A Class of Periodic Function Spaces and Interpolation on Sparse Grids

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Abstract

We present a unified approach to error estimates of periodic interpolation on equidistant, full, and sparse grids for functions from a scale of function spaces which includes L_2–Sobolev spaces, the Wiener algebra and the Korobov spaces.

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1 Introduction

The interpolation of periodic functions on an equidistant grid by translates of a given function and the corresponding error estimates have been analyzed by several authors (e.g. [4, 10, 11]) in the univariate as well as in the multivariate case. A unified approach to error estimates in the L_2–norm using periodic Strang–Fix conditions can be found in [3, 11] for the first time.

The investigation of sparse grids and the related field of hyperbolic approximation has already a long tradition (e.g. [5, 6, 18]). In the bivariate case, the number of degrees of freedom is reduced to $\mathcal{O}(N \log_2 N)$ for the sparse grids where the equidistant grid has $\mathcal{O}(N^2)$ points. Nevertheless, the interpolation on sparse grids yields error estimates for functions with dominating mixed smoothness properties which are asymptotically only by a logarithmic term worse than the error estimates for the interpolation on the corresponding equidistant grids.

We introduce spaces $A_q^\alpha(\mathbb{T}^n)$ of periodic functions defined via weighted ℓ_q–summability of Fourier coefficients. Beside the isotropic function spaces, we also consider the corresponding spaces $S_{q,q}^{\alpha,\beta}A(\mathbb{T}^2)$ of bivariate functions with dominating mixed smoothness properties. They turn out to be tensor products of univariate function spaces. Their norms are uniform crossnorms. This fact allows us to carry over error estimates from the univariate to the bivariate case as in [12].

We generalize the periodic Strang–Fix conditions in a way that we can find error estimates for the interpolation on equidistant, full, and sparse grids in the norm of these new function spaces. This scale is best suited for the use of such simple conditions on the modulus of the Fourier coefficients of the fundamental interpolant. By this method, we can generalize known results for functions from L_2–Sobolev spaces (e.g. [11]), Wiener algebra–type spaces [4] and Korobov spaces (e.g. [5]). The constants occurring in the error estimates are given explicitly.

Using imbedding theorems, one can derive similar error estimates also for functions from L_p–Sobolev spaces $H_p^\alpha(\mathbb{T}^n)$ and $S_{p,p}^{\alpha,\beta}H(\mathbb{T}^2)$ for $1 < p < \infty$ and $\alpha, \beta \geq 0$.

2 Function Spaces

Let $\mathbb{T}^n$ denote the n–dimensional torus represented by the cube

$$\mathbb{T}^n := \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,;\ |x_k| \le \pi,\ k = 1, \dots, n\right\}.$$

By $D'(\mathbb{T}^n)$, we denote the linear space of tempered periodic distributions (or generalized periodic functions) (see [7, 14]). The Fourier coefficients of a distribution $g \in D'(\mathbb{T}^n)$ are $c_k(g) := g(\mathrm{e}^{-\mathrm{i}k\cdot})$ for $k \in \mathbb{Z}^n$. With the help of the inner product in $L_2(\mathbb{T}^n)$,

$$\langle f, g\rangle_{\mathbb{T}^n} := \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} f(x)\overline{g(x)}\,\mathrm{d}x,$$

the Fourier coefficients for functions $g \in L_p(\mathbb{T}^n)$ can be written as $c_k(g) = \langle g, \mathrm{e}^{\mathrm{i}k\cdot}\rangle_{\mathbb{T}^n}$.

Definition 1 *Let $1 \le q \le \infty$ and $\alpha \in \mathbb{R}$. The space $A_q^\alpha(\mathbb{T}^n)$ of order α is defined as*

$$A_q^\alpha(\mathbb{T}^n) := \left\{f \in D'(\mathbb{T}^n) \,;\ \|f \mid A_q^\alpha(\mathbb{T}^n)\| := \left\|\{(1+|k|_2^2)^{\alpha/2} c_k(f)\}_{k\in\mathbb{Z}^n} \ \middle|\ \ell_q(\mathbb{Z}^n)\right\| < \infty\right\}.$$

The scale of the spaces $A_q^\alpha(\mathbb{T}^n)$ of generalized functions includes the well–known Wiener algebra $A_1^0(\mathbb{T}^n) = A(\mathbb{T}^n)$ of functions with absolutely summable Fourier series, the L_2–Sobolev spaces $A_2^\alpha(\mathbb{T}^n) = H_2^\alpha(\mathbb{T}^n)$ and for $n = 1$ the Korobov spaces $A_\infty^\alpha(\mathbb{T}) = E^\alpha(\mathbb{T})$ (see [8]). The next theorem provides imbeddings of spaces $A_q^\alpha(\mathbb{T}^n)$ into Sobolev spaces $H_p^\mu(\mathbb{T}^n)$. It can be proved (cf. [16]) by using Haussdorff–Young's and Hölder's inequality and then interpolation between the function spaces by the Riesz–Thorin Theorem.

Theorem 1 *Let $1/p + 1/q = 1$, $\alpha, \beta, \mu \in \mathbb{R}$, $\varepsilon > 0$, $\beta = (n+\varepsilon)(1/p - 1/q)$, $\mu = (n+\varepsilon)(1/q - 1/p)$ and*

$$\mathrm{K}_n(\varepsilon, p) := \|\{(1+|k|_2^2)^{-(n+\varepsilon)/2}\}_{k\in\mathbb{Z}^n} \mid \ell_1(\mathbb{Z}^n)\|^{1/p-1/q}.$$

(i) *For $1 < p \le 2$, it holds that $A_q^{\alpha+\beta}(\mathbb{T}^n) \subset H_p^\alpha(\mathbb{T}^n) \subset A_q^\alpha(\mathbb{T}^n)$ with*

$$\|f \mid A_q^\alpha(\mathbb{T}^n)\| \le \|f \mid H_p^\alpha(\mathbb{T}^n)\| \le \mathrm{K}_n(\varepsilon, p)\ \|f \mid A_q^{\alpha+\beta}(\mathbb{T}^n)\|\,.$$

(ii) *For $2 \le p < \infty$, it holds that $A_q^\alpha(\mathbb{T}^n) \subset H_p^\alpha(\mathbb{T}^n) \subset A_q^{\alpha-\mu}(\mathbb{T}^n)$ with*

$$\frac{1}{\mathrm{K}_n(\varepsilon, q)}\ \|f \mid A_q^{\alpha-\mu}(\mathbb{T}^n)\| \le \|f \mid H_p^\alpha(\mathbb{T}^n)\| \le \|f \mid A_q^\alpha(\mathbb{T}^n)\|\,.$$

Remark: For $n < \beta < \infty$, the constant appearing in Theorem 1 can be estimated by

$$\|\{(1+|k|_2^2)^{-\beta/2}\}_{k\in\mathbb{Z}^n} \mid \ell_1(\mathbb{Z}^n)\| \le 1 + \sum_{s=1}^{n} \binom{n}{s} \frac{1}{s^{\beta/2}} \left(\frac{2\beta}{\beta - s}\right)^s.$$

Additionally, we define spaces of generalized functions with dominating mixed smoothness properties.

Definition 2 *Let $1 \le q \le \infty$ and $\alpha, \beta \in \mathbb{R}$. The spaces $S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)$ of bivariate periodic (generalized) functions with dominating mixed smoothness of order α, β with the exponent q are defined by*

$$\begin{aligned} S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2) \;&:=\; \Big\{ f \in D'(\mathbb{T}^2) \;;\; \|f \mid S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)\| := \\ &\quad \big\| \{(1+k^2)^{\alpha/2}(1+\ell^2)^{\beta/2} c_{k,\ell}(f)\}_{(k,\ell)\in\mathbb{Z}^2} \mid \ell_q(\mathbb{Z}^2) \big\| < \infty \Big\}. \end{aligned}$$

Again we have closed relations to known function spaces. By construction, it holds that $S^{\alpha,\beta}_{2,2}A(\mathbb{T}^2) = S^{\alpha,\beta}_{2,2}H(\mathbb{T}^2)$ as well as $S^{\alpha,\alpha}_{\infty,\infty}A(\mathbb{T}^2) = E^{\alpha}(\mathbb{T}^2)$ and $S^{0,0}_{1,1}A(\mathbb{T}^2) = A(\mathbb{T}^2)$. Similarly to Theorem 1, one can prove imbeddings of the spaces $S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)$ into spaces of the scale of the Sobolev spaces $S^{\mu,\nu}_{p,p}H(\mathbb{T}^2)$ with dominating mixed smoothness properties (see [14, 16]).

Lemma 1 *Let $1 \le q \le \infty$, $1/p + 1/q = 1$ and $\alpha, \beta \in \mathbb{R}$. The spaces $A^{\alpha}_q(\mathbb{T}^n)$ and $S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)$ are reflexive Banach spaces with the dual spaces $\big(A^{\alpha}_q(\mathbb{T}^n)\big)' = A^{-\alpha}_p(\mathbb{T}^n)$ and $\big(S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)\big)' = S^{-\alpha,-\beta}_{p,p}A(\mathbb{T}^2)$, respectively.*

Proof: This lemma can be proved by a straight–forward computation using the duality of the underlying ℓ_p–sequence spaces and the separability of the A^{α}_q–spaces. ■

3 Interpolation on Equidistant Grids

In this section, we give a generalization of the periodic Strang–Fix conditions given in [3, 11] in order to obtain error estimates for interpolation on equidistant grids in the scale of the $A^{\alpha}_q(\mathbb{T}^n)$–spaces.

With the help of the index set $J_N = \{k \in \mathbb{Z}^n \;;\; -N/2 \le k_s < N/2,\ s = 1, \dots, n\}$, we define the equidistant grid $\mathcal{T}_N := \{2\pi k/N \;;\; k \in J_N\}$, the set of trigonometric polynomials $T_N := \{\sum_{k\in J_N} \eta_k \,\mathrm{e}^{\mathrm{i}kx} \;;\; \eta_k \in \mathbb{C}\}$, and the discrete Fourier coefficients

$$c^N_k(f) := \frac{1}{N^n} \sum_{\ell \in J_N} f\Big(\frac{2\pi\ell}{N}\Big)\, \mathrm{e}^{2\pi \mathrm{i} k\ell/N}$$

of a pointwise given function. For $f \in A(\mathbb{T}^n)$, we can use the aliasing formula

$$c^N_k(f) = \sum_{\ell\in\mathbb{Z}^n} c_{k+\ell N}(f) \tag{1}$$

for computation. The N–th Fourier partial sum of a function $f \in L_1(\mathbb{T}^n)$ is denoted by

$$S_N f := \sum_{k\in J_N} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot}.$$

Let a continuous 2π–periodic fundamental interpolant Λ_N for the grid $\mathcal{T}_N$ be given, i.e., Λ_N satisfies the interpolatory conditions

$$\Lambda_N\Big(\frac{2\pi k}{N}\Big) = \delta_{0,k}, \qquad k \in J_N.$$

The corresponding Lagrange interpolation operator is of the form

$$L_N f := \sum_{k\in J_N} f\Big(\frac{2\pi k}{N}\Big)\, \Lambda_N\Big(\cdot - \frac{2\pi k}{N}\Big).$$

The Fourier coefficients of the interpolant of a function f can be computed as

$$c_k(L_N f) = N^n\, c_k^N(f)\, c_k(\Lambda_N)\,, \qquad k \in \mathbb{Z}^n. \tag{2}$$

We are interested in estimating the error of interpolation of 2π–periodic functions by L_N in the norm of the spaces $A_q^\alpha(\mathbb{T}^n)$. These error estimates will use the smoothness of the function to be interpolated on one hand. On the other hand, the order of the interpolation error depends on the behaviour of the fundamental interpolant. This behaviour can be characterized by a certain decay of the Fourier coefficients of Λ_N.

