OPTIMAL SHADOWS AND IDEALS IN SUBMATRIX ORDERS

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We consider the poset of all submatrices of a given matrix, ordered by containment. The unique rank function for this poset is given by $r(M) = R(M) + C(M) - 1$, where $R(M)$ and $C(M)$ denote the number of rows and columns of a nonempty matrix M, respectively, and the rank of the empty matrix is 0. For fixed k and i our objective is to find a set $\mathcal{M}$ of submatrices $M_1, M_2, \ldots, M_k$ such that $r(M_j) = i$ for all j and the shadow of $\mathcal{M}$ is minimal, that is, the number of submatrices M with $r(M) = i - 1$ contained in a member of $\mathcal{M}$ should be smallest possible.

Partial results concerning the shadow minimization problem for our poset were obtained by Sali. In general, he conjectured a theorem of Kruskal–Katona type to hold. We show that this conjecture is true. In fact, our result covers a slightly more general case (the cartesian product of a Submatrix Order and a Boolean lattice).

The mentioned result can be formulated in terms of finite sets: Let A, B be two disjoint subsets of a finite set N. We consider all subsets $F \subseteq N$ satisfying $A \not\subseteq F$ and $B \not\subseteq F$. For fixed i, k we determine a family $\mathcal{F} = \{F_1, \ldots, F_k\}$ of such i–subsets of N which has minimum number of $(i-1)$–sets contained in some $F \in \mathcal{F}$.

Finally, as an application we give a solution to the problem of finding an ideal of given size and maximum weight in Submatrix Orders and in their duals.

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1. Introduction

In this paper we solve the *Shadow Minimization Problem* for the poset of submatrices of a matrix, i.e. we prove a theorem for this poset which is analogous to the classical Kruskal–Katona theorem [7, 8] for Boolean lattices. For all definitions not included in this article we refer to Engel's book [2].

1.1. **Submatrix Orders.** Let M be a matrix and R and C its sets of rows and columns, respectively. Clearly, every nonempty submatrix L of M can be considered as a pair (S,T), where $\emptyset \neq S \subseteq R$, $\emptyset \neq T \subseteq C$ and L consists of all cells lying in a row from S and a column from T. The empty matrix can be considered as the pair $(\emptyset,\emptyset)$. Now a submatrix $L_1=(S_1,T_1)$ is contained in another submatrix $L_2=(S_2,T_2)$ iff $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$.

Introducing the set $N := R \cup C$, the poset of all nonempty submatrices of M, ordered by containment, is isomorphic to the poset $SM(R,C)$ of all subsets F of N satisfying $F \cap R \neq \emptyset$ and $F \cap C \neq \emptyset$, ordered by the usual set inclusion. (To include the empty matrix we could give the empty set into $SM(R,C)$. In the sequel, we will consider it without the empty set since it is more convenient and our results can be extended to $SM(R,C)$ with the empty set in a trivial manner.) Obviously, $SM(R,C)$ is isomorphic to the cartesian product of $B'_{|R|}$ and $B'_{|C|}$, where B'_n denotes the Boolean lattice of order n without its minimal element. Furthermore, the unique rank function for $SM(R,C)$ is given by $r(F)=|F|-2$.

1.2. **Kruskal–Katona type theorems.** Sali [11, 12] proved for $SM(R,C)$ several analogues to classical theorems on extremal sets. In [11] he conjectured a *theorem of Kruskal–Katona type* for $SM(R,C)$. Let P be ranked poset with the associated partial order $\leq$, and denote its i–th *level* (the set of all elements of rank i) by $N_i(P)$. If there is no danger of ambiguity, we will simply write N_i. For an element $x \in P$ the *shadow* $\Delta(x)$ of x is the set of all $y \in N_{i-1}$ such that $y \leq x$. The shadow of a subset $X \subseteq N_i$ is defined to be the set $\Delta(X)=\bigcup_{x\in X}\Delta(x)$. Let further $\prec$ be a (total) linear order of the elements of P, and for $X \subseteq N_i$ denote the set of the smallest (with respect to $\prec$) $|X|$ elements of N_i by $C(X)$ and call it the *compression* of X (w.r.t. $\prec$). A subset $X \subseteq N_i$ is said to be *compressed* if $X = C(X)$. If the statement

$$\Delta(C(X)) \subseteq C(\Delta(X)) \text{ for all } i,\ X \subseteq N_i \tag{1.1}$$

is true, then it is called a theorem of Kruskal–Katona type for P.

