Some Error Estimates for Periodic Interpolation of Functions from Besov Spaces

Winfried Sickel and Frauke Sprengel

Abstract

Using periodic Strang–Fix conditions, we can give an approach to error estimates for periodic interpolation on equidistant and sparse grids for functions from certain Besov spaces.

1 Introduction

We investigate the L_2–error of interpolation on equidistant and sparse grids for periodic functions from isotropic L_2–Besov spaces and L_2–Besov spaces of functions with dominating mixed smoothness properties.

The interpolation of periodic functions by translates of a given function and the corresponding error estimates have been analyzed by several authors (e.g. [3, 8, 14]) in the univariate as well as in the multivariate case. The periodic Strang–Fix conditions were introduced in [2, 14]. There, they were used to find L_2–error estimates for functions from isotropic L_2–Sobolev spaces.

The approximation of functions on sparse grids and the related field of hyperbolic approximation have a fairly long tradition (e.g. [4, 5, 27]) as well. For bivariate functions, the number of interpolation knots can be reduced to $\mathcal{O}(N\,\log_2 N)$ for the sparse grids where the equidistant grid has $\mathcal{O}(N^2)$ points. Nevertheless, the interpolation on sparse grids yields error estimates for functions with dominating mixed smoothness properties which are asymptotically only by a logarithmic term worse than the error estimates for the interpolation on the corresponding equidistant grids.

The aim of this paper is to give error estimates for periodic interpolation for functions from L_2–Besov spaces which extend the results for the L_2–Sobolev spaces [2, 14, 16] on one hand. On the other hand, there already exist error estimates for interpolation on sparse grids for functions from Nikol'skij–Besov spaces [22]. But there, for the general L_p–case, we needed conditions on the cardinal fundamental interpolant from which the periodic fundamental interpolant was constructed via periodization. In the L_2–case, we don't need the long way around with cardinal interpolation but can use conditions on the periodic fundamental interpolant directly.

2 Besov Spaces

We start with recalling the definition and some basic properties of the function spaces to be dealt with. For this, we follow [19, Chap. 3]. By $\mathbb{T}^n$, we denote the n–dimensional torus represented by the cube

$$\mathbb{T}^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,;\ |x_r| \leq \pi,\ r = 1, \ldots, n\}.$$

Let $D(\mathbb{T}^n)$ and $D'(\mathbb{T}^n)$ denote the set of all complex–valued, 2π–periodic (in each component), and infinitely differentiable functions and its dual space, respectively. The Fourier coefficients of a distribution $g \in D'(\mathbb{T}^n)$ are

$$c_k(g) := g(\mathrm{e}^{-\mathrm{i}k\cdot})$$

for $k \in \mathbb{Z}^n$. With the help of the inner product in $L_2(\mathbb{T}^n)$,

$$\langle f, g\rangle_{\mathbb{T}^n} := \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} f(x)\overline{g(x)}\,\mathrm{d}x,$$

the Fourier coefficients for functions $g \in L_1(\mathbb{T}^n)$ can be written as $c_k(g) = \langle g, \mathrm{e}^{\mathrm{i}k\cdot}\rangle_{\mathbb{T}^n}$. Then any $f \in D'(\mathbb{T}^n)$ can be represented by its Fourier series

$$f = \sum_{k\in\mathbb{Z}^n} c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot} \qquad (\text{convergence in } D'(\mathbb{T}^n))$$

and the Fourier coefficients satisfy an inequality of the type

$$|\,c_k(f)\,| \le C_M\,(1+|k|_2)^M\,, \qquad k \in \mathbb{Z}^n, \tag{1}$$

for some $M \in \mathbb{N}$. Here and in the sequel, $|k|_2 := (k_1^2+k_2^2+\cdots+k_n^2)^{1/2}$ is the euclidian norm. Conversely, each formal Fourier series with polynomially bounded Fourier coefficients as in (1) can be interpreted as a periodic distribution in $D'(\mathbb{T}^n)$.

The Wiener algebra of functions with absolutely summable Fourier series we denote by $A(\mathbb{T}^n)$.

In the following, we restrict our definitions to the L_2–case because all the estimates in the forthcoming sections hold in this case only. We need the index sets

$$\begin{aligned} Q_0^n &= \{0\},\\ Q_j^n &= \{k\in\mathbb{Z}^n\,;\ |k_r| < 2^j, r=1,\ldots,n\}\setminus\{k\in\mathbb{Z}^n\,;\ |k_r|<2^{j-1}, r=1,\ldots,n\}. \end{aligned}$$

Definition 1 *Let $1 \le q \le \infty$ and $s \in \mathbb{R}$. Then we define the isotropic periodic L_2–Besov space $B^s_{2,q}(\mathbb{T}^n)$ as*

$$\begin{aligned} B^s_{2,q}(\mathbb{T}^n) &:= \Big\{ f \in D'(\mathbb{T}^n)\,;\ \|f \mid B^s_{2,q}(\mathbb{T}^n)\| \\ &\qquad = \Big(\sum_{\ell=0}^{\infty} 2^{\ell sq} \Big\| \sum_{k\in Q_j^n} c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot} \,\Big|\, L_2(\mathbb{T}^n)\Big\|^q\Big)^{1/q} < \infty \Big\} \end{aligned}$$

for $q<\infty$ and

$$\begin{aligned} B^s_{2,\infty}(\mathbb{T}^n) &:= \Big\{ f \in D'(\mathbb{T}^n)\,;\ \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\| \\ &\qquad = \sup_{\ell\in\mathbb{N}_0} 2^{\ell s} \Big\| \sum_{k\in Q_j^n} c_k(f)\,\mathrm{e}^{\mathrm{i}k\cdot} \,\Big|\, L_2(\mathbb{T}^n)\Big\| < \infty \Big\}, \end{aligned}$$

respectively.