Definition 3 *The 2π–periodic fundamental interpolant $\Lambda_N \in A(\mathbb{T}^n)$ satisfies the* periodic Strang–Fix conditions *of order $m > 0$ for the exponents $q \geq 1$ and $\alpha \geq 0$, if for all $k \in J_N$ the inequalities*

$$\begin{aligned} |1 - N^n c_k(\Lambda_N)| &\leq b_0\ |2\pi k|_2^m\ N^{-m}, \\ |N^n c_{k+\ell N}(\Lambda_N)| &\leq b_\ell\ |2\pi k|_2^m\ N^{-m-\alpha}, \quad \ell \in \mathbb{Z}^n \setminus \{0\} \end{aligned} \tag{3}$$

hold for some sequence of non–negative constants $\{b_\ell\}_{\ell\in\mathbb{Z}^n}$ with

$$\gamma_1 := \|\{(1 + |\ell|_2^2)^{\alpha/2}\, b_\ell\}_{\ell\in\mathbb{Z}^n} \mid \ell_q(\mathbb{Z}^n)\| < \infty.$$

These periodic Strang–Fix conditions are a generalization of the periodic Strang–Fix conditions given in [3, 11] for the special case $q = 2,\ \alpha = 0$ to the more general situation $q \geq 1,\ \alpha \geq 0$. Originally, the Strang–Fix conditions for cardinal interpolation by translates were given in [15] for the univariate and in [17] for the multivariate case.

The norm of the interpolation error can be estimated by applying the triangle inequality

$$\begin{aligned} \|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\| &\leq \|S_N f - L_N S_N f \mid A_q^\alpha(\mathbb{T}^n)\| \\ &\quad + \|f - S_N f \mid A_q^\alpha(\mathbb{T}^n)\| + \|L_N(f - S_N f) \mid A_q^\alpha(\mathbb{T}^n)\|. \end{aligned} \tag{4}$$

The next three theorems give error estimates for the error terms on the right–hand side of the inequality.

Theorem 2 *Let the 2π–periodic fundamental interpolant $\Lambda_N \in A(\mathbb{T}^n)$ satisfy the periodic Strang–Fix conditions (3) of order $m > 0$ for the exponents $q \geq 1$ and $\alpha \geq 0$. Then for all trigonometric polynomials $f \in T_N$, it holds that*

$$\|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\| \leq \gamma_1 \left(\frac{2\pi}{N}\right)^m \|f \mid A_q^{\alpha+m}(\mathbb{T}^n)\|,$$

where γ_1 is the constant from the Strang–Fix conditions.

Proof: We can apply the aliasing formula (1) for $f \in T_N$ and obtain the Fourier coefficients of the interpolant from (2) as

$$c_k(L_N f) = N^n \sum_{\nu\in\mathbb{Z}^n} c_{k+\nu N}(f)\, c_k(\Lambda_N), \qquad k \in \mathbb{Z}^n.$$

The error of interpolation can be written as

$$\begin{aligned} &\|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\| \\ &= \Big\| \sum_{k\in\mathbb{Z}^n} (c_k(f) - c_k(L_N f))\ \mathrm{e}^{\mathrm{i}k\cdot} \ \Big|\ A_q^\alpha(\mathbb{T}^n) \Big\| \\ &= \Big\| \sum_{k\in\mathbb{Z}^n} \Big((1 - N^n c_k(\Lambda_N))\, c_k(f) - \sum_{\nu\in\mathbb{Z}^n\setminus\{0\}} c_{k+\nu N}(f) N^n c_k(\Lambda_N) \Big)\ \mathrm{e}^{\mathrm{i}k\cdot}\ \Big|\ A_q^\alpha(\mathbb{T}^n) \Big\|. \end{aligned}$$

Because $c_k(f) = 0$ for $k \notin J_N$ these sums can be simplified

$$\begin{aligned}
&\|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\| \\
&\quad = \Big\| \sum_{k \in J_N} (1 - N^n c_k(\Lambda_N))\, c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \\
&\qquad\qquad - \sum_{k \in J_N} \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \sum_{\nu \in \mathbb{Z}^n \setminus \{0\}} c_{k+(\ell+\nu)N}(f) N^n c_{k+\ell N}(\Lambda_N)\, \mathrm{e}^{\mathrm{i}(k+\ell N)\cdot} \ \Big| \ A_q^\alpha(\mathbb{T}^n) \Big\| \\
&\quad = \Big\| \sum_{k \in J_N} (1 - N^n c_k(\Lambda_N))\, c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \\
&\qquad\qquad - \sum_{k \in J_N} \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} c_k(f) N^n c_{k+\ell N}(\Lambda)\, \mathrm{e}^{\mathrm{i}k\cdot}\, \mathrm{e}^{\mathrm{i}\ell N\cdot} \ \Big| \ A_q^\alpha(\mathbb{T}^n) \Big\| \\
&\quad = \Big\| \sum_{k \in J_N} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \Big((1 - N^n c_k(\Lambda_N)) - \sum_{\ell \in \mathbb{Z}^d \setminus \{0\}} N^n c_{k+\ell N}(\Lambda)\, \mathrm{e}^{\mathrm{i}\ell N\cdot} \Big) \ \Big| \ A_q^\alpha(\mathbb{T}^n) \Big\|.
\end{aligned}$$

As a next step, we apply the Definition 1 of the norm of the space $A_q^\alpha(\mathbb{T}^n)$ (here for $q < \infty$, the case $q = \infty$ is analoguous)

$$\begin{aligned}
\|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\|^q &= \sum_{k \in J_N} |c_k(f)|^q \Big(\big| (1 - N^n c_k(\Lambda_N))(1 + |k|_2^2)^{\alpha/2} \big|^q \\
&\qquad + \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \big| N^n c_{k+\ell N}(\Lambda_N)(1 + |k + \ell N|_2^2)^{\alpha/2} \big|^q \Big) \\
&\le \sum_{k \in J_N} \big| (1 + |k|_2^2)^{\alpha/2}\, c_k(f) \big|^q \Big(|1 - N^n c_k(\Lambda_N)|^q \\
&\qquad + \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} \big| N^n c_{k+\ell N}(\Lambda_N)(1 + |\ell|_2^2)^{\alpha/2} N^\alpha \big|^q \Big).
\end{aligned}$$

The periodic Strang–Fix conditions (3) yield the estimates

$$\begin{aligned}
&\|f - L_N f \mid A_q^\alpha(\mathbb{T}^n)\|^q \\
&\quad \le \Big(\sum_{k \in J_N} |2\pi k|_2^{mq}\ N^{-mq}\ \big| (1 + |k|_2^2)^{\alpha/2}\, c_k(f) \big|^q \Big) \Big(\sum_{\ell \in \mathbb{Z}^n} \big((1 + |\ell|_2^2)^{\alpha/2} b_\ell \big)^q \Big) \\
&\quad \le \gamma_1^q \left(\frac{2\pi}{N} \right)^{mq} \ \|f \mid A_q^{m+\alpha}(\mathbb{T}^n)\|^q,
\end{aligned}$$

which give us the desired result. ■

Theorem 3 *Let $f \in A_q^\mu(\mathbb{T}^n)$ with $q \ge 1$ and $\mu \ge \alpha \ge 0$. Then,*

$$\|f - S_N f \mid A_q^\alpha(\mathbb{T}^n)\| \le \pi^{\alpha-\mu} \left(\frac{2\pi}{N} \right)^{\mu-\alpha} \ \|f \mid A_q^\mu(\mathbb{T}^n)\|.$$

Proof: Using Hölder's inequality, we can compute

$$\begin{aligned}
\|f - S_N f \mid A_q^\alpha(\mathbb{T}^n)\| &= \big\| \{(1 + |k|_2^2)^{\alpha/2}\, c_k(f)\}_{k \in \mathbb{Z}^n \setminus J_N} \mid \ell_q(\mathbb{Z}^n \setminus J_N) \big\| \\
&= \big\| \{(1 + |k|_2^2)^{(\alpha-\mu)/2}\, (1 + |k|_2^2)^{\mu/2}\, c_k(f)\}_{k \in \mathbb{Z}^n \setminus J_N} \mid \ell_q(\mathbb{Z}^n \setminus J_N) \big\| \\
&\le \max_{k \in \mathbb{Z}^n \setminus J_N} (1 + |k|_2^2)^{(\alpha-\mu)/2}\ \|f \mid A_q^\mu(\mathbb{T}^n)\| \\
&\le \pi^{\alpha-\mu} \left(\frac{2\pi}{N} \right)^{\mu-\alpha} \ \|f \mid A_q^\mu(\mathbb{T}^n)\|.
\end{aligned}$$

■

Theorem 4 *Let $f \in A_q^\mu(\mathbb{T}^n)$ with $q \geq 1$, $\mu \geq \alpha \geq 0$ and $\mu > n(1-1/q)$. Let*

$$\gamma_2 := N^n \begin{cases} \max\limits_{k\in J_N}\Big(|c_k(\Lambda_N)|^q + N^{\alpha q} \sum\limits_{\ell\in\mathbb{Z}^n\setminus\{0\}} \big|(1+|\ell|_2^2)^{\alpha/2}\, c_{k+\ell N}(\Lambda_N)\big|^q\Big)^{1/q} & \\ & \textit{for } 1 \leq q < \infty, \\ \max\limits_{k\in J_N}\ \sup\Big\{|c_k(\Lambda_N)|,\ N^\alpha \big|(1+|\ell|_2^2)^{\alpha/2}\, c_{k+\ell N}(\Lambda_N)\big|\ ;\ \ell\in\mathbb{Z}^n\setminus\{0\}\Big\} & \\ & \textit{for } q = \infty \end{cases}$$

be finite for the 2π–periodic fundamental interpolant Λ_N. Then the estimate

$$\|L_N(f - S_N f) \mid A_q^\alpha(\mathbb{T}^n)\| \leq \gamma_2\ \gamma_3\ \left(\frac{2\pi}{N}\right)^{\mu-\alpha}\ \|f \mid A_q^\mu(\mathbb{T}^n)\|$$

holds with

$$\gamma_3 := (1+n)^{\alpha/2}\ 2^{-\mu}\ \pi^{\alpha-\mu} \begin{cases} \max\limits_{k\in J_N}\left(\sum\limits_{\ell\in\mathbb{Z}^n\setminus\{0\}} \left|\dfrac{k}{N}+\ell\right|_2^{-\mu p}\right)^{1/p} & \textit{for } q>1,\ 1/p+1/q=1, \\ \max\limits_{k\in J_N}\ \sup\limits_{\ell\in\mathbb{Z}^n\setminus\{0\}} \left|\dfrac{k}{N}+\ell\right|_2^{-\mu} & \textit{for } q=1. \end{cases}$$