The original Kruskal–Katona theorem says that the above inclusion holds for the *reverse–lexicographic order* (to be defined later) on Boolean lattices. Posets with the property that there is an order $\prec$ satisfying (1.1) often are called *Macaulay posets* since the first example (cartesian powers of an infinite chain) is due to Macaulay [10]. Other examples can be found in [2] for instance.

It is easy to observe (see [2] for instance) that (1.1) holds for some poset P and some total order $\prec$ of its elements iff the conditions

$$|\Delta(C(X))| \leq |\Delta(X)| \text{ for all } i,\ X \subseteq N_i \tag{1.2}$$

and

$$C(\Delta(C(X))) = \Delta(C(X)) \text{ for all } i,\ X \subseteq N_i \tag{1.3}$$

hold. These conditions are called *nestedness* resp. *continuity* of the solutions for the Shadow Minimization Problem.

It is also well–known (cf. [1, 2]) that (1.1) is true for P and $\prec$ iff the same condition is satisfied for the *dual* of P and the *reverse* of $\prec$. The dual P^* of P is the poset on the same set of elements with the partial order given by $x \leq^* y \iff y \leq x$, and the reverse of $\prec$ is defined by $x \prec^* y \iff y \prec x$.

1.3. **The poset $SM^*(N;A,B)$.** By this equivalence, instead of $SM(R,C)$ we can consider its dual $SM^*(R,C)$. For convenience, replace every element $L=(S,T)$ of $SM(R,C)$ by $L^*=(R\setminus S)\cup(C\setminus T)$ when turning to the dual. (As well we could say that a submatrix L of M is now represented by those rows and columns that have to be deleted from M to obtain L.) Now $SM^*(R,C)$ is the poset of all subsets F of $N=R\cup C$ satisfying $R\not\subseteq F$ and $C\not\subseteq F$. The partial order on the subsets of N is the usual set inclusion, and the rank of $F\in SM^*(R,C)$ equals its cardinality. The advantage of working with $SM^*(R,C)$ rather than with $SM(R,C)$ is that for $F\in N_i(SM^*(R,C))$ *all* $(i-1)$–subsets of F are elements of $N_{i-1}(SM^*(R,C))$. Therefore, $\Delta(F)$ is the set of all $(i-1)$–subsets of F, independent of the concrete choice of N,R,C.

In fact, we will study a slightly more general class of posets. Namely, we will not demand that the "forbidden" subsets form a partition of N. So let $SM^*(N;A,B)$ be the poset of all subsets F of a finite set N such that $A\not\subseteq F$ and $B\not\subseteq F$, where A and B are nonempty subsets of N with $A\cap B=\emptyset$. Clearly, $SM^*(N;A,B)$ is isomorphic to the cartesian product of a Boolean lattice and two Boolean lattices without their maximal elements. (You could also think of it in terms of matrices: Given a matrix Q with $m\geq|A|$ rows and $|N|-m\geq|B|$ columns and a $(|A|\times|B|)$–submatrix M, $SM^*(N;A,B)$ is isomorphic to the dual of the poset of all submatrices of Q which share at least one cell with M.)