For the definition of the spaces of functions with dominating mixed smoothness properties, we restrict ourselves to the two–dimensional situation. We put the index sets

$$P_{j_1,j_2} = Q^1_{j_1} \times Q^1_{j_2}, \quad j_1, j_2 \in \mathbb{N}_0.$$

As a consequence, we have the splitting

$$\mathbb{Z}^2 = \bigcup_{j_2=0}^{\infty}\bigcup_{j_1=0}^{\infty} P_{j_1,j_2} \qquad \text{with} \qquad P_{j_1,j_2}\cap P_{j_1',j_2'} = \emptyset \quad \text{if} \quad (j_1,j_2)\ne(j_1',j_2').$$

Definition 2 *Let $1 \le q \le \infty$ and $r_1, r_2 \in \mathbb{R}$. Then the L_2–Besov space $S^{r_1,r_2}_{2,q}B(\mathbb{T}^2)$ of bivariate periodic functions with dominating mixed smoothness properties is defined as*

$$\begin{aligned} S^{r_1,r_2}_{2,q}B(\mathbb{T}^2) \;&:=\; \Big\{ f \in D'(\mathbb{T}^2)\;;\; \|f \mid S^{r_1,r_2}_{2,q}B(\mathbb{T}^2)\| \\ &= \Big(\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 2^{(j_1 r_1 + j_2 r_2) q} \Big\| \sum_{k \in P_{j_1,j_2}} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \;\Big|\; L_2(\mathbb{T}^2) \Big\|^q \Big)^{1/q} < \infty \Big\} \end{aligned}$$

for $q < \infty$ and

$$\begin{aligned} S^{r_1,r_2}_{2,\infty}B(\mathbb{T}^2) \;&:=\; \Big\{ f \in D'(\mathbb{T}^2)\;;\; \|f \mid S^{r_1,r_2}_{2,\infty}B(\mathbb{T}^2)\| \\ &= \sup_{j_1 \in \mathbb{N}_0} \sup_{j_2 \in \mathbb{N}_0} 2^{(j_1 r_1 + j_2 r_2)} \Big\| \sum_{k \in P_{j_1,j_2}} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \;\Big|\; L_2(\mathbb{T}^2) \Big\| < \infty \Big\}, \end{aligned}$$

respectively.

Equivalent definitions of the Besov spaces using the moduli of smoothness and further characterizations can be found in [18, 19, 21].
By construction, it holds that

$$B^0_{2,2}(\mathbb{T}^n) = L_2(\mathbb{T}^n) \qquad \text{and} \qquad S^{0,0}_{2,2}B(\mathbb{T}^2) = L_2(\mathbb{T}^2). \tag{2}$$

The Besov spaces of bivariate functions with dominating mixed smoothness properties can be characterized as tensor products

$$B^{s_1}_{2,q}(\mathbb{T}) \otimes_\lambda B^{s_2}_{2,q}(\mathbb{T}) = S^{s_1,s_2}_{2,q}B(\mathbb{T}^2) \tag{3}$$

of the corresponding univariate Besov spaces (equivalent norms). Here, the norm λ which was used for the completion of the algebraic tensor product is the injective tensor norm for $1 \le q < \infty$ and a certain modification thereof for $q = \infty$, cf. [21] for details. These norms have the main advantage to be uniform crossnorms, cf. [7, 21]. That means (together with (2)) in particular, for two operators $P \in \mathcal{L}(B^{s_1}_{2,q}(\mathbb{T}), L_2(\mathbb{T}))$ and $Q \in \mathcal{L}(B^{s_2}_{2,q}(\mathbb{T}), L_2(\mathbb{T}))$, the tensor product operator $P \otimes Q$ given by

$$(P \otimes Q)(f \otimes g) := P(f) \otimes Q(g)$$

is bounded, i.e. $P \otimes Q \in \mathcal{L}(S^{s_1,s_2}_{2,q}B(\mathbb{T}^2), L_2(\mathbb{T}^2))$, and its norm can be estimated as

$$\|P \otimes Q \mid \mathcal{L}(S^{s_1,s_2}_{2,q}B(\mathbb{T}^2), L_2(\mathbb{T}^2))\| \le C\, \|P \mid \mathcal{L}(B^{s_1}_{2,q}(\mathbb{T}), L_2(\mathbb{T}))\|\, \|Q \mid \mathcal{L}(B^{s_2}_{2,q}(\mathbb{T}), L_2(\mathbb{T}))\| \tag{4}$$

with some constant C independent of P and Q.