Proof: The assumption $\mu > n(1-1/q)$ insures $f \in A(\mathbb{T}^n)$. Therefore, interpolation is well–defined and aliasing is applicable. We only prove the case $q < \infty$. The case $q = \infty$ can be handled analogously. The Definition 1 of the norm in $A_q^\alpha(\mathbb{T}^n)$ yields

$$\begin{aligned} \|L_N(f - S_N f) \mid A_q^\alpha(\mathbb{T}^n)\|^q &= \Big\|\sum_{k\in\mathbb{Z}^n} c_k^N(f - S_N f)\, N^n c_k(\Lambda_N)\, \mathrm{e}^{\mathrm{i}k\cdot}\ \Big|\ A_q^\alpha(\mathbb{T}^n)\Big\|^q \\ &= \sum_{k\in\mathbb{Z}^n} \big|(1+|k|_2^2)^{\alpha/2}\, c_k^N(f-S_Nf)\, N^n c_k(\Lambda_N)\big|^q \\ &\leq \sum_{k\in J_N} \big|(1+|k|_2^2)^{\alpha/2}\, c_k^N(f-S_Nf)\big|^q\ N^{nq}\Big(|c_k(\Lambda_N)|^q \\ &\qquad + N^{\alpha q}\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} \big|(1+|\ell|_2^2)^{\alpha/2}\, c_{k+\ell N}(\Lambda_N)\big|^q\Big) \\ &\leq N^{nq}\max_{k\in J_N}\Big(|c_k(\Lambda_N)|^q + N^{\alpha q}\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} \big|(1+|\ell|_2^2)^{\alpha/2}\, c_{k+\ell N}(\Lambda_N)\big|^q\Big) \\ &\qquad \cdot\Big(\sum_{k\in J_N} \big|(1+|k|_2^2)^{\alpha/2}\, c_k^N(f-S_Nf)\big|^q\Big). \end{aligned}$$

Now we want to estimate the sum in the last factor. We use the aliasing formula (1) and Hölder's inequality for the case $1 < q < \infty$, $1/p + 1/q = 1$. Analogous computation can be done for $q = 1$. We obtain

$$\begin{aligned} \sum_{k\in J_N} \big|(1+|k|_2^2)^{\alpha/2}\, c_k^N(f-S_Nf)\big|^q &\leq \sum_{k\in J_N}(1+|k|_2^2)^{\alpha q/2}\Big(\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} |c_{k+\ell N}(f)|\Big)^q \\ &\leq \sum_{k\in J_N}(1+|k|_2^2)^{\alpha q/2}\Big(\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} (1+|k+\ell N|_2^2)^{-\mu p/2}\Big)^{q/p} \\ &\qquad \cdot\Big(\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}} \big|(1+|k+\ell N|_2^2)^{\mu/2} c_{k+\ell N}(f)\big|^q\Big). \end{aligned}$$

The assumption $\mu > n(1-1/q) = n/p$ insures the convergence of the series for γ_3. Hence,

$$
\begin{aligned}
&\sum_{k\in J_N} \left|(1+|k|_2^2)^{\alpha/2}\, c_k^N(f-S_Nf)\right|^q \\
&\le \max_{k\in J_N}(1+|k|_2^2)^{\alpha q/2}\; N^{-\mu q}\; \max_{k\in J_N}\left(\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}}\left|\frac{k}{N}+\ell\right|_2^{-\mu p}\right)^{q/p} \\
&\qquad\cdot\left(\sum_{k\in\mathbb{Z}^n\setminus J_N}\left|(1+|k+\ell N|_2^2)^{\mu/2}c_{k+\ell N}(f)\right|^q\right) \\
&\le (1+n)^{\alpha q/2}\; 2^{-\alpha q}\; N^{(\alpha-\mu)q}\; \max_{k\in J_N}\left(\sum_{\ell\in\mathbb{Z}^n\setminus\{0\}}\left|\frac{k}{N}+\ell\right|_2^{-\mu p}\right)^{q/p} \|f\mid A_q^\mu(\mathbb{T}^n)\|^q \\
&\le \gamma_3^q\left(\frac{2\pi}{N}\right)^{(\mu-\alpha)q} \|f\mid A_q^\mu(\mathbb{T}^n)\|^q.
\end{aligned}
$$

Together with the first estimate this proves the theorem. ■

Remark: While the constant γ_2 again depends on the behaviour of the fundamental interpolant Λ_N, the order of the error and the constant γ_3 are only influenced by the smoothness μ of the function to be interpolated. We give upper bounds for the constants.

(i) If $\Lambda_N \in A(\mathbb{T}^n)$ satisfies the periodic Strang–Fix conditions (3) of order m for the exponents $q \ge 1$ and $\alpha \ge 0$, then the constant γ_2 from Theorem 4 can be estimated by

$$
\gamma_2 \le \pi^m\; n^{m/2}\begin{cases}\left(\gamma_1^q - b_0^q + (b_0+\pi^{-m}\; n^{-m/2})^q\right)^{1/q} & \text{for } 1\le q<\infty,\\ \max\left\{\gamma_1, b_0+\pi^{-m}\; n^{-m/2}\right\} & \text{for } q=\infty,\end{cases}
$$

with γ_1, b_0 being the constants from the Strang–Fix conditions.

(ii) For $\mu > n/p,\ 1/p+1/q=1$, the constant γ_3 from Theorem 4 satisfies

$$
\gamma_3 \le (1+n)^{\alpha/2}\,\pi^{\alpha-\mu}\begin{cases}\left(\displaystyle\sum_{s=1}^n 2^s\binom{n}{s}\frac{1}{s^{\mu p/2}}\left(1+\frac{s}{2(\mu p-s)}\right)^s\right)^{1/p} & \text{for } 1\le p<\infty,\\ 1 & \text{for } p=\infty.\end{cases}
$$

Now we can use the Theorems 2–4 to give an error estimate for the interpolation of a function $f \in A_q^\mu(\mathbb{T}^n)$.

Theorem 5 *Let $f \in A_q^\mu(\mathbb{T}^n)$ with $q \ge 1,\ \mu \ge \alpha \ge 0$ and $\mu > n(1-1/q)$. Let the 2π–periodic fundamental interpolant $\Lambda_N \in A(\mathbb{T}^n)$ satisfy the periodic Strang–Fix conditions* (3) *of order m for the exponents q and α. Then, for $\varrho := \min\{\mu, m+\alpha\}$, the estimate*

$$
\|f - L_Nf \mid A_q^\alpha(\mathbb{T}^n)\| \le \mathrm{C}_\varrho\;\left(\frac{2\pi}{N}\right)^{\varrho-\alpha}\;\|f\mid A_q^\mu(\mathbb{T}^n)\|
$$

holds with

$$
\mathrm{C}_\varrho := \begin{cases}\gamma_1+\pi^{\alpha-\mu}+\gamma_2\,\gamma_3 & \textit{for } \varrho = m+\alpha,\\ \pi^{m+\alpha-\mu}\,(n+1)^{(m+\alpha-\mu)/2}\,\gamma_1+\pi^{\alpha-\mu}+\gamma_2\,\gamma_3 & \textit{for } \varrho=\mu.\end{cases}
$$

The constants γ_1, γ_2, γ_3 are chosen as in the Theorems 2 and 4.

Proof: We use the triangle inequality (4). In case $\varrho = m+\alpha$, we have

$$\|f \mid A_q^{m+\alpha}(\mathbb{T}^n)\| \le \|f \mid A_q^{\mu}(\mathbb{T}^n)\|$$

for $m+\alpha \le \mu$. The assertion follows directly from the above Theorems 2–4.

If $\varrho = \mu$ we can not apply Theorem 2 in that direct way. We need the following estimate for the norm of order μ

$$\begin{aligned}\|S_N f - L_N S_N f \mid A_q^{\alpha}(\mathbb{T}^n)\| &\le \gamma_1 \left(\frac{2\pi}{N}\right)^m \left\|\{(1+|k|_2^2)^{(m+\alpha)/2}\, c_k(f)\}_{k\in J_N} \mid \ell_q(J_N)\right\| \\ &\le \gamma_1\, \pi^{m+\alpha-\mu}\, (n+1)^{(m+\alpha-\mu)/2} \left(\frac{2\pi}{N}\right)^{\mu-\alpha} \|f \mid A_q^{\mu}(\mathbb{T}^n)\|.\end{aligned}$$

Combined with Theorems 3 and 4 this proves the theorem. ■

Example: Periodized B–Splines

As an example, we may use the interpolation by the 2π–periodized centered B–Spline $\mathcal{M}_{N,r}$ of order $r \in \mathbb{N}$. Its Fourier coefficients are known as $c_k(\mathcal{M}_{N,r}) = 1/N\,(\mathrm{sinc}(\pi k/N))^r$, $k \in \mathbb{Z}$, with sinc $t := \sin t/t$. The corresponding fundamental interpolant $\Lambda_{N,r}$ can be computed from

$$c_k(\Lambda_{N,r}) := \frac{c_k(\mathcal{M}_{N,r})}{N\, c_k^N(\mathcal{M}_{N,r})}, \qquad k \in \mathbb{Z}. \tag{5}$$

The following theorem gives the order and the constants of the periodic Strang–Fix conditions satisfied by the B–splines. It was proved in [11] for $q=2$, $\alpha=0$ and in [16] as below. The order of Strang–Fix conditions in case $q=2$, $\alpha=0$ coincides with the approximation order of the cardinal B–splines (cf. [2]).