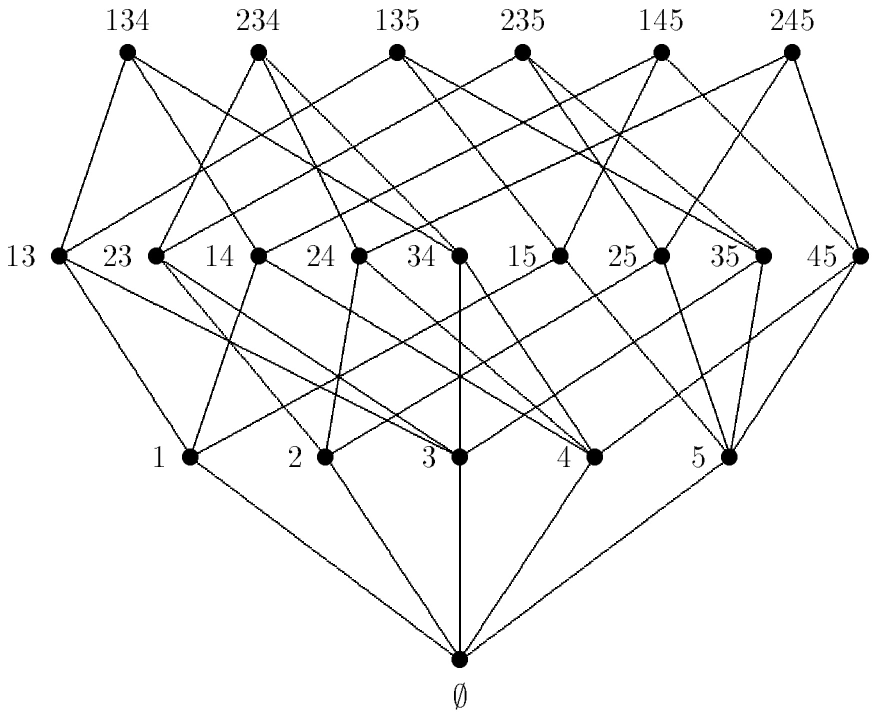

FIGURE 1. $SM^*(\{1,2,3,4,5\};\{1,2\},\{3,4,5\})$

2. THE TOTAL ORDER OF THE ELEMENTS OF $SM^*(N;A,B)$

We will now introduce a linear order $\prec$ of the elements of $SM^*(N;A,B)$. In section 4 we will show that (1.1) is satisfied for $\prec$.

Throughout we assume $|N|=n$ and $A=\{a_1,a_2,\dots,a_s\}$, $B=\{b_1,b_2,\dots,b_t\}$ such that

$$1\leq s\leq t,\ a_1<a_2<\cdots<a_s<b_1<b_2<\cdots<b_t$$

and

$$e<a_1 \text{ for all } e\in N\setminus(A\cup B).$$

Furthermore, throughout i is an integer with $0\leq i\leq n-2$. For brevity we put $\mathcal{S}:=SM^*(N;A,B)$.

2.1. **Definition of the order.** Our definition of $\prec$ involves the reverse–lexicographic order which for $F, G \in \mathcal{S}$ is given by

$$F \prec_{rl} G \iff \max[(F \cup G) \setminus (F \cap G)] \in G.$$

For $F \in \mathcal{S}$ let

$$a(F) := \max\{h : a_h \notin F\}.$$

The following two conditions for $F, G \in \mathcal{S}$ establish the order $\prec$ on $\mathcal{S}$:

$$F \prec G \quad \text{if} \quad a(F) > a(G), \tag{2.1}$$

$$F \prec G \quad \text{if} \quad a(F) = a(G) \text{ and } F \prec_{rl} G. \tag{2.2}$$

2.2. **Properties of $\prec$.** Condition (2.1) provides a partition of $\mathcal{S}$ into *blocks*, where for $h = 1, 2, \ldots, s$ the block $\mathcal{B}(a_h)$ is defined by

$$\mathcal{B}(a_h) := \{F \in \mathcal{S} : a(F) = h\}.$$

We divide every block into *subblocks*. For $F \in \mathcal{S}$ let

$$b(F) := \max\{j : b_j \notin F\},$$

and for $h = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$ the subblock $\mathcal{B}(a_h, b_j)$ is defined by

$$\mathcal{B}(a_h, b_j) := \{F \in \mathcal{B}(a_h) : b(F) = j\}.$$

For $\mathcal{F}, \mathcal{G} \subseteq \mathcal{S}$ let us write $\mathcal{F} \prec \mathcal{G}$ if the greatest element of $\mathcal{F}$ (with respect to $\prec$) is smaller than the smallest element of $\mathcal{G}$.

In the next proposition we list some facts that follow immediately from the above definitions.