Remark The Besov spaces of bivariate functions with dominating mixed smoothness properties are tensor products

$$B^{s_1}_{p,q}(\mathbb{T}) \otimes_\lambda B^{s_2}_{p,q}(\mathbb{T}) = S^{s_1,s_2}_{p,q}B(\mathbb{T}^2)$$

of univariate Besov spaces for general p, $1 \le p \le \infty$. The case $p = 2$ is the only one where for $s_1 = s_2 = 0$, this reduces with (2) to

$$L_2(\mathbb{T}) \otimes_\lambda L_2(\mathbb{T}) = L_2(\mathbb{T}^2).$$

In all other cases, $B^0_{p,q}(\mathbb{T})$ does not coincide with $L_p(\mathbb{T})$. As a consequence, to take advantage of the tensor product property one has to deal with error estimates in the norm $\|\cdot \mid B^0_{p,q}\|$ which is more complicated as in the L_p–setting, see [22]. Whether the estimate (4) extends to $p \ne 2$ seems to be open.
Because of the imbeddings $B^s_{2,q}(\mathbb{T}^n) \hookrightarrow B^s_{2,\infty}(\mathbb{T}^n)$ for $1 \le q < \infty$ we may restrict our error estimates in the following sections to the most interesting case $q = \infty$.

3 Interpolation on Equidistant Grids

This section is devoted to error estimates for periodic interpolation on equidistant grids. We can apply the concept of periodic Strang–Fix conditions on the fundamental interpolant in order to find such error estimates.
Let N be a natural number and denote by

$$J_N = \Big\{ k \in \mathbb{Z}^n\,;\; -\frac{N}{2} \le k_r < \frac{N}{2},\; r = 1, \dots, n \Big\}$$

a related set of indices. Further

$$T_N = \Big\{ \sum_{k \in J_N} \eta_k \, \mathrm{e}^{\mathrm{i}k\cdot}\,;\; \eta_k \in \mathbb{C} \Big\}$$

denotes a corresponding set of trigonometric polynomials. The discrete Fourier coefficients of a continuous function f are given by

$$c_k^N(f) = \frac{1}{N} \sum_{\ell \in J_N} f\Big(\frac{2\pi\ell}{N}\Big)\, \mathrm{e}^{2\pi \mathrm{i} k\ell/N}, \qquad k \in J_N.$$

Discrete Fourier coefficients and Fourier coefficients are connected by aliasing

$$c_k^N(f) = \sum_{\ell \in \mathbb{Z}^n} c_{k+\ell N}(f),$$

as long as $f \in A(\mathbb{T}^n)$. We consider interpolation on equidistant grids of type

$$\mathcal{T}_N = \Big\{ \frac{2\pi k}{N}\,;\; k \in J_N \Big\}.$$

The continuous and 2π–periodic function Λ_N is called a fundamental interpolant for $\mathcal{T}_N$ if

$$\Lambda_N\Big(\frac{2\pi k}{N}\Big) = \delta_{0,k}\,, \qquad k \in J_N.$$

The associated Lagrange interpolation operator L_N is defined as

$$L_N f = \sum_{k \in J_N} f\Big(\frac{2\pi k}{N}\Big)\, \Lambda_N\Big(\cdot - \frac{2\pi k}{N}\Big).$$

The Fourier coefficients of $L_N f$ can be easily computed

$$c_k(L_N f) = N^n\, c_k^N(f)\, c_k(\Lambda_N) = N^n\, c_k(\Lambda_N) \sum_{\ell \in \mathbb{Z}} c_{k+\ell N}(f)$$

for $f \in A(\mathbb{T}^n)$. Finally, we denote the N–th Fourier partial sum by

$$S_N f = \sum_{k \in J_N} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot}.$$

For cardinal interpolation, one can use the Strang–Fix conditions [20, 25] on the fundamental interpolant in order to characterize the reproduction of polynomials and therefore the order of interpolation, too. Up to now there is no complete periodic counterpart.
But we can use the concept of periodic Strang–Fix conditions introduced by Pöplau [2, 14] for L_2–error estimates. Here, the behaviour of the fundamental interpolant is characterized by a certain decay of the Fourier coefficients of Λ_N.

Definition 3 *Let $\Lambda_N \in A(\mathbb{T}^n)$ be a fundamental interpolant with respect to $\mathcal{T}_N$. Then Λ_N satisfies the periodic Strang–Fix conditions of order $m > 0$ if for all $k \in J_N$ the inequalities*

$$\begin{aligned} |1 - N^n c_k(\Lambda_N)| &\leq b_0\, |k|_2^m\, N^{-m}, \\ |N^n c_{k+\ell N}(\Lambda_N)| &\leq b_\ell\, |k|_2^m\, N^{-m}, \qquad \ell \in \mathbb{Z}^n \setminus \{0\}, \end{aligned}$$

hold for some sequence $\{b_\ell\}_{\ell \in \mathbb{Z}^n} \in \ell_2(\mathbb{Z}^n)$ of non–negative numbers.

The periodic Strang–Fix conditions can be seen as the periodic counterpart of the strong Strang–Fix conditions for cardinal interpolation [6].