Theorem 6 *Let $\Lambda_{N,r}$ be the fundamental interpolant (5) corresponding to the 2π–periodic centered B–spline of order $r \in \mathbb{N}$. Let $q \ge 1$, $\alpha \ge 0$ and $r-\alpha > 1/q$. Then, $\Lambda_{N,r}$ satisfies the periodic Strang–Fix conditions of order $r-\alpha$ for the exponents q and α with the constants*

$$b_0 = (2\pi)^{-\alpha} \begin{cases} \dfrac{1}{2^{r+1}} & \text{for } r \text{ odd,} \\ \dfrac{1}{2(2^r-1)} & \text{for } r \text{ even,} \end{cases}$$

$$b_\ell = \frac{1}{2^\alpha \pi^{r+\alpha}(2|\ell|-1)^r} \begin{cases} 1 & \text{for } r=1, \\ \dfrac{(r-1)!}{E_{(r-1)/2}} & \text{for } r>1,\ \text{odd,} \\ \dfrac{r!}{2^r(2^r-1)B_{r/2}} & \text{for } r \text{ even,} \end{cases}$$

for $\ell \neq 0$ and

$$\gamma_1^q \le (2\pi)^{-\alpha q} \begin{cases} \left[\dfrac{1}{4^q} + \dfrac{2^{(\alpha+1)q}}{\pi^q}\left(1 + \dfrac{1}{2(q-\alpha q-1)}\right)\right] & \text{for } r=1, \\ \left[\dfrac{1}{2^{(r+1)q}} + \left(\dfrac{2^{\alpha+1}(r-1)!}{\pi^r E_{(r-1)/2}}\right)^q \left(1 + \dfrac{1}{2((r-\alpha)q-1)}\right)\right] & \text{for } r>1,\ \text{odd,} \\ \left[\dfrac{1}{2^q(2^r-1)^q} + \left(\dfrac{2^{\alpha-r+1}r!}{\pi^r(2^r-1)B_{r/2}}\right)^q \left(1 + \dfrac{1}{2((r-\alpha)q-1)}\right)\right] & \text{for } r \text{ even,} \end{cases}$$

for $1 \le q < \infty$ *and*

$$\gamma_1 \le (2\pi)^{-\alpha} \begin{cases} \dfrac{2^\alpha}{\pi} & \textit{for } r = 1, \\ \max\left\{\dfrac{1}{2^{r+1}}, \dfrac{2^\alpha (r-1)!}{\pi^r E_{(r-1)/2}}\right\} & \textit{for } r > 1, \textit{ odd}, \\ \max\left\{\dfrac{1}{2(2^r-1)}, \dfrac{2^{\alpha-r} r!}{\pi^r (2^r-1) B_{r/2}}\right\} & \textit{for } r \textit{ even}, \end{cases}$$

for $q = \infty$, *respectively. There,* B_s *and* E_s $(s \in \mathbb{N})$ *denote the corresponding Bernoulli and Euler numbers.*

We note that if we let the exponent α (the order of the norm in which we measure the interpolation error) grow then the order of the Strang–Fix conditions goes down.

In the special case $q = 1$ and $\alpha = 0$, the Theorems 5 and 6 yield the estimate of the interpolation error for spline interpolation contained in [4].

Example: Trigonometric Polynomials

Another example is the trigonometric interpolation. The de la Vallée Poussin means $\mathcal{V}_N^K$ $(N, K \in \mathbb{N}, N > K)$ of the Dirichlet are given by

$$\mathcal{V}_N^K(x) := \frac{1}{4KN} \sum_{\ell=N-K}^{N+K-1} \left(\sum_{k=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}kx} \right).$$

They are fundamental interpolants for the grid $\mathcal{T}_{2N}$. So, we have a lot of different fundamental interpolants for the grids $\mathcal{T}_N$ (N even) belonging to different parameters K. We denote them by $\Lambda_{N,K} := \mathcal{V}_{N/2}^K$ for $N/2, K \in \mathbb{N}, N/2 > K$. Since the de la Vallée Poussin means are trigonometric polynomials they of course satisfy Strang–Fix conditions of arbitrary order. The next theorem gives the corresponding constants. It can be proved by some simple calculations (cf. [16]).

Theorem 7 *The de la Vallée Poussin mean* $\Lambda_{N,K}$ *satisfies the periodic Strang–Fix conditions of arbitrary order* m *for the exponents* $q \ge 1$ *and* $\alpha \ge 0$ *with the constants*

$$b_\ell = \frac{1}{2\,\pi^m} \begin{cases} (1 - 2(K-1)/N)^{-m} & \textit{for } \ell = 0, \\ 2^\alpha (1 - 2(K-1)/N)^{-m-\alpha} & \textit{for } \ell = -1, 1, \\ 0 & \textit{otherwise}, \end{cases}$$

and

$$\gamma_1 = \frac{1}{2\,\pi^m} \left(1 - \frac{2(K-1)}{N}\right)^{-m} \begin{cases} \left(1 + 2^{3\alpha q/2+1} (1 - 2(K-1)/N)^{-\alpha q}\right)^{1/q} & \textit{for } 1 \le q < \infty, \\ 2^{3\alpha/2} (1 - 2(K-1)/N)^{-\alpha} & \textit{for } q = \infty. \end{cases}$$

The constants of the Strang–Fix conditions depend on the quotient of the parameters K and N. For $K = 1$, we obtain the best constants since $\Lambda_{N,1}$ is only a slight modification of the Dirichlet kernel whose Fourier coefficients are compared with the Fourier coefficients of the fundamental interpolant. For $K = N/2 - 1$, the corresponding de la Vallée Poussin mean is already very closed to the Fejér kernel and the constants are much bigger.

In addition to these univariate examples one can find examples of bivariate functions (3– and 4–direction box splines, see [11]) satisfying periodic Strang–Fix conditions of some order.

4 Chains of Interpolation Operators

In the next sections, we provide some preparation to construct interpolation operators on full and sparse grids and to give error estimates for them. In the following, we investigate chains of interpolation operators with certain common properties.

The ordering relation $P \leq Q$ for projectors holds if $PQ = QP = P$. A family of projectors $\{P_j\}_{j=0}^{\infty}$ forms a chain if $P_j \leq P_{j+1},\ j \in \mathbb{N}_0$. For two interpolation projectors L_K and L_N, the ordering $L_K \leq L_N$ holds if and only if the images $\operatorname{Im} L_K \subset \operatorname{Im} L_N$ as well as the grids $\mathcal{T}_K \subset \mathcal{T}_N$ are ordered (cf. [1]).

Fix $d \in \mathbb{N}$. By the choice $N_j := d\, 2^j$, we immediately insure $\mathcal{T}_{N_j} \subset \mathcal{T}_{N_{j+1}}$. Furthermore, we assume

$$\operatorname{Im} L_{N_j} \subset \operatorname{Im} L_{N_{j+1}}. \tag{6}$$

This property has to be proved for every example by hand. Then, we have a chain

$$L_{N_0} \leq L_{N_1} \leq \cdots \leq L_{N_j} \leq L_{N_{j+1}} \leq \cdots \tag{7}$$

of interpolation operators. The members of such a chain should have similar approximation properties.

Definition 4 *A family $\{\Lambda_{N_j}\}_{j=0}^{\infty}$ of 2π–periodic fundamental interpolants $\Lambda_{N_j} \in A(\mathbb{T}^n)$ satisfies the* uniform periodic Strang–Fix conditions *of order $m > 0$ for the exponents $q \geq 1$ and $\alpha \geq 0$, if all the fundamental interpolants $\Lambda_{N_j}, j \in \mathbb{N}_0$ satisfy the Strang–Fix conditions (3) of order $m > 0$ for the exponents q and α with the same constants $b_\ell, \ell \in \mathbb{Z}^n$, and γ_1.*

These conditions can be used to formulate error estimates for the interpolation L_{N_j} with constants independent of j.

Theorem 8 *Let $f \in A_q^\mu(\mathbb{T}^n)$ with $q \geq 1$, $\mu \geq \alpha \geq 0$ and $\mu > n(1 - 1/q)$. The family $\{\Lambda_{N_j}\}_{j=0}^{\infty}$ of fundamental interpolants $\Lambda_{N_j} \in A(\mathbb{T}^n)$ satisfies the uniform periodic Strang–Fix conditions of order $m > 0$ for the exponents q and α. Let $\varrho := \min\{\mu, m + \alpha\}$, γ_1 from the Strang–Fix conditions, $\gamma_4 := \sup_{j\in\mathbb{N}_0} \gamma_2(j)$, and $\gamma_5 := \sup_{j\in\mathbb{N}_0} \gamma_3(j)$ where $\gamma_2(j), \gamma_3(j)$ denote the constants from Theorem 4 belonging to Λ_{N_j}. Then, for all $j \in \mathbb{N}_0$, the estimate*

$$\|f - L_{N_j} f \mid A_q^\alpha(\mathbb{T}^n)\| \leq \mathrm{C}_\varrho \left(\frac{2\pi}{N_j}\right)^{\varrho-\alpha} \|f \mid A_q^\mu(\mathbb{T}^n)\|$$

holds with

$$\mathrm{C}_\varrho := \begin{cases} \gamma_1 + \pi^{\alpha-\mu} + \gamma_4\,\gamma_5 & \text{for } \varrho = m + \alpha, \\ \pi^{m+\alpha-\mu}(n+1)^{(m+\alpha-\mu)/2}\gamma_1 + \pi^{\alpha-\mu} + \gamma_4\,\gamma_5 & \text{for } \varrho = \mu. \end{cases}$$

Example: Periodized B–Splines

The fundamental interpolants $\Lambda_{N_j,r}$ belonging to the 2π–periodic centered B–spline of even order $r \in \mathbb{N}$ satisfy (6) automatically since at the step from j to $j+1$ only some new spline knots are added. Therefore, the corresponding interpolation operators form a chain (7). The constants for Strang–Fix conditions given in Theorem 6 do not depend on N_j. That means these spline interpolants satisfy uniform periodic Strang–Fix conditions of the order given in Theorem 6.

The fundamental interpolants $\Lambda_{N_j,r}$ belonging to the 2π–periodic centered B–spline of odd order $r \in \mathbb{N}$ do not satisfy (6). For splines for the grid $\mathcal{T}_{N_{j+1}}$ only totally new knots are used compared to the j–th grid.

Example: Trigonometric Polynomials

The de la Vallée Poussin means Λ_{N_j,K_j} satisfy (6) only under certain restrictions on K_j and N_j. In [13], it was shown that for N_j as before and

$$K_j := \begin{cases} 2^{j-\kappa-1} & \text{for } j > \kappa, \\ 1 & \text{for } j \leq \kappa, \end{cases} \qquad \kappa \in \mathbb{N},\ 3 \leq d\, 2^\kappa, \tag{8}$$

condition (6) is fulfiled. The case $\kappa = \infty$ is allowed. With this choice of the parameters N_j and K_j, uniform Strang–Fix conditions are satisfied, too. The constants of the Strang–Fix conditions of order m for the exponents $q \geq 1$ and $\alpha \geq 0$ can be computed as

$$b_\ell = \frac{1}{2(2\pi)^m} \begin{cases} 3^m & \text{for } \ell = 0, \\ 3^{\alpha m} & \text{for } \ell = -1, 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_1 = \frac{1}{2} \left(\frac{3}{2\pi}\right)^m \begin{cases} \left(1 + 2\,\left(3\sqrt{2}\right)^{\alpha q}\right)^{1/q} & \text{for } 1 \leq q < \infty, \\ \left(3\sqrt{2}\right)^\alpha & \text{for } q = \infty. \end{cases}$$

5 Tensor Products of Reflexive Banach Spaces

We want to use some general theory on operators in tensor product spaces for our error estimates. Because the function spaces considered here are reflexive, we restrict our following introduction to tensor products to this case. In [9], one can find a general introduction.