Proposition 2.1. *The order $\prec$, the blocks and subblocks, respectively, have the following properties.*

(a) $\mathcal{B}(a_s) \prec \mathcal{B}(a_{s-1}) \prec \cdots \prec \mathcal{B}(a_1)$,

(b) $\mathcal{B}(a, b_t) \prec \mathcal{B}(a, b_{t-1}) \prec \cdots \prec \mathcal{B}(a, b_1)$ *for all* $a \in A$,

(c) *The block* $\mathcal{B}(a_h)$ *is isomorphic to the poset obtained from the Boolean lattice* $B_{n-s+h-1}$ *by deleting its reverse–lexicographically greatest* $2^{n-s+h-1-t}$ *elements (which form a Boolean lattice of order* $n - s + h - 1 - t$*), and the order* $\prec$ *on* $\mathcal{B}(a_h)$ *is isomorphic to the reverse–lexicographic order on this poset.*

(d) *The subblock* $\mathcal{B}(a_h, b_j)$ *is isomorphic to the Boolean lattice* $B_{n-s-t+h+j-2}$*, and the order* $\prec$ *on* $\mathcal{B}(a_h, b_j)$ *is isomorphic to the reverse–lexicographic order on* $B_{n-s-t+h+j-2}$.

(e) *Let* $J \in \{A, B, N \setminus A, N \setminus B\}$ *and* $F, G \in \mathcal{S}$ *such that* $F \cap J = G \cap J$*. We have* $F \prec G$ *iff* $F \prec_{rl} G$.

(f) *If* $s \geq 2$*, then* $\mathcal{S} \setminus \mathcal{B}(a_1) = SM(N; A \setminus \{a_1\}, B)$ *holds. Moreover, the order* $\prec$ *restricted to* $\mathcal{S} \setminus \mathcal{B}(a_1)$ *is the order* $\prec$ *on* $SM(N; A \setminus \{a_1\}, B)$.

(g) *If* $s \geq 2$*, then* $\mathcal{S} \setminus \mathcal{B}(a_s) = \{F \cup \{a_s\} : F \in SM(N \setminus \{a_s\}; A \setminus \{a_s\}, B)\}$ *holds, where the order* $\prec$ *restricted to* $\mathcal{S} \setminus \mathcal{B}(a_s)$ *is isomorphic to the order* $\prec$ *on* $SM(N \setminus \{a_s\}; A \setminus \{a_s\}, B)$. ∎

2.3. **Properties of the new–shadow.** We say that a subset $\mathcal{G} \subseteq N_i(\mathcal{S})$ is a *segment* (of $N_i(\mathcal{S})$) if $\mathcal{G}$ consists of consecutive elements of $N_i(\mathcal{S})$ w.r.t. $\prec$. A segment $\mathcal{G}$ is called *initial* (resp. *final*) if it contains the smallest (resp. greatest) element of $N_i(\mathcal{S})$ in the order $\prec$. We extend this definition saying that some segment $\mathcal{G}_1$ is an initial segment of another segment $\mathcal{G}_2$ if $\mathcal{G}_1 \subseteq \mathcal{G}_2$ and the smallest elements of $\mathcal{G}_1$, $\mathcal{G}_2$ w.r.t. $\prec$ are equal.

Let $F \in N_i(\mathcal{S})$, and let further $\mathcal{F} \subseteq N_i(\mathcal{S})$ be the initial segment whose greatest element w.r.t. $\prec$ is F. The *new–shadow* of F is defined by

$$\Delta_{new}(F) := \Delta(\mathcal{F}) \setminus \Delta(\mathcal{F} \setminus \{F\}).$$

The new–shadow of a subset $\mathcal{G} \in N_i(\mathcal{S})$ is the family

$$\Delta_{new}(\mathcal{G}) := \bigcup_{F \in \mathcal{G}} \Delta_{new}(F).$$

We easily derive the following properties of the new–shadow operator.