Theorem 4 *Let the fundamental interpolant $\Lambda_N \in A(\mathbb{T}^n)$ satisfy the periodic Strang–Fix conditions of order $m > 0$. Let $n/2 < s < m$. Then there exists a constant C (independent of N) such that*

$$\|f - L_N f \mid L_2(\mathbb{T}^n)\| \leq C\, N^{-s}\, \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|$$

holds for all $f \in B^s_{2,\infty}(\mathbb{T}^n)$.

Proof: Step 1. We investigate the case $f \in T_N$ first. Some computations and the periodic Strang–Fix conditions yield

$$\begin{aligned} &\|f - L_N f \mid L_2(\mathbb{T}^n)\|^2 \\ &= \Big\| \sum_{k \in \mathbb{Z}^n} (c_k(f) - N^n c_k^N(f) c_k(\Lambda_N))\, \mathrm{e}^{\mathrm{i}k\cdot} \,\Big|\, L_2(\mathbb{T}^n) \Big\|^2 \\ &= \Big\| \sum_{k \in J_N} c_k(f)\, \mathrm{e}^{\mathrm{i}k\cdot} \Big((1 - N^n c_k(\Lambda_N)) - \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} N^n c_{k+\ell N}(\Lambda_N)\, \mathrm{e}^{\mathrm{i}\ell N\cdot} \Big) \,\Big|\, L_2(\mathbb{T}^n) \Big\|^2 \\ &= \sum_{k \in J_N} |c_k(f)|^2 \Big(|1 - N^n c_k(\Lambda_N)|^2 + \sum_{\ell \in \mathbb{Z}^n \setminus \{0\}} |N^n c_{k+\ell N}(\Lambda_N)|^2 \Big) \\ &\leq \sum_{k \in J_N} |c_k(f)|^2 |k|_2^{2m} N^{-2m} \sum_{\ell \in \mathbb{Z}^n} b_\ell^2 \end{aligned}$$

Let $2^r \leq N < 2^{r+1}$. Then

$$\begin{aligned} \sum_{k \in J_N} |k|_2^{2m} |c_k(f)|^2 &= \sum_{\ell=0}^{r} \sum_{k \in Q_\ell^n} |k|_2^{2m} 2^{-\ell s} 2^{\ell s} |c_k(f)|^2 \\ &\leq 2^{2m} n^m \sum_{\ell=0}^{r} 2^{2(m-s)\ell}\; 2^{2ls} \sum_{k \in Q_\ell^n} |c_k(f)|^2 \end{aligned}$$

We apply Hölder's inequality and obtain

$$\begin{aligned} \sum_{k \in J_N} |k|_2^{2m} |c_k(f)|^2 &\leq 2^{2m} n^m \Big(\sum_{\ell=0}^{r} 2^{2(m-s)\ell} \Big) \sup_{\ell=0,\ldots,r} 2^{2ls} \sum_{k \in Q_\ell^n} |c_k(f)|^2 \\ &\leq C_1 N^{2(m-s)} \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|^2. \end{aligned}$$

That means, for $f \in T_N$, we proved

$$\|f - L_N f \mid L_2(\mathbb{T}^n)\| \leq C_2\, N^{-s} \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|, \tag{5}$$

where C_2 does not depend on f.

STEP 2. We investigate the general case $f \in B^s_{2,\infty}(\mathbb{T}^n)$. Because of $s > n/2$ it holds that $B^s_{2,\infty}(\mathbb{T}^n) \hookrightarrow B^{n/2}_{2,1}(\mathbb{T}^n) \hookrightarrow A(\mathbb{T}^n)$. The interpolation is well–defined and aliasing is applicable. Using the periodic Strang–Fix conditions with $n/2 < s' < s$ and Cauchy–Schwarz inequality, it follows

$$\begin{aligned}
&\|L_N(f - S_N f) \mid L_2(\mathbb{T}^n)\|^2 \\
&\quad = \sum_{k\in\mathbb{Z}^n} \Big| N^n c_k(\Lambda_N) \sum_{\ell\in\mathbb{Z}^n} c_{k+\ell N}(f - S_N f)\Big|^2 \\
&\quad = \sum_{k\in J_N} \sum_{r\in\mathbb{Z}^n} \Big| N^n c_{k+rN}(\Lambda_N) \sum_{\ell\in\mathbb{Z}^n} c_{k+rN+\ell N}(f - S_N f)\Big|^2 \\
&\quad \le C_3 \sum_{k\in J_N} \sum_{r\in\mathbb{Z}^n} b_r^2\, |k|_2^{2s'}\, N^{-2s'} \Big|\sum_{\ell\in\mathbb{Z}^n} c_{k+\ell N}(f - S_N f)\Big|^2 \\
&\quad \le C_3 N^{-2s'} \|\{b_r\} \mid \ell_2(\mathbb{Z}^n)\|^2 \sum_{k\in J_N} |k|_2^{2s'} \sum_{\ell\in\mathbb{Z}^n} |k+\ell N|_2^{2s'} |c_{k+\ell N}(f - S_N f)|^2 \sum_{r\in\mathbb{Z}^n} |k+nN|_2^{-2s'}.
\end{aligned}$$

Next we use that for $s' > n/2$

$$\sup_{k\in J_N} |k|_2^{2s'} \sum_{r\in\mathbb{Z}^n} |k+rN|_2^{-2s'} = \sup_{k\in J_N} \Big|\frac{k}{N}\Big|_2^{2s'} \sum_{r\in\mathbb{Z}^n} \Big|\frac{k}{N} + r\Big|_2^{-2s'} = C_4 < \infty.$$

