(w) = y\}.$$

It is easy to check, that $d(\mathfrak{G})$ is a multiple of $d^{\pm}(\vec{\mathfrak{H}})$ of the new digraph $\vec{\mathfrak{H}}$.

Choose $q = p^s$ for some integer s such that $m = nq$ is sufficiently large. By $q \equiv p \equiv 1 \bmod 2e(\mathfrak{G})$, we obtain $m = nq \equiv n \bmod 2e(\mathfrak{G})$. In addition, this implies $n(m-1) = n(nq-1) \equiv n(n-1) \equiv 0 \bmod e(\mathfrak{G})$ and, thus,

$$m(m-1) = qn(m-1) \equiv 0 \bmod e(\vec{\mathfrak{H}}),$$

where $e(\vec{\mathfrak{H}}) = pe(\mathfrak{G})$. Further, we also have $m - 1 = nq - 1 \equiv n - 1 \equiv 0 \bmod d(\mathfrak{G})$ which implies

$$m - 1 \equiv 0 \bmod d^{\pm}(\vec{\mathfrak{H}}).$$

This gives us the necessary conditions (3, 4) for a $\vec{\mathfrak{H}}$-decomposition of $\mathfrak{K}_m$ which imply $m \in \mathcal{S}(\vec{\mathfrak{H}})$ for sufficiently large m. On the other hand, we have $\mathcal{S}(\vec{\mathfrak{H}}) \subseteq \mathcal{S}(\mathfrak{G})$ which concludes the proof. $\square$

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