Proposition 2.2. *Let $a \in A$, $b \in B$ and $i \geq 1$.*

(a) *If $\mathcal{I}$ is an initial segment of $N_i(\mathcal{B}(a,b))$, then $\Delta_{new}(\mathcal{I})$ is an initial segment of $N_{i-1}(\mathcal{B}(a,b))$.*

(b) *$\Delta_{new}(N_i(\mathcal{B}(a,b))) = N_{i-1}(\mathcal{B}(a,b))$ holds.*

Proof. The validity of (b) follows immediately from Proposition 2.1(a),(b),(d), and (a) is implied by (b) together with Proposition 2.1(a),(b),(d). ∎

The poset $\mathcal{S}$ is called *additive* (w.r.t. $\prec$) if for all $i \in \{1, \ldots, n-2\}$ and all initial segments $\mathcal{I} \subseteq N_i(\mathcal{S})$, segments $\mathcal{T} \subseteq N_i(\mathcal{S})$ and final segments $\mathcal{F} \subseteq N_i(\mathcal{S})$ with $|\mathcal{I}| = |\mathcal{T}| = |\mathcal{F}|$ the following inequality is satisfied:

$$|\Delta(\mathcal{I})| \geq |\Delta_{new}(\mathcal{T})| \geq |\Delta_{new}(\mathcal{F})|.$$

The next lemma yields a corollary which will be needed in the proof of our main theorem.

Lemma 2.3. *If $s = 1$ holds, then $\mathcal{S}$ is additive.*

Proof. We proceed by induction on t. If $t = 1$, then $\mathcal{S}$ is a Boolean lattice of order $n-2$, and $\prec$ is the reverse–lexicographic order. It is well–known that Boolean lattices are additive w.r.t. $\prec_{rl}$ (see [2] for instance).

Let $t > 1$ and let the assertion be true for $t' < t$. Furthermore, let $i, \mathcal{I}, \mathcal{T}, \mathcal{F}$ be like in the above definition. First note that $|\Delta(\mathcal{I})| \geq |\Delta_{new}(\mathcal{T})|$ follows immediately by $s = 1$, Proposition 2.1(c) and the additivity of Boolean lattices.

It remains to show $|\Delta_{new}(\mathcal{T})| \geq |\Delta_{new}(\mathcal{F})|$. We proceed by induction on $m := |\mathcal{T}|$. If $m = 1$, then the assertion is easily verified. Let $m > 1$ and let the assertion be true for $m' < m$. By the induction hypothesis (induction on m), without loss of generality we can assume that $\mathcal{T} \cap \mathcal{F} = \emptyset$.

If $\mathcal{T} \cap \mathcal{B}(a_1, b_1) \neq \emptyset$, then let $\mathcal{T}' = \mathcal{T} \cap \mathcal{B}(a_1, b_1)$ and $\mathcal{T}'' = \mathcal{T} \setminus \mathcal{B}(a_1, b_1)$. Furthermore, let let $\mathcal{F}'$ and $\mathcal{F}''$ be sets of the smallest $|\mathcal{T}'|$ and the greatest $|\mathcal{T}''|$ elements of $\mathcal{F}$ w.r.t. $\prec$, respectively. Now $|\Delta_{new}(\mathcal{T}')| \geq |\Delta_{new}(\mathcal{F}')|$ follows from the fact that $\mathcal{B}(a_1, b_1)$ is isomorphic to a Boolean lattice and $|\Delta_{new}(\mathcal{T}'')| \geq |\Delta_{new}(\mathcal{F}'')|$ is implied by the induction hypothesis (induction on m).

If $\mathcal{T} \cap \mathcal{B}(a_1, b_1) = \emptyset$, then partition $\mathcal{F}$ into $\mathcal{F}_1 = \mathcal{F} \cap \mathcal{B}(a_1, b_1)$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{B}(a_1, b_1)$. Furthermore, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be sets of the greatest $|\mathcal{F}_1|$ and the smallest $|\mathcal{F}_2|$ elements of $\mathcal{T}$ w.r.t. $\prec$, respectively. Now $|\Delta_{new}(\mathcal{T}_2)| \geq |\Delta_{new}(\mathcal{F}_2)|$ is implied by $\mathcal{S} \setminus \mathcal{B}(a_1, b_1) = SM(N; \{a_1\}, B \setminus \{b_1\})$ and the induction hypothesis (induction on t). Finally, define $\mathcal{F}_3 := \{(F \setminus \{b_2\}) \cup \{b_1\} : F \in \mathcal{F}_1\}$. It is easily observed that $\mathcal{F}_3$ is a final segment of $SM(N; \{a_1\}, B \setminus \{b_1\})$ and that $|\Delta_{new}(\mathcal{F}_3)| = |\Delta_{new}(\mathcal{F})|$. Consequently, the induction hypothesis (induction on t) yields $|\Delta_{new}(\mathcal{T}_1)| \geq |\Delta_{new}(\mathcal{F}_1)|$. ∎