This proves

$$\begin{aligned}
&\|L_N(f - S_N f) \mid L_2(\mathbb{T}^n)\|^2 \\
&\quad \le C_3 C_4\, N^{-2s'} \|\{b_r\} \mid \ell_2(\mathbb{Z}^n)\|^2 \sum_{k\in J_N} \sum_{\ell\in\mathbb{Z}^n} |k+\ell N|^{2s'}\, |c_{k+\ell N}(f - S_N f)|^2 \\
&\quad \le C_5\, N^{-2s'}\, \|f - S_N f \mid H_2^{s'}(\mathbb{T}^n)\|^2,
\end{aligned}$$

where $H_2^{s'}(\mathbb{T}^n)$ denotes the fractional order Sobolev space with the norm

$$\|f \mid H_2^{s'}(\mathbb{T}^n)\|^2 := \sum_{k\in\mathbb{Z}^n} (1 + |k|_2^2)^{s'}\, |c_k(f)|^2.$$

In case $s > s'$ one knows

$$\|f - S_N f \mid H_2^{s'}(\mathbb{T}^n)\| \le C_6\, N^{s'-s}\, \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|, \tag{6}$$

cf. e.g. [11]. This yields

$$\|L_N(f - S_N f) \mid L_2(\mathbb{T}^n)\| \le C_7\, N^{-s}\, \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|, \tag{7}$$

where again the constant C_5 does not depend on N and f. To finish the proof observe that (5), (6) and (7) (applied with $s' = 0$) imply

$$\begin{aligned}
\|f - L_N f \mid L_2(\mathbb{T}^n)\| &\le \|f - S_N f \mid L_2(\mathbb{T}^n)\| + \|S_N f - L_N(S_N f) \mid L_2(\mathbb{T}^n)\| \\
&\qquad + \|L_N(f - S_N f) \mid L_2(\mathbb{T}^n)\| \\
&\le C\, N^{-s}\, \|f \mid B^s_{2,\infty}(\mathbb{T}^n)\|.
\end{aligned}$$

This proves the theorem. ■

Remark We note that the most constants appearing in the proof only depend on the dimension n and on the smoothness s of the function to be interpolated. The dependency on the used fundamental interpolant Λ_N is reflected in the constants by the term $\|\{b_r\} \mid \ell_2(\mathbb{Z}^n)\|$ from the Strang–Fix conditions.

Remark Recall, if X is a Banach space and W a subspace of X, then the linear N–width is defined as

$$\lambda_N(W, X) = \inf_{\substack{U_N \in Lin_N(X) \\ P \in \mathcal{L}(X, U_N)}} \sup_{f \in W} \|f - Pf \mid X\|,$$

where the infimum is taken over all subspaces U_N of X of finite dimension $\leq N$ and all linear operators P from X to U_N. Here we are interested in $X = L_2(\mathbb{T})$ and W the unit ball in the Nikol'skij–Besov space $B^s_{2,\infty}(\mathbb{T})$, denoted by $B^s_2(\mathbb{T})$. If $s > 0$, then

$$\lambda_N(B^s_2(\mathbb{T}), L_2(\mathbb{T})) \sim N^{-s}$$

cf. [9, Theorem 14.3.8]. In this sense, approximation of univariate functions with those interpolation operators L_N is nearly optimal (nearly optimal means the order of approximation is correct but may be not the constants). More details about widths may be found in [9, 26].

Remark An interesting limiting case has been observed by Pöplau [2, 14]. If Λ_N is a fundamental interpolant which satisfies the periodic Strang–Fix condition of order $m > n/2$, then there exists a constant C (independent of N) such that

$$\|f - L_N f \mid L_2(\mathbb{T}^n)\| \leq C\, N^{-m}\, \|f \mid B^m_{2,2}(\mathbb{T}^n)\|$$

holds for all $f \in B^m_{2,2}(\mathbb{T}^n)$. For a generalization in various directions, including different function spaces (defined by using decay properties of the Fourier coefficients), we refer to [23, 24].

Corollary 5 *Let the univariate fundamental interpolant $\Lambda_N \in A(\mathbb{T})$ satisfy the periodic Strang–Fix conditions of order $m > 0$. Let $L_N \otimes L_N$ be the interpolation operator associated with the bivariate fundamental interpolant $\Lambda_N \otimes \Lambda_N$. Let $1 < s < m$. Then there exists a constant C (independent of N) such that*

$$\|f - (L_N \otimes L_N) f \mid L_2(\mathbb{T}^2)\| \leq C\, N^{-s}\, \|f \mid B^s_{2,\infty}(\mathbb{T}^2)\|$$

holds for all $f \in B^s_{2,\infty}(\mathbb{T}^2)$.