Given two reflexive Banach spaces X, Y, we form the linear space of formal linear combinations

$$F(X,Y) := \Big\{ \sum_{k=1}^r \alpha_k\,(x_k, y_k)\ ;\ \alpha_k \in \mathbb{C},\ x_k \in X,\ y_k \in Y,\ k = 1, \ldots, r,\ r \in \mathbb{N} \Big\}$$

of pairs $(x_k, y_k) \in X \times Y$. Furthermore, we denote by G the subspace of $F(X,Y)$ spanned by elements of the form

$$\sum_{k=1}^r \sum_{\ell=1}^s \alpha_k\, \beta_\ell\, (x_k, y_\ell) - \Big(\sum_{k=1}^r \alpha_k\, x_k\,,\ \sum_{\ell=1}^s \beta_\ell\, y_\ell \Big).$$

Then, the algebraic tensor product of X and Y is the quotient space

$$X \otimes Y := F(X,Y)/G.$$

The equivalence class generated by (x, y) is denoted by $x \otimes y$. In general, the algebraic tensor product $X \otimes Y$ equipped with a suitable norm is not complete. The completion of $X \otimes Y$ with respect to the norm λ yields the tensor product $X \otimes_\lambda Y$ of the spaces X and Y. Two Banach spaces will be identified if they are isomorphic to each other. A norm λ on $X \otimes Y$ with $\lambda(x \otimes y) = \|x \mid X\| \cdot \|y \mid Y\|$ for all $x \otimes y \in X \otimes Y$, is termed a *crossnorm.* A pair of crossnorms λ_1 on $X_1 \otimes Y_1$ and λ_2 on $X_2 \otimes Y_2$ is *uniform,* if for all operators $P \in \mathcal{L}(X_1, X_2)$ and $Q \in \mathcal{L}(Y_1, Y_2)$ and all $x_k \in X_1,\ y_k \in Y_1$, it holds that

$$\lambda_2\Big(\sum_{k=1}^r P x_k \otimes Q y_k\Big) \leq\ \|P \mid \mathcal{L}(X_1, X_2)\| \cdot \|Q \mid \mathcal{L}(Y_1, Y_2)\|\ \lambda_1\Big(\sum_{k=1}^r x_k \otimes y_k\Big).$$

In the case of uniform crossnorms the tensor product $P \otimes Q$ has a unique continuous extension $P \hat{\otimes} Q$ with

$$\|P \hat{\otimes} Q \mid \mathcal{L}(X_1 \otimes_{\lambda_1} Y_1, X_2 \otimes_{\lambda_2} Y_2)\| = \|P \mid \mathcal{L}(X_1, X_2)\| \cdot \|Q \mid \mathcal{L}(Y_1, Y_2)\|.$$

Using this property of uniform crossnorms, one is able to obtain results for the tensor product spaces directly from the results for the underlying single spaces. An example for spaces with uniform crossnorms is a pair of tensor product spaces both equipped with the p–nuclear norm (see [9, 16]).

For $1 \le p \le \infty$, $1/p + 1/q = 1$, the *p–nuclear norm* of $z \in X \otimes Y$ is given by

$$\tilde{\lambda}_p(z) := \inf\Big\{ \Big(\sum_{j=1}^{n} \|x_j \mid X\|^p\Big)^{1/p} \theta_q(y_1, \ldots, y_n) \; ; \; z = \sum_{j=1}^{n} x_j \otimes y_j \Big\},$$

where

$$\theta_q(y_1, \ldots, y_n) := \sup\Big\{ \Big\| \sum_{j=1}^{n} \eta_j \, y_j \;\Big|\; Y \Big\| \; ; \; \|\{\eta_j\}_{j=1,\ldots,n} \mid \ell_p(\{1, \ldots, n\})\| = 1 \Big\}.$$

The infimum is taken with respect to all representations of z.

6 Tensor Products of Function Spaces

We discuss the relationship between isotropic function spaces and the spaces of functions with dominating mixed smoothness. The spaces of functions with dominating mixed smoothness turn out to be tensor products of univariate function spaces. We check whether their norms are uniform crossnorms. This will be a powerful tool in order to obtain error estimates for the interpolation problems.

Theorem 9 *Let*

$$\lambda_{\alpha,\beta}(h) := \|\{(1+k^2)^{\alpha/2}(1+\ell^2)^{\beta/2} c_{k,\ell}(h)\}_{(k,\ell)\in\mathbb{Z}^2} \mid \ell_q(\mathbb{Z}^2)\| \tag{9}$$

denote the norm in $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$.

(i) *$\lambda_{\alpha,\beta}$ is a crossnorm on $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$.*

(ii) *Let $\tilde{\lambda}_q$ be the q–nuclear norm on $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$. Then $\lambda_{\alpha,\beta}(h) = \tilde{\lambda}_q(h)$ holds for all $h \in A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$.*

(iii) *The pair of norms $\lambda_{\alpha,\beta}$ and $\lambda_{\mu,\nu}$ on the spaces $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$ and $A_q^\mu(\mathbb{T}) \otimes A_q^\nu(\mathbb{T})$ given by (9) is uniform.*

(iv) *It holds that $S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2) = A_q^\alpha(\mathbb{T}) \otimes_{\lambda_{\alpha,\beta}} A_q^\beta(\mathbb{T})$.*

Proof: (i) For dyads $f \otimes g \in A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$, we have $c_{k,\ell}(f \otimes g) = c_k(f)\, c_\ell(g)$, $(k, \ell \in \mathbb{Z})$. This yields immediately that the norm $\lambda_{\alpha,\beta}$ is a crossnorm.

(ii) Let $h = \sum_{r=1}^{s} f_r \otimes g_r$ be a representation of $h \in A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$. For simplicity, let $1 < q < \infty$, $1/p + 1/q = 1$. In case of $q = \infty$ or $p = \infty$, the sums have to be substituted by the supremum. Then, we can write

$$\begin{aligned}
\lambda_{\alpha,\beta}(h) &= \Big(\sum_{k,\ell\in\mathbb{Z}}(1+k^2)^{\alpha q/2}(1+\ell^2)^{\beta q/2}|c_{k,\ell}(h)|^q\Big)^{1/q} \\
&= \Big(\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha q/2}\Big(\sum_{\ell\in\mathbb{Z}}(1+\ell^2)^{\beta q/2}\Big|\sum_{r=1}^{s} c_k(f_r)c_\ell(g_r)\Big|^q\Big)^{q/q}\Big)^{1/q} \\
&= \Big\|\,\Big\|\sum_{r=1}^{s} c(f_r)c(g_r)\ \mid\ Y\Big\|\ \mid\ X\Big\|
\end{aligned}$$

with the sequences $c(f_r) = \{c_k(f_r)\}_{k\in\mathbb{Z}}$ and $c(g_r) = \{c_\ell(g_r)\}_{\ell\in\mathbb{Z}}$ of Fourier coefficients and the weighted ℓ_q–spaces X and Y with the norms

$$\begin{aligned}
\|\{a_k\}_{k\in\mathbb{Z}} \mid X\| &= \|\{(1+k^2)^{\alpha q/2}a_k\}_{k\in\mathbb{Z}} \mid \ell_q(\mathbb{Z})\|, \\
\|\{a_\ell\}_{\ell\in\mathbb{Z}} \mid Y\| &= \|\{(1+\ell^2)^{\beta q/2}a_\ell\}_{\ell\in\mathbb{Z}} \mid \ell_q(\mathbb{Z})\|.
\end{aligned}$$

We follow the idea of proof of Lemma 1.47 in [9]. In the sequel, the supremum on ψ is taken on all $\psi\in Y'$ with $\|\psi \mid Y'\| = 1$. With this, we have

$$\lambda_{\alpha,\beta}(h) = \Big\|\sup_{\psi}\Big|\sum_{r=1}^{s} c(f_r)\psi(c(g_r))\Big|\ \mid\ X\Big\|.$$

Hölder's inequality (the absolute value of the sequence is taken componentwise) yields

$$\begin{aligned}
\lambda_{\alpha,\beta}(h) &\le \Big\|\sup_{\psi}\Big(\sum_{r=1}^{s}|c(f_r)|^q\Big)^{1/q}\Big(\sum_{u=1}^{s}|\psi(c(g_u))|^p\Big)^{1/p}\ \mid\ X\Big\| \\
&\le \sup_{\psi}\Big(\sum_{u=1}^{s}|\psi(c(g_u))|^p\Big)^{1/p}\Big(\sum_{k\in\mathbb{Z}}(1+k^2)^{\alpha q/2}\sum_{r=1}^{s}|c_k(f_r)|^q\Big)^{1/q} \\
&= \sup_{\psi}\Big(\sum_{u=1}^{s}|\psi(c(g_u))|^p\Big)^{1/p}\Big(\sum_{r=1}^{s}\|c(f_r) \mid X\|^q\Big)^{1/q}.
\end{aligned}$$

Using an equivalent definition of $\theta_p(y_1,\dots,y_s) := \sup_\psi \|\{\psi(y_u)\}_{u=1,\dots,s} \mid \ell_p(\{1,\dots,s\}\|$ (cf. [9], Definition 1.43), leads to

$$\begin{aligned}
\lambda_{\alpha,\beta}(h) &\le \Big(\sum_{r=1}^{s}\|f_r \mid A_q^\alpha(\mathbb{T})\|^q\Big)^{1/q}\ \theta_p(c(g_1),\dots,c(g_s)) \\
&= \Big(\sum_{r=1}^{s}\|f_r \mid A_q^\alpha(\mathbb{T})\|^q\Big)^{1/q}\ \sup_{\eta}\Big\|\sum_{r=1}^{s}\eta_r c(g_r)\ \mid\ Y\Big\| \\
&= \Big(\sum_{r=1}^{s}\|f_r \mid A_q^\alpha(\mathbb{T})\|^q\Big)^{1/q}\ \sup_{\eta}\Big\|\sum_{r=1}^{s}\eta_r g_r\ \mid\ A_q^\beta(\mathbb{T})\Big\| \\
&= \Big(\sum_{r=1}^{s}\|f_r \mid A_q^\alpha(\mathbb{T})\|^q\Big)^{1/q}\ \theta_p(g_1,\dots,g_s).
\end{aligned}$$

There the supremum on η is taken on all $\|\{\eta_r\}_{r=1,\dots,s} \mid \ell_q(\{1,\dots,s\})\| \le 1$. Because this holds for all representations of h we proved $\lambda_{\alpha,\beta}(h) \le \tilde{\lambda}_q(h)$.