Corollary 2.4. *Let $s \geq 2$ and $i \geq t-1$.*

(a) *Let $\mathcal{G}$ be a family of w.r.t. $\prec$ consecutive elements of $\bigcup_{h=2}^{s} N_i(\mathcal{B}(a_h, b_1))$. Furthermore, let $\mathcal{H}$ be an initial segment of $N_i(\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1))$ such that $|\mathcal{G}| = |\mathcal{H}|$. The inequality $|\Delta_{new}(\mathcal{G})| \leq |\Delta_{new}(\mathcal{H})|$ holds.*

(b) *Let $\mathcal{G}$ be a family of w.r.t. $\prec$ consecutive elements of $\bigcup_{h=2}^{s} N_i(\mathcal{B}(a_h, b_1))$ which contains the w.r.t. $\prec$ greatest element of $\bigcup_{h=2}^{s} N_i(\mathcal{B}(a_h, b_1))$. Furthermore, let $\mathcal{H}$ be a segment of $N_i(\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1))$ such that $|\mathcal{G}| = |\mathcal{H}|$. The inequality $|\Delta_{new}(\mathcal{G})| \leq |\Delta_{new}(\mathcal{H})|$ holds.*

Proof. By Proposition 2.1(d), the subblocks $\mathcal{B}(a_h, b_1)$ and $\mathcal{B}(a_1, b_h)$ are isomorphic for $h = 2, \ldots, s$, as well as the orders of the elements. Consequently, in case (a) the family $\mathcal{G}$ is isomorphic to a segment of $N_i(\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1))$ and in case (b) to a final segment of $N_i(\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1))$.

Let $\mathcal{S}' = SM(N \setminus (A \setminus \{a_1\}); \{a_1\}, B \setminus \{b_1\})$. Trivially, $\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1) = \{F \cup (A \setminus \{a_1\} : F \in \mathcal{S}'\}$ holds, where the order $\prec$ on $\mathcal{B}(a_1) \setminus \mathcal{B}(a_1, b_1)$ is the reverse–lexicographic one as well as the order $\prec$ on $\mathcal{S}'$.

By Lemma 2.3, $\mathcal{S}'$ is additive. Together with Proposition 2.2 this impies the claim. ∎

3. Compression techniques and stable families

In the following we will introduce three types of operations, *partial compression*, *partial shifting* and a *move of segments*. These operations transform a subset $\mathcal{F} \subseteq N_i(\mathcal{S})$ into another subset of $N_i(\mathcal{S})$ preserving the cardinality. It is known that partial compression does not increase the shadow. Furthermore, we will prove that partial shifting does not increase the shadow of a subset which is stable under application of partial compression. Moreover, we will show that the move of segments does not increase the shadow of a subset which is stable with respect to partial compression and partial shifting. Finally, we will investigate the structure of families which are stable under application of all three operations.

3.1. **Partial compression.** Applying compression in lower dimensions is an efficient standard method for proving Kruskal–Katona type theorems for cartesian products of Macaulay posets. Here we will make use of the fact that $\mathcal{S}$ is isomorphic to the cartesian product $B_{n-s-t} \times (B'_s)^* \times (B'_t)^*$.

Let $J \subseteq N$ and let $\mathcal{F}$ be a family of i–subsets of N (not necessarily a subset of $\mathcal{S}$). For $I \subseteq N \setminus J$ we introduce the subfamily

$$\mathcal{F}_I := \{F \in \mathcal{F} : F \cap (N \setminus J) = I\},$$

and let $C_J(\mathcal{F}_I)$ be the family of the $|\mathcal{F}_I|$ reverse–lexicographically smallest i–subsets of N whose intersection with $N \setminus J$ is I. Eventually, let

$$C_J(\mathcal{F}) := \bigcup_{I \subseteq N \setminus J} C_J(\mathcal{F}_I).$$

Using the fact that, by the Kruskal–Katona theorem, (1.1) is satisfied for the poset of all subsets of N (ordered by inclusion) and the reverse–lexicographic order, it is easy to verify that the inclusion

$$\Delta(C_J(\mathcal{F})) \subseteq C_J(\Delta(\mathcal{F})) \tag{3.1}$$

holds (see [9] for an explicite proof).