Proof: Because $c_k(\Lambda_N \otimes \Lambda_N) = c_{k_1}(\Lambda_N) c_{k_2}(\Lambda_N)$ one proves easily that also the bivariate fundamental interpolant $\Lambda_N \otimes \Lambda_N$ satisfies periodic Stang–Fix conditions of order m. Then, Theorem 4 is applicable. ∎

Example: B–Splines

As an example, we may use the interpolation by the 2π–periodized centered B–Spline $\mathcal{M}_{N,r}$ of order $r \in \mathbb{N}$. Its Fourier coefficients are known as

$$c_k(\mathcal{M}_{N,r}) = \frac{1}{N} \left(\operatorname{sinc} \frac{\pi k}{N} \right)^r, \qquad k \in \mathbb{Z},$$

with sinc $t := \sin t/t$. The fundamental interpolant $\Lambda_{N,r}$ corresponding to the 2π–periodic centered B–spline of order r can be computed from

$$c_k(\Lambda_{N,r}) := \frac{c_k(\mathcal{M}_{N,r})}{N\, c_k^N(\mathcal{M}_{N,r})}, \qquad k \in \mathbb{Z}. \tag{8}$$

Then, $\Lambda_{N,r}$ satisfies the periodic Strang–Fix conditions of order r (cf. [14]) with the constants

$$b_0 = \begin{cases} \dfrac{1}{2^{r+1}} & \text{for } r \text{ odd,} \\ \dfrac{1}{2(2^r-1)} & \text{for } r \text{ even,} \end{cases}$$

and

$$b_\ell = \frac{1}{\pi^r(2|\ell|-1)^r} \begin{cases} 1 & \text{for } r = 1, \\ \dfrac{(r-1)!}{E_{(r-1)/2}} & \text{for } r > 1, \text{ odd,} \\ \dfrac{r!}{2^r(2^r-1)B_{r/2}} & \text{for } r \text{ even,} \end{cases}$$

for $\ell \neq 0$. Here, B_s and E_s ($s \in \mathbb{N}$) denote the corresponding Bernoulli and Euler numbers.

Example: Trigonometric Interpolation

Another example is the trigonometric interpolation. The de la Vallée Poussin means $\mathcal{V}_N^K$ ($N, K \in \mathbb{N}, N > K$) of the Dirichlet are given by

$$\mathcal{V}_N^K(x) := \frac{1}{4KN} \sum_{\ell=N-K}^{N+K-1} \left(\sum_{k=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}kx} \right).$$

They are fundamental interpolants for the grid $\mathcal{T}_{2N}$. So, we have a lot of different fundamental interpolants for the grids $\mathcal{T}_N$ (N even) belonging to different parameters K. We denote them by $\Lambda_{N,K} := \mathcal{V}_{N/2}^K$ for $N/2, K \in \mathbb{N}, N/2 > K$. Since the de la Vallée Poussin means are trigonometric polynomials they of course satisfy Strang–Fix conditions of arbitrary order. But the constants of the Strang–Fix conditions depend on the quotient of the parameters K and N. For $K = 1$, we obtain the best constants since $\Lambda_{N,1}$ is only a slight modification of the Dirichlet kernel whose Fourier coefficients are compared with the Fourier coefficients of the fundamental interpolant. For $K = N/2 - 1$, the corresponding de la Vallée Poussin mean is already very closed to the Fejér kernel and the constants are much bigger (for details we refer to [23, 24]).

Example: Radial Basis Functions

A nice n–variate example can be found in [15]. Let the n–variate radial basis function φ be given by its Fourier coefficients

$$N^n c_k(\varphi) := |k|_2^{-\alpha}, \qquad k \in \mathbb{Z}^n \setminus \{0\},$$

for a fixed $\alpha > d$. In case $\alpha \in 2\mathbb{N}$, we obtain the periodized version of the cardinal polyharmonic splines [10]. The associated fundamental interpolant can be constructed analogously to the spline case (8) from its Fourier coefficients

$$N^n c_k(\Lambda_{N,\varphi}) := \begin{cases} \dfrac{|k|_2^{-\alpha}}{\sum_{\ell \in \mathbb{Z}^n} |k + \ell N|_2^{-\alpha}} & \text{for } k \in \mathbb{Z}^n \setminus N\mathbb{Z}^n, \\ 1 & \text{for } k = 0, \\ 0 & \text{for } k \in N\mathbb{Z}^n \setminus \{0\}. \end{cases}$$

Because of $\alpha > d$ this fundamental interpolant $\Lambda_{N,\varphi}$ belongs to the Wiener algebra. Furthermore, it satisfies the periodic Strang–Fix conditions of order α with the constants

$$b_0 = 2^\alpha \sum_{r=1}^{n} \binom{n}{r} \frac{1}{r^{\alpha/2}} \binom{2\alpha - r}{\alpha - r}$$

and

$$b_\ell = 2^\alpha |v(\ell)|_2^{-\alpha}, \qquad \ell \in \mathbb{Z}^n \setminus \{0\}$$

where the vector v has the components $v_r(\ell) = \delta_{0,\ell_r}(2|\ell_r| - 1)$ for $r = 1, \ldots, n$.
In addition to these examples one can find more examples of bivariate functions in [14, 13] (3– and 4–direction box splines) satisfying periodic Strang–Fix conditions of certain order.