Let

$$h_N := \sum_{\ell=-N}^{N}\Big((1+\ell^2)^{\beta/2}\sum_{k\in\mathbb{Z}}\sum_{r=1}^{s} c_\ell(g_r)\, c_k(f_r)\,\mathrm{e}^{\mathrm{i}k\cdot}\Big)\otimes(1+\ell^2)^{-\beta/2}\,\mathrm{e}^{\mathrm{i}\ell\cdot}$$

be the partial sum of the Fourier series of h. For all N, we can estimate

$$\begin{aligned}
\tilde{\lambda}_q(h_N) &\le \left(\sum_{\ell=-N}^{N} \left\| (1+\ell^2)^{\beta/2} \sum_{k\in\mathbb{Z}} \sum_{r=1}^{s} c_\ell(g_r)\, c_k(f_r)\, \mathrm{e}^{\mathrm{i}k\cdot} \;\middle|\; A_q^\alpha(\mathbb{T}) \right\|^q \right)^{1/q} \\
&\qquad \cdot\, \theta_p\left((1+(-N)^2)^{-\beta/2}\, \mathrm{e}^{-\mathrm{i}N\cdot}, \ldots, (1+N^2)^{-\beta/2}\, \mathrm{e}^{\mathrm{i}N\cdot} \right) \\
&= \left(\sum_{\ell=-N}^{N} \left((1+\ell^2)^{\beta/2} \left\| \Big\{ \sum_{k\in\mathbb{Z}} \sum_{r=1}^{s} c_\ell(g_r)\, c_k(f_r) \Big\}_{k\in\mathbb{Z}} \;\middle|\; \ell_q(\mathbb{Z}) \right\| \right)^q \right)^{1/q} \cdot\, 1 \\
&\le \lambda_{\alpha,\beta}(h)
\end{aligned}$$

which gives $\tilde{\lambda}_q(h) \le \lambda_{\alpha,\beta}(h)$.

(iii) The assertion follows from the uniformity of the q–nuclear norms and (ii).

(iv) The definitions of the function spaces immediately yield $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T}) \subset S_{q,q}^{\alpha,\beta}A(\mathbb{T}^2)$. The functions h_N of the form

$$h_N = \sum_{k,\ell=-N}^{N} \eta_{k,\ell}\, \mathrm{e}^{\mathrm{i}k\cdot} \otimes \mathrm{e}^{\mathrm{i}\ell\cdot} \tag{10}$$

with $\lambda_{\alpha,\beta}(h_N) < \infty$ belong to $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$. Therefore, the algebraic tensor product $A_q^\alpha(\mathbb{T}) \otimes A_q^\beta(\mathbb{T})$ is a dense subset of the Banach space $S_{q,q}^{\alpha,\beta}A(\mathbb{T}^2)$ (Lemma 1). The completion of a dense subset is unique up to isomorphy. This proves (iv). ■

The isotropic function spaces $A_q^\alpha(\mathbb{T}^2)$ can be represented as the intersection of certain spaces of functions with dominating mixed smoothness. The norm inequalities given below will be used for the error estimates in such isotropic spaces.

Theorem 10 *Let $1 \le q \le \infty$ and $\alpha \ge \beta \ge 0$.*

(i) *It holds that $A_q^\alpha(\mathbb{T}^2) = S_{q,q}^{\alpha,0}A(\mathbb{T}^2) \cap S_{q,q}^{0,\alpha}A(\mathbb{T}^2)$ with the equivalence of norms*

$$\begin{aligned}
\mathrm{C}_1(\alpha,q)\, \|\cdot \mid A_q^\alpha(\mathbb{T}^2)\| &\le \left(\|\cdot \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\|^q + \|\cdot \mid S_{q,q}^{0,\alpha}A(\mathbb{T}^2)\|^q \right)^{1/q} \\
&\le \mathrm{C}_2(\alpha,q)\, \|\cdot \mid A_q^\alpha(\mathbb{T}^2)\|
\end{aligned}$$

where

$$\mathrm{C}_1(\alpha,q) := \begin{cases} 2^{1/q} & \text{for } 1 \le q < \infty,\ \alpha = 0, \\ 1 & \text{for } 1 \le q < \infty,\ 0 < \alpha \le 2/q, \\ 2^{1/q-\alpha/2} & \text{for } 1 \le q < \infty,\ \alpha > 2/q, \\ 1 & \text{for } q = \infty,\ \alpha = 0, \\ 2^{-1} & \text{for } q = \infty,\ \alpha > 0, \end{cases}$$

$$\mathrm{C}_2(\alpha,q) := \begin{cases} 2^{1/q} & \text{for } 1 \le q < \infty,\ \alpha \ge 0, \\ 1 & \text{for } q = \infty,\ \alpha \ge 0. \end{cases}$$

(ii) *We have $A_q^\alpha(\mathbb{T}^2) \subset S_{q,q}^{\beta,\alpha-\beta}A(\mathbb{T}^2)$ with $\|\cdot \mid S_{q,q}^{\beta,\alpha-\beta}A(\mathbb{T}^2)\| \le \|\cdot \mid A_q^\alpha(\mathbb{T}^2)\|$.*

(iii) *It holds that $S_{q,q}^{\alpha,\alpha}A(\mathbb{T}^2) \subset A_q^\alpha(\mathbb{T}^2)$ with $\|\cdot \mid A_q^\alpha(\mathbb{T}^2)\| \le \|\cdot \mid S_{q,q}^{\alpha,\alpha}A(\mathbb{T}^2)\|$.*

Proof: (i) For $\alpha = 0$, the Definitions 1 and 2 immediately yield $A^0_q(\mathbb{T}^2) = S^{0,0}_{q,q}A(\mathbb{T}^2)$ and the equality of norms. Let $1 \le q < \infty$, $\alpha > 0$. Using $1 + k^2 \le 1 + k^2 + \ell^2$ and $1 + \ell^2 \le 1 + k^2 + \ell^2$, one finds $(1 + k^2)^{\alpha q/2} + (1 + \ell^2)^{\alpha q/2} \le 2\ (1 + k^2 + \ell^2)^{\alpha q/2}$. On the other hand, we can use Hölder's inequality for $\alpha \ge 2/q$ to obtain $(1 + k^2 + \ell^2)^{\alpha q/2} \le 2^{\alpha q/2\beta}((1 + k^2)^{\alpha q/2} + (1 + \ell^2)^{\alpha q/2})$ with $2/\alpha q + 1/\beta = 1$. In case of $0 < \alpha \le 2/q$, we can directly use $(1 + k^2 + \ell^2)^{\alpha q/2} \le (1 + k^2)^{\alpha q/2} + (1 + \ell^2)^{\alpha q/2}$. For $q = \infty$, assertion (i) follows from $\frac{1}{2}(1 + k^2 + \ell^2) \le \max\{1 + k^2, 1 + \ell^2\} \le 1 + k^2 + \ell^2$.

(ii) The assertion follows from $(1 + k^2)^\beta(1 + \ell^2)^{\alpha-\beta} \le (1 + k^2 + \ell^2)^\alpha$.

(iii) This assertion holds because of $(1 + k^2 + \ell^2) \le (1 + k^2)(1 + \ell^2)$. ■

7 Parametric Extensions of Interpolation Operators

As a preparation to the next sections, we consider the parametric extensions of interpolation operators of univariate functions in y– and x–direction

$$\begin{aligned} L^x_K f(x,y) &:= \sum_{\ell\in J_K} f\Big(\frac{2\pi\ell}{K}, y\Big)\ \Lambda_K\Big(x - \frac{2\pi\ell}{K}\Big), \\ L^y_N f(x,y) &:= \sum_{\ell\in J_N} f\Big(x, \frac{2\pi\ell}{N}\Big)\ \Lambda_N\Big(y - \frac{2\pi\ell}{N}\Big). \end{aligned}$$

Obviously, for continuous functions $f \in C(\mathbb{T}^2) \cap S^{\mu,\nu}_{q,q}A(\mathbb{T}^2)$ from spaces of functions with dominating mixed smoothness properties, we can write these operators as $L^x_K f = (L_K \hat{\otimes} I)\, f$ and $L^y_N f = (I \hat{\otimes} L_N)\, f$ with the identity operator I. In the sequel, we use the operators L^x_K and L^y_N in this sense for all functions from the tensor product spaces. The complementary operator of a projector P is denoted by $P^c := I - P$. Error estimates for the interpolation operators carry over to their parametric extensions in the following way.

Lemma 2 *Let $1 \le q \le \infty$, $\mu \ge \alpha \ge 0$ and $\nu \ge \beta \ge 0$. We have*

(i) *if $\mu > 1 - 1/q$ then* $\|(L^x_K)^c \mid \mathcal{L}(S^{\mu,\nu}_{q,q}A(\mathbb{T}^2), S^{\alpha,\nu}_{q,q}A(\mathbb{T}^2))\| = \|L^c_K \mid \mathcal{L}(A^\mu_q(\mathbb{T}), A^\alpha_q(\mathbb{T}))\|$,

(ii) *if $\nu > 1 - 1/q$ then* $\|(L^y_N)^c \mid \mathcal{L}(S^{\mu,\nu}_{q,q}A(\mathbb{T}^2), S^{\mu,\beta}_{q,q}A(\mathbb{T}^2))\| = \|L^c_N \mid \mathcal{L}(A^\nu_q(\mathbb{T}), A^\beta_q(\mathbb{T}))\|$,

(iii) *if $\mu, \nu > 1 - 1/q$ then*
$$\|(L^x_K)^c(L^y_N)^c \mid \mathcal{L}(S^{\mu,\nu}_{q,q}A(\mathbb{T}^2), S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2))\| = \|L^c_K \mid \mathcal{L}(A^\mu_q(\mathbb{T}), A^\alpha_q(\mathbb{T}))\| \cdot \|L^c_N \mid \mathcal{L}(A^\nu_q(\mathbb{T}), A^\beta_q(\mathbb{T}))\|.$$

Proof: Because of $(L^x_K)^c = I\hat{\otimes}I - L_K\hat{\otimes}I = (L^x_K)^c\hat{\otimes}I$ assertion (i) follows from the uniformity of the norms (Theorem 9 (iii)). Assertion (ii) holds by symmetry and yields together with (i) assertion (iii). ■

8 Interpolation on Full Grids

A full grid is the cartesian product $\mathcal{T}_K \times \mathcal{T}_N$ of two equidistant grids for $n = 1$. The step sizes $2\pi/K$ in x–direction and $2\pi/N$ in y–direction may differ.