Since we want to apply partial compression to transform a subset $\mathcal{F} \subseteq N_i(\mathcal{S})$ into another subset of $N_i(\mathcal{S})$ we have to be a little careful with choosing J. The next lemma provides some possible choices of J.

Lemma 3.1. *If $\mathcal{F} \subseteq N_i(\mathcal{S})$ and $J \in \{A, B, N \setminus A, N \setminus B\}$, then $C_J(\mathcal{F}) \subseteq N_i(\mathcal{S})$ and $|\Delta(C_J(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ hold.*

Proof. The inequality $|\Delta(C_J(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ follows immediately from (3.1). Furthermore, for $J \in \{A, B\}$ we have $C_J(\mathcal{F}) \subseteq N_i(\mathcal{S})$ by the definition of the operator C_J.

Let $J \in \{N \setminus A, N \setminus B\}$ and $I \subseteq N \setminus J$. In this case, $C_J(\mathcal{F}_I) \subseteq N_i(\mathcal{S})$ follows from the definition of $C_J(\mathcal{F}_I)$ and the fact that, by the definition of $\prec_{rl}$, for any $F, G \subseteq J$ with $N \setminus J \not\subseteq F$ and $N \setminus J \subseteq G$ we have $I \cup F \prec_{rl} I \cup G$. ∎

Note that partial compression does not give us a possibility to replace in $F \in \mathcal{F}$ elements of B by elements of A. That is why we also need partial shifting.

3.2. **Partial shifting.** Shifting techniques are an important tool in the theory of extremal sets and have many applications (cf. [2, 4, 5, 9]). Here we will give a definition of the shifting operator s_{xy} that was introduced for Boolean lattices in the classical paper [3]. Then we will modify it to make it applicable to $SM(N;A,B)$. Our modification is reasonable in the sense that we still can use it for solving the Shadow Minimization Problem (see Lemma 3.2).

Let $\mathcal{F}$ be a family of i–subsets of N (not necessarily a subset of $N_i(\mathcal{S})$). For $\{x,y\} \subseteq N$ and $F \in \mathcal{F}$ define

$$s_{xy}(F) := \begin{cases} (F \setminus \{y\}) \cup \{x\}, & \text{if } y \in F,\ x \notin F,\ (F \setminus \{y\}) \cup \{x\} \notin \mathcal{F}, \\ F, & \text{else} \end{cases}$$

and

$$s_{xy}(\mathcal{F}) := \{s_{xy}(F) : F \in \mathcal{F}\}.$$

Katona [6] observed that

$$|\Delta(s_{xy}(\mathcal{F}))| \leq |\Delta(\mathcal{F})| \tag{3.2}$$

holds for all $\mathcal{F}, x, y$ like above. One even has the inclusion $\Delta(s_{xy}(\mathcal{F})) \subseteq s_{xy}(\Delta(\mathcal{F}))$ (see [4] for instance).

Now let $\mathcal{F} \subseteq N_i(\mathcal{S})$. If $a \in A$ and $b \in B$, then $s_{ab}(\mathcal{F})$ is not necessarily a subset of $\mathcal{S}$. Therefore, we modify the operation defining for $a \in A$, $b \in B$ and $F \in \mathcal{F} \subseteq N_i(\mathcal{S})$

$$s'_{ab}(\mathcal{F}) := s_{ab}(\mathcal{F} \setminus \mathcal{B}(a)) \cup (\mathcal{F} \cap \mathcal{B}(a)).$$

By this definition, $s'_{ab}(\mathcal{F})$ is a subset of $N_i(\mathcal{S})$ such that $|s'_{ab}(\mathcal{F})| = |\mathcal{F}|$. Moreover, we have the following analogue to (3.2) for partially compressed families.