4 Interpolation on Sparse Grids

Now we want to define the interpolation operators for interpolation on sparse grids and give error estimates. The definition of the blending interpolation operator and its basic properties can be found e.g. in [1, 4]. This definition needs the notation of a chain of projectors.
The ordering relation $P \leq Q$ for projectors holds if $PQ = QP = P$. A family of projectors $\{P_j\}_{j=0}^{\infty}$ forms a chain if $P_j \leq P_{j+1}$, $j \in \mathbb{N}_0$. For two interpolation projectors L_K and L_N, the ordering $L_K \leq L_N$ holds if and only if the images Im $L_K \subset$ Im L_N as well as the grids $\mathcal{T}_K \subset \mathcal{T}_N$ are ordered.
Fix $d \in \mathbb{N}$. By the choice $N_j := d\, 2^j$, we immediately insure $\mathcal{T}_{N_j} \subset \mathcal{T}_{N_{j+1}}$. Furthermore, we assume

$$\text{Im } L_{N_j} \subset \text{Im } L_{N_{j+1}}. \tag{9}$$

This property has to be proved for every example by hand. Then, we have a chain

$$L_{N_0} \leq L_{N_1} \leq \cdots \leq L_{N_j} \leq L_{N_{j+1}} \leq \cdots \tag{10}$$

of interpolation operators.
Given a chain (10) of interpolation operators $L_{N_j}, j \in \mathbb{N}_0$, for univariate functions. For bivariate functions, we will consider the j–th order blending operator defined by the j–th order Boolean sum

$$B_j := \bigoplus_{r=0}^{j} L_{N_r} \otimes L_{N_{j-r}},$$

where $A \oplus B := A + B - AB$. The representation of B_j in terms of ordinary sums is known to be

$$B_j = \sum_{r=0}^{j} L_{N_r} \otimes L_{N_{j-r}} - \sum_{r=0}^{j-1} L_{N_r} \otimes L_{N_{j-r-1}}.$$

The Boolean sums also form a chain. They have the range

$$\text{Im } B_j = \sum_{r=0}^{j} \text{Im } L_{N_r} \otimes \text{Im } L_{N_{j-r}}$$

and the interpolate on the sparse grid

$$\mathcal{T}_j^{\mathrm{B}} := \bigcup_{r=0}^{j} \mathcal{T}_{N_r} \times \mathcal{T}_{N_{j-r}}.$$

The sparse grid $\mathcal{T}_j^{\mathrm{B}}$ has $d^2(j2^{j-1} + 2^j)$ nodes which is essentially less than the $d^2\, 2^{2j}$ nodes in the equidistant grid $\mathcal{T}_{N_j} \times \mathcal{T}_{N_j}$.

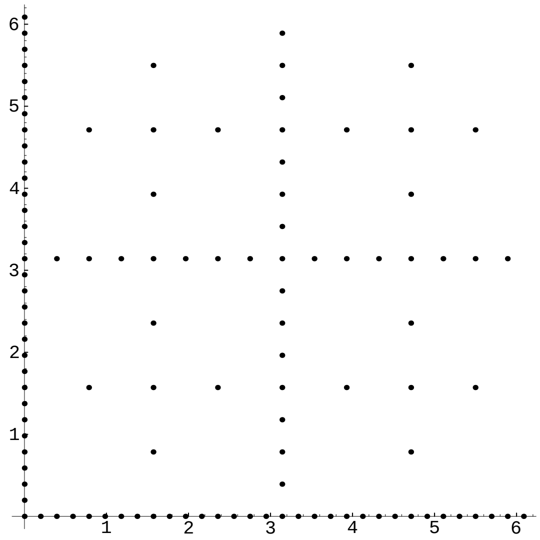

Figure 1: Sparse grid $\mathcal{T}_5^{\mathrm{B}}$ for $d = 1$.

Theorem 6 *Suppose that the interpolation operators L_{N_j}, $j \in \mathbb{N}_0$, form a chain* (10) *and satisfy*

$$\sup_{j \in \mathbb{N}_0} N_j^{s_k} \, \|f - L_{N_j} f \mid L_2(\mathbb{T})\| \leq C_k \, \|f \mid B^{s_k}_{2,\infty}(\mathbb{T})\|$$

with some constants C_k independent of f and for some fixed s_1, s_2 with $s_1, s_2 > 1/2$.
Then in case $s_1 = s_2 = s$, we find

$$\|f - B_j f \mid L_2(\mathbb{T}^2)\| \leq C \, (j+1) \, N_j^{-s} \, \|f \mid S^{s,s}_{2,\infty} B(\mathbb{T}^2)\|$$

for all $f \in S^{s,s}_{2,\infty} B(\mathbb{T}^2)$, whereas in case $s_1 \neq s_2$, it holds that

$$\|f - B_j f \mid L_2(\mathbb{T}^2)\| \leq C \, N_j^{-\min(s_1,s_2)} \, \|f \mid S^{s_1,s_2}_{2,\infty} B(\mathbb{T}^2)\|$$

for all $f \in S^{s_1,s_2}_{2,\infty} B(\mathbb{T}^2)$. In both situations C denotes a constant independent on j and f.