The interpolation on such a grid can be realized by

$$L^x_K L^y_N f(x,y) = \sum_{r\in J_K} \sum_{s\in J_N} f\Big(\frac{2\pi r}{K}, \frac{2\pi s}{N}\Big)\ \Lambda_K\Big(x - \frac{2\pi r}{K}\Big)\ \Lambda_N\Big(y - \frac{2\pi s}{N}\Big).$$

The Boolean sum $P \oplus Q$ of two commuting projectors P and Q is the projector $P \oplus Q := (P^c Q^c)^c = P + Q - PQ$ (cf. [5]). With this, we can write the remainder of interpolation as

$$(L_K^x L_N^y)^c = (L_K^x)^c \oplus (L_N^y)^c = (L_K^x)^c + (L_N^y)^c - (L_K^x)^c (L_N^y)^c. \tag{11}$$

We estimate the error of interpolation for functions with dominating mixed smoothness properties first.

Theorem 11 *Let $f \in S_{q,q}^{\mu,\nu} A(\mathbb{T}^2)$ with $1 \le q \le \infty$, $\mu \ge \alpha \ge 0$, $\nu \ge \beta \ge 0$, and $\mu, \nu > 1 - 1/q$. Let the 2π–periodic fundamental interpolants Λ_K, $\Lambda_N \in A(\mathbb{T})$ satisfy the periodic Strang–Fix conditions* (3) *of order k for the exponents q and α and of order m for the exponents q and β, respectively. Let $\varrho := \min\{\mu, k + \alpha\}$, $\sigma := \min\{\nu, m + \beta\}$. With the constants C_ϱ for Λ_K and C_σ for Λ_N from Theorem* 5*, it holds that*

$$\begin{aligned}
&\|f - L_K^x L_N^y f \mid S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2)\| \\
&\le \left(\mathrm{C}_\varrho \left(\frac{2\pi}{K}\right)^{\varrho - \alpha} + \mathrm{C}_\sigma \left(\frac{2\pi}{N}\right)^{\sigma - \beta} + \mathrm{C}_\varrho \, \mathrm{C}_\sigma \left(\frac{2\pi}{K}\right)^{\varrho - \alpha} \left(\frac{2\pi}{N}\right)^{\sigma - \beta} \right) \|f \mid S_{q,q}^{\mu,\nu} A(\mathbb{T}^2)\|.
\end{aligned}$$

Proof: The assumption $\mu, \nu > 1 - 1/q$ insures $f \in A(\mathbb{T}^2)$. So the interpolant is well–defined. We use the representation (11) of the remainder and the estimates from Lemma 2 and Theorem 5. We obtain

$$\begin{aligned}
&\|f - L_K^x L_N^y f \mid S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2)\| \\
&\le \|(L_K^x)^c f \mid S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2)\| + \|(L_N^y)^c f \mid S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2)\| + \|(L_K^x)^c (L_N^y)^c f \mid S_{q,q}^{\alpha,\beta} A(\mathbb{T}^2)\| \\
&\le \mathrm{C}_\varrho \left(\frac{2\pi}{K}\right)^{\varrho - \alpha} \|f \mid S_{q,q}^{\mu,\beta} A(\mathbb{T}^2)\| + \mathrm{C}_\sigma \left(\frac{2\pi}{N}\right)^{\sigma - \beta} \|f \mid S_{q,q}^{\alpha,\nu} A(\mathbb{T}^2)\| \\
&\qquad + \mathrm{C}_\varrho \, \mathrm{C}_\sigma \left(\frac{2\pi}{K}\right)^{\varrho - \alpha} \left(\frac{2\pi}{N}\right)^{\sigma - \beta} \|f \mid S_{q,q}^{\mu,\nu} A(\mathbb{T}^2)\|.
\end{aligned}$$

With $\|\cdot \mid S_{q,q}^{\eta,\kappa} A(\mathbb{T}^2)\| \le \|\cdot \mid S_{q,q}^{\mu,\nu} A(\mathbb{T}^2)\|$ for $\eta \le \mu$ and $\kappa \le \nu$, this proves the theorem. ■

From the estimate in Theorem 11, we can see that it can make sense to use different step sizes of different kinds of fundamental interpolants for the different directions. This case occurs if the smoothness parameter μ and ν differ. For functions from isotropic function spaces, only equidistant interpolation makes sense (cf. [16]). Therefore, the next theorem gives error estimates for this case only.

Theorem 12 *Let $f \in A_q^\mu(\mathbb{T}^2)$ with $1 \le q \le \infty$, $\mu \ge 2\alpha \ge 0$ and $\mu > 2 - 2/q$. Let the 2π–periodic fundamental interpolant $\Lambda_N \in A(\mathbb{T})$ satisfy the periodic Strang–Fix conditions* (3) *of order m for the exponents q and α. Let $\varrho := \min\{\mu, m + \alpha\}$, $\eta := \min\{\mu/2, m + \alpha\}$, $\kappa := \min\{\mu/2, m\}$ and $N > 2\pi$. The constants $\mathrm{C}_\varrho, \mathrm{C}_{\varrho-\alpha}, \mathrm{C}_\eta, \mathrm{C}_\kappa$, and $\mathrm{C}_1(\alpha, q)$, $\mathrm{C}_2(\mu, q)$ correspond to those given in Theorem* 5 *and in Theorem* 10*, respectively. With the constant $\tilde{\mathrm{C}}_\varrho := (\mathrm{C}_1(\alpha, q))^{-1} \; 3^{q-1} \; \left((\mathrm{C}_2(\mu, q))^q \mathrm{C}_\varrho^q + 2\mathrm{C}_{\varrho-\alpha}^q + 2\mathrm{C}_\eta^q \mathrm{C}_\kappa^q\right)^{1/q}$, it holds the estimate*

$$\|f - L_N^x L_N^y f \mid A_q^\alpha(\mathbb{T}^2)\| \le \tilde{\mathrm{C}}_\varrho \left(\frac{2\pi}{N}\right)^{\varrho - \alpha} \|f \mid A_q^\mu(\mathbb{T}^2)\|.$$

Proof: First, we apply Theorem 10 (i)

$$\begin{aligned}
&\|f - L_N^x L_N^y f \mid A_q^\alpha(\mathbb{T}^2)\| \\
&\le (\mathrm{C}_1(\alpha, q))^{-1} \left(\|(L_N^x L_N^y)^c f \mid S_{q,q}^{\alpha,0} A(\mathbb{T}^2)\|^q + \|(L_N^x L_N^y)^c f \mid S_{q,q}^{0,\alpha} A(\mathbb{T}^2)\|^q \right)^{1/q}
\end{aligned}$$

and estimate each term separately. The remainder representation (11) and the triangle inequality yield for the first term

$$\begin{aligned}
&\|(L_N^x L_N^y)^c f \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\| \\
&\quad \le \ \|(L_N^x)^c f \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\| + \|(L_N^y)^c f \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\| + \|(L_N^x)^c (L_N^y)^c f \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\|.
\end{aligned}$$

From the Strang–Fix conditions, Lemma 2 and Theorem 5, we obtain

$$\begin{aligned}
\|(L_N^x L_N^y)^c f \mid S_{q,q}^{\alpha,0}A(\mathbb{T}^2)\| &\le \mathrm{C}_\varrho \left(\frac{2\pi}{N}\right)^{\varrho-\alpha} \|f \mid S_{q,q}^{\mu,0}A(\mathbb{T}^2)\| \\
&+ \mathrm{C}_{\varrho-\alpha}\left(\frac{2\pi}{N}\right)^{\varrho-\alpha} \|f \mid S_{q,q}^{\alpha,\mu-\alpha}A(\mathbb{T}^2)\| + \mathrm{C}_\eta \mathrm{C}_\kappa \left(\frac{2\pi}{N}\right)^{\eta-\alpha}\left(\frac{2\pi}{N}\right)^{\kappa} \|f \mid S_{q,q}^{\mu/2,\mu/2}A(\mathbb{T}^2)\|.
\end{aligned}$$

An analogous estimate holds for $\|(L_N^x L_N^y)^c f \mid S_{q,q}^{0,\alpha}A(\mathbb{T}^2)\|$. We add the q–th powers of both estimates and apply Hölder's inequality. This leads to

$$\begin{aligned}
\|f - L_N^x L_N^y f \mid A_q^\alpha(\mathbb{T}^2)\| &\le (\mathrm{C}_1(\alpha,q))^{-1}\, 3^{q(1-1/q)} \left(\left(\frac{2\pi}{N}\right)^{(\varrho-\alpha)q} \Big(\mathrm{C}_\varrho^q\big(\|f \mid S_{q,q}^{\mu,0}A(\mathbb{T}^2)\|^q \right. \\
&+ \|f \mid S_{q,q}^{0,\mu}A(\mathbb{T}^2)\|^q\big) + \mathrm{C}_{\varrho-\alpha}^q\big(\|f \mid S_{q,q}^{\mu-\alpha,\alpha}A(\mathbb{T}^2)\|^q + \|f \mid S_{q,q}^{\alpha,\mu-\alpha}A(\mathbb{T}^2)\|^q\big)\Big) \\
&\left. + 2\mathrm{C}_\eta^q \mathrm{C}_\kappa^q \left(\frac{2\pi}{N}\right)^{(\kappa+\eta-\alpha)q} \|f \mid S_{q,q}^{\mu/2,\mu/2}A(\mathbb{T}^2)\|^q \right)^{1/q}.
\end{aligned}$$

Now, the norm inequalities from Theorem 10 (i), (ii) prove the estimate. ■

9 Interpolation on Sparse Grids

We want to define the interpolation operators for interpolation on sparse grids and give error estimates. The following definition of the blending interpolation operator and its basic properties can be found e.g. in [1, 5].

Given a chain (7) of interpolation operators L_{N_j}, $j \in \mathbb{N}_0$, for univariate functions. Their parametric extensions $L_{N_j}^x$ and $L_{N_j}^y$ also form chains. The *j–th order Boolean sum* (or j–th order blending interpolation operator) is defined as

$$B_j := \bigoplus_{r=0}^{j} L_{N_r}^x \, L_{N_{j-r}}^y. \tag{12}$$

It can be written as a difference of ordinary sums

$$B_j = \sum_{r=0}^{j} L_{N_r}^x \, L_{N_{j-r}}^y - \sum_{r=0}^{j-1} L_{N_r}^x \, L_{N_{j-r-1}}^y. \tag{13}$$

The j–th order Boolean sums $\{B_j\}_{j\in\mathbb{N}_0}$ form a chain and interpolate on the sparse grids $\mathcal{T}_j^{\mathrm{B}} := \bigcup_{r=0}^{j} \mathcal{T}_{N_r} \times \mathcal{T}_{N_{j-r}}$. The sparse grid $\mathcal{T}_j^{\mathrm{B}}$ has $d^2(j2^{j-1}+2^j)$ nodes which is essentially less than the $d^2\,2^{2j}$ nodes in the equidistant grid $\mathcal{T}_{N_j} \times \mathcal{T}_{N_j}$.