Lemma 3.2. *If $a \in A$, $b \in B$ and $\mathcal{F} = C_A(\mathcal{F}) \subseteq N_i(\mathcal{S})$, then $|\Delta(s'_{ab}(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ holds.*

Proof. For convenience, let $\mathcal{F}_1 := \mathcal{F} \setminus \mathcal{B}(a)$ and $\mathcal{F}_2 := \mathcal{F} \cap \mathcal{B}(a)$. We have

$$|\Delta(\mathcal{F})| = |\Delta(\mathcal{F}_1)| + |\Delta(\mathcal{F}_2)| - |\Delta(\mathcal{F}_1) \cap \Delta(\mathcal{F}_2)|$$

and

$$|\Delta(s'_{ab}(\mathcal{F}))| = |\Delta(s_{ab}(\mathcal{F}_1))| + |\Delta(\mathcal{F}_2)| - |\Delta(s_{ab}(\mathcal{F}_1)) \cap \Delta(\mathcal{F}_2)|.$$

By (3.2), it suffices to show that

$$\Delta(\mathcal{F}_1) \cap \Delta(\mathcal{F}_2) \subseteq \Delta(s_{ab}(\mathcal{F}_1)) \cap \Delta(\mathcal{F}_2),$$

i.e. for arbitrary $G \in \Delta(\mathcal{F}_1) \cap \Delta(\mathcal{F}_2)$ we have to show that $G \in \Delta(s_{ab}(\mathcal{F}_1))$. This follows immediately if there is a set F_1 such that $a \in F_1 \in \mathcal{F}_1$ and $G \subset F_1$. We show that such an F_1 can always be found.

Let $e \in F_2 \in \mathcal{F}_2$ such that $G = F_2 \setminus \{e\}$. The definition of $\mathcal{F}_2$ implies $a \notin F_2$ and $A' := \{a' \in A : a' > a\} \subseteq F_2$.

Case 1. If $e \notin A'$, then there is a set F_1 such that $A' \subseteq G \subset F_1 \in \mathcal{F}_1$, and $F_1 \notin \mathcal{F}_2$ yields $a \in F_1$.

Case 2. If $e \in A'$, then $F_2 \in \mathcal{F}$ and $C_A(\mathcal{F}) = \mathcal{F}$ imply $F_1 := (F_2 \setminus \{e\}) \cup \{a\} = G \cup \{a\} \in \mathcal{F}_1$. ∎

The next lemma is a statement about the structure of families which are stable under application of partial compression and partial shifting. Obviously, it would not be true if we would just demand that the families are partially compressed.

Lemma 3.3. *Let $\mathcal{F} \subseteq N_i(\mathcal{S})$ such that $C_J(\mathcal{F}) = s'_{ab}(\mathcal{F}) = \mathcal{F}$ for all $J \in \{A, B, N \setminus A, N \setminus B\}$ and $a \in A$, $b \in B$. If $F = \{f_1, \ldots, f_i\} \in \mathcal{F} \cap \mathcal{B}(a_s)$, $G \in N_i(\mathcal{B}(a_s)) \setminus \mathcal{F}$ such that $g_j \leq f_j$ for $j = 1, \ldots, i$, then $G \in \mathcal{F}$.*

Proof. Let $F \in \mathcal{F} \cap N_i(\mathcal{B}(a_s))$ and $e_1, e_2 \in N \setminus \{a_s\}$ such that $e_1 \notin F$, $e_2 \in F$ and $e_1 < e_2$. If $\{e_1, e_2\} \subseteq J \in \{A, B, N \setminus A, N \setminus B\}$, then $C_J(\mathcal{F}) = \mathcal{F}$ implies $(F \setminus \{e_2\}) \cup \{e_1\} \in \mathcal{F}$. Otherwise $e_1 \in A \setminus \{a_s\}$ and $e_2 \in B$ must hold, and $(F \setminus \{e_2\}) \cup \{e_1\} \in \mathcal{F}$ is implied by $s'_{e_1 e_2}(\mathcal{F}) = \mathcal{F}$.

Iterating this argument, we obtain that $F = \{f_1, \dots, f_i\} \in \mathcal{F} \cap N_i(\mathcal{B}(a_s))$ implies $G \in \mathcal{F}$ for all $G = \{g_1, \dots, g_i\} \in N_i(\mathcal{B}(a_s))$ with $g_j \leq f_j$ for $j = 1, \dots, i$. ∎