Proof: Because of $S^{s_1,s_2}_{2,\infty} B(\mathbb{T}^2) \hookrightarrow A(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ for $s_1, s_2 > 1/2$ interpolation is well–defined. The remainder ($P^c := I - P$) of the blending interpolation has the representation

$$B_j^c = L^c_{N_j} \otimes I + I \otimes L^c_{N_j} - \sum_{r=0}^{j} L^c_{N_r} \otimes L^c_{N_{j-r}} + \sum_{r=0}^{j-1} L^c_{N_r} \otimes L^c_{N_{j-r-1}},$$

cf. [4]. With this, the assertion follows from the triangle inequality, the uniformity of the norms (see (4)) and the assumption on the error for the univariate interpolation. ■

Corollary 7 *Let the 2π–periodic fundamental interpolants $\Lambda_{N_j} \in A(\mathbb{T})$ satisfy the periodic Strang–Fix conditions of order $m > 0$ with same sequence $\{b_\ell\}$ of constants. The corresponding interpolation operators L_{N_j}, $j \in \mathbb{N}_0$, form a chain* (10). *Let $1/2 < s_1, s_2 < m$. Then, in case $s_1 = s_2 = s$, we can estimate*

$$\|f - B_j f \mid L_2(\mathbb{T}^2)\| \leq C \, (j+1) \, N_j^{-s} \, \|f \mid S^{s,s}_{2,\infty} B(\mathbb{T}^2)\|,$$

for all $f \in S^{s,s}_{2,\infty} B(\mathbb{T}^2)$.
In case $s_1 \neq s_2$, it holds that

$$\|f - B_j f \mid L_2(\mathbb{T}^2)\| \leq C \, N_j^{-\min(s_1,s_2)} \, \|f \mid S^{s_1,s_2}_{2,\infty} B(\mathbb{T}^2)\|$$

for all $f \in S^{s_1,s_2}_{2,\infty} B(\mathbb{T}^2)$. In both situations C denotes a constant independent on j and f.

The same ideas as before yield the following estimate. It shows that the order of the interpolation error for equidistant grids does not improve for the smoother functions with dominating mixed smoothness properties in comparison to the isotropic case. For the functions with dominating mixed smoothness the error of interpolation on sparse grids is only by a logarithmic factor worse the result for equidistant grids.

Corollary 8 *Let the univariate fundamental interpolant $\Lambda_N \in A(\mathbb{T})$ satisfy the periodic Strang–Fix conditions of order $m > 0$. Let $L_N \otimes L_N$ be the interpolation operator associated with the bivariate fundamental interpolant $\Lambda_N \otimes \Lambda_N$. Let $1/2 < s_1, s_2 < m$. Then there exists a constant C (independent of N) such that*

$$\|f - (L_N \otimes L_N)f \mid L_2(\mathbb{T}^2)\| \leq C\, N^{-\min(s_1,s_2)}\, \|f \mid S^{s_1,s_2}_{2,\infty}B(\mathbb{T}^2)\|$$

holds for all $f \in S^{s_1,s_2}_{2,\infty}B(\mathbb{T}^2)$.

Example: B–Splines

The fundamental interpolants $\Lambda_{N_j,r}$ belonging to the 2π–periodic centered B–spline of even order $r \in \mathbb{N}$ satisfy (9) automatically since at the step from j to $j+1$ only some new spline knots are added. Therefore, the corresponding interpolation operators form a chain (10). The constants for Strang–Fix conditions given in the previous section do not depend on N_j. The fundamental interpolants $\Lambda_{N_j,r}$ belonging to the 2π–periodic centered B–spline of odd order $r \in \mathbb{N}$ do not satisfy (9). For splines for the grid $\mathcal{T}_{N_{j+1}}$ only totally new spline knots are used compared to the j–th grid.

Example: Trigonometric Interpolation

The de la Vallée Poussin means Λ_{N_j,K_j} satisfy the chain condition (9) only under certain restrictions on K_j and N_j. In [17], it was shown that for N_j as before and

$$K_j := \begin{cases} 2^{j-\kappa-1} & \text{for } j > \kappa, \\ 1 & \text{for } j \leq \kappa, \end{cases} \qquad \kappa \in \mathbb{N},\ 3 \leq d\, 2^{\kappa},$$

condition (9) is satisfied. The case $\kappa = \infty$ is allowed. With this choice of the parameters N_j and K_j, one can estimate the constants of the Strang–Fix conditions of order m uniformly by

$$b_\ell = \begin{cases} \dfrac{3^m}{2(2\pi)^m} & \text{for } \ell = -1, 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example: Radial Basis Functions

The fundamental interpolants $\Lambda_{N_j,\varphi}$ constructed from the radial basis function φ satisfy the periodic Strang–Fix conditions with constants not depending on N_j. Now we restrict ourselves to the univariate case. One can find constants a_k, $k = 0, \ldots, N_{j+1}-1$, such that

$$c_{k+\ell N_{j+1}}(\Lambda_{N_j,\varphi}) = a_k\, c_{k+\ell N_{j+1}}(\Lambda_{N_{j+1},\varphi}), \qquad \ell \in \mathbb{Z},\ k = 0, \ldots, N_{j+1}-1.$$

These constants are $a_0 = 1$, $a_{N_j} = 0$, and $a_k = 1/2\,(\sum_{\ell\in\mathbb{Z}} |k + 2\ell N_j|^{-\alpha})/(\sum_{\ell\in\mathbb{Z}} |k + \ell N_j|^{-\alpha})$, otherwise. This yields the chain property (9), cf. [12].

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