The remainder has the representation

$$B_j^c = (L_{N_j}^x)^c + (L_{N_j}^y)^c - \sum_{r=0}^{j} (L_{N_r}^x)^c \, (L_{N_{j-r}}^y)^c + \sum_{r=0}^{j-1} (L_{N_r}^x)^c \, (L_{N_{j-r-1}}^y)^c. \tag{14}$$

We now proceed to error estimates for the interpolation by j–th order Boolean sums. We start with the estimate for functions with dominating mixed smoothness properties.

Theorem 13 *Let $f \in S^{\mu,\nu}_{q,q}A(\mathbb{T}^2)$ with $1 \le q \le \infty$, $\mu \ge \alpha \ge 0$, $\nu \ge \beta \ge 0$, and $\mu,\nu > 1-1/q$. Let the family $\{\Lambda_{N_j}\}_{j\in\mathbb{N}_0}$ of 2π–periodic fundamental interpolants $\Lambda_{N_j} \in A(\mathbb{T})$ satisfy the uniform periodic Strang–Fix conditions of order k for the exponents q and α and of order m for the exponents q and β. The corresponding interpolation operators form a chain (7). Let $\varrho := \min\{\mu, k+\alpha\}$, $\sigma := \min\{\nu, m+\beta\}$ and the constants C_ϱ and C_σ as in Theorem 8.*

(i) *Then, we can estimate*

$$\begin{aligned} &\|f - B_j f \mid S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)\| \\ &\le \left(\mathrm{C}_\varrho \left(\frac{2\pi}{N_j}\right)^{\varrho-\alpha} + \mathrm{C}_\sigma \left(\frac{2\pi}{N_j}\right)^{\sigma-\beta} + \mathrm{C}_\varrho\,\mathrm{C}_\sigma \sum_{r=0}^{j} \left(\frac{2\pi}{N_r}\right)^{\varrho-\alpha} \left(\frac{2\pi}{N_{j-r}}\right)^{\sigma-\beta}\right. \\ &\qquad \left. + \mathrm{C}_\varrho\,\mathrm{C}_\sigma \sum_{r=0}^{j-1} \left(\frac{2\pi}{N_r}\right)^{\varrho-\alpha} \left(\frac{2\pi}{N_{j-r}}\right)^{\sigma-\beta}\right) \|f \mid S^{\mu,\nu}_{q,q}A(\mathbb{T}^2)\|. \end{aligned}$$

(ii) *Let $\tau := \min\{\varrho-\alpha, \sigma-\beta\}$, $\tilde{\mathrm{C}}_\tau := \max\limits_{0\le j\le 3}\{(2\pi/N_j)^{\varrho-\alpha-\tau}, (2\pi/N_j)^{\sigma-\beta-\tau}\}$, and $\mathrm{C}_\tau = \mathrm{C}_\tau(j) := \tilde{\mathrm{C}}_\tau\,(\mathrm{C}_\varrho + \mathrm{C}_\sigma + (3j-2)\,(2\pi/d)^\tau\,\mathrm{C}_\varrho\,\mathrm{C}_\sigma)$. Altogether, it holds that*

$$\|f - B_j f \mid S^{\alpha,\beta}_{q,q}A(\mathbb{T}^2)\| \le \mathrm{C}_\tau \left(\frac{2\pi}{N_j}\right)^{\tau} \|f \mid S^{\mu,\nu}_{q,q}A(\mathbb{T}^2)\|.$$

Proof: We can argue in the same way as in the proof of Theorem 11. From the representation (14) of the remainder B^c_j, we obtain the estimate (i). The factor C_τ depends linearly on j. With that definition of C_τ, assertion (i) yields (ii), immediately. ∎

We note that $\mathrm{C}_\tau(j)$ depends on j. For $\varrho-\alpha = \sigma-\beta$, we obtain $\mathrm{C}_\tau(j) = \mathcal{O}(j)$. That means that in this case, we lose only a logarithmic term in approximation order if we compare the interpolation by B_j on a sparse grid with $\mathcal{O}(jN_j)$ points and the interpolation by $L^x_{N_j}L^y_{N_j}$ on an equidistant grid with N_j^2 points. Similarly to Theorem 12, we can give an error estimate for the case of isotropic function spaces.

Theorem 14 *Let $f \in A^\mu_q(\mathbb{T}^2)$ with $1 \le q \le \infty$, $\mu \ge 2\alpha \ge 0$, and $\mu > 2 - 2/q$. Let the family $\{\Lambda_{N_j}\}_{j\in\mathbb{N}_0}$ of 2π–periodic fundamental interpolants $\Lambda_{N_j} \in A(\mathbb{T})$ satisfy the uniform periodic Strang–Fix conditions of order m for the exponents q and α. The corresponding interpolation operators form a chain (7). Let $\varrho := \min\{\mu, m+\alpha\}$, $\eta := \min\{\mu/2, m+\alpha\}$, $\kappa := \min\{\mu/2, m\}$, $\tau := \min\{\eta-\alpha, \kappa\}$ and $N_j > 2\pi$. The constants $\mathrm{C}_\varrho, \mathrm{C}_{\varrho-\alpha}, \mathrm{C}_\eta, \mathrm{C}_\kappa$ and $\mathrm{C}_1(\alpha,q), \mathrm{C}_2(\mu,q)$ are chosen as in Theorem 8 and 10, respectively.*
With $\tilde{\mathrm{C}}_\tau = \max\limits_{0\le j\le 3}\{(2\pi/N_j)^{\eta-\alpha-\tau}, (2\pi/N_j)^{\kappa-\tau}\}$, and

$$\mathrm{C}_\tau = \mathrm{C}_\tau(j) := \frac{3^{q-1}\tilde{\mathrm{C}}_\tau}{\mathrm{C}_1(\alpha,q)} \left((\mathrm{C}_2(\mu,q))^q \mathrm{C}^q_\varrho + 2\mathrm{C}_{\varrho-\alpha} + 2(3j-2)^q \left(\frac{2\pi}{d}\right)^{\tau q} \mathrm{C}^q_\eta \mathrm{C}^q_\kappa\right)^{1/q},$$

we can estimate

$$\|f - B_j f \mid A^\alpha_q(\mathbb{T}^2)\| \le \mathrm{C}_\tau \left(\frac{2\pi}{N_j}\right)^{\tau} \|f \mid A^\mu_q(\mathbb{T}^2)\|.$$

Proof: We follow the proof of Theorem 12. We apply Theorem 10 (i) and obtain

$$\|f - B_j f \mid A^\alpha_q(\mathbb{T}^2)\| \le (\mathrm{C}_1(\alpha,q))^{-1} \left(\|B^c_j f \mid S^{\alpha,0}_{q,q}A(\mathbb{T}^2)\|^q + \|B^c_j f \mid S^{0,\alpha}_{q,q}A(\mathbb{T}^2)\|^q\right)^{1/q}.$$

The representation of the remainder (14) and the triangle inequality give

$$\|B_j^c f \mid S^{0,\alpha}_{q,q}A(\mathbb{T}^2)\| \leq \mathrm{C}_{\varrho-\alpha}\left(\frac{2\pi}{N_j}\right)^{\varrho-\alpha}\|f \mid S^{\mu-\alpha,\alpha}_{q,q}A(\mathbb{T}^2)\| + \mathrm{C}_{\varrho}\left(\frac{2\pi}{N_j}\right)^{\varrho-\alpha}\|f \mid S^{0,\mu}_{q,q}A(\mathbb{T}^2)\|$$
$$+\,\mathrm{C}_\eta \mathrm{C}_\kappa\left(\sum_{r=0}^{j}\left(\frac{2\pi}{N_r}\right)^{\eta-\alpha}\left(\frac{2\pi}{N_{j-r}}\right)^{\kappa} + \sum_{r=0}^{j-1}\left(\frac{2\pi}{N_r}\right)^{\eta-\alpha}\left(\frac{2\pi}{N_{j-r-1}}\right)^{\kappa}\right)\|f \mid S^{\mu/2,\mu/2}_{q,q}A(\mathbb{T}^2)\|.$$

Theorem 8 and Lemma 2 yield the estimate

$$\begin{aligned}\|B_j^c f \mid S^{0,\alpha}_{q,q}A(\mathbb{T}^2)\| \;&\leq\; \tilde{\mathrm{C}}_\tau\left(\frac{2\pi}{N_j}\right)^{\tau}\Big(\mathrm{C}_{\varrho-\alpha}\|f \mid S^{\mu-\alpha,\alpha}_{q,q}A(\mathbb{T}^2)\| + \mathrm{C}_{\varrho}\|f \mid S^{0,\mu}_{q,q}A(\mathbb{T}^2)\| \\ &\qquad + (3j-2)\left(\frac{2\pi}{d}\right)^{\tau}\mathrm{C}_\eta \mathrm{C}_\kappa\|f \mid S^{\mu/2,\mu/2}_{q,q}A(\mathbb{T}^2)\|\Big).\end{aligned}$$

The remaining steps can be carried out analogously to the proof of Theorem 12. ■

This result shows us that we really need the mixed smoothness properties to benefit from the sparse grid approach. In comparision to the equidistant grid, we lose roughly half the approximation order (in terms of powers of N_j) using interpolation on sparse grids. If one compares in terms of the number N_{G} of grid points one obtains the following.

Smoothness of f	$(L_{N_j}\otimes L_{N_j})f-f$	$B_j f - f$
	in the norm of $A^\alpha_q(\mathbb{T}^2)$	
$f\in A^\mu_q(\mathbb{T}^2)$, before saturation	$\mathcal{O}(N_{\mathrm{G}}^{-(\mu-\alpha)/2})$	$\mathcal{O}(N_{\mathrm{G}}^{-\mu/2+\alpha}\,(\log_2 N_{\mathrm{G}})^{\mu/2-\alpha+1})$
$f\in A^\mu_q(\mathbb{T}^2)$, after saturation	$\mathcal{O}(N_{\mathrm{G}}^{-m/2})$	$\mathcal{O}(N_{\mathrm{G}}^{-m}\,(\log_2 N_{\mathrm{G}})^{m+1})$
	in the norm of $S^{\alpha,\alpha}_{q,q}A(\mathbb{T}^2)$	
$f\in S^{\mu,\mu}_{q,q}A(\mathbb{T}^2)$, before saturation	$\mathcal{O}(N_{\mathrm{G}}^{-(\mu-\alpha)/2})$	$\mathcal{O}(N_{\mathrm{G}}^{-\mu+\alpha}\,(\log_2 N_{\mathrm{G}})^{\mu-\alpha+1})$
$f\in S^{\mu,\mu}_{q,q}A(\mathbb{T}^2)$, after saturation	$\mathcal{O}(N_{\mathrm{G}}^{-m/2})$	$\mathcal{O}(N_{\mathrm{G}}^{-m}\,(\log_2 N_{\mathrm{G}})^{m+1})$

As one can see from this table interpolation on full grids beats interpolation on sparse grids only for functions from isotropic spaces (compared in terms of grid points). In this case, it is better only by a logarithmic term. For functions with dominating mixed smoothness properties, interpolation on sparse grids is essentially better suited.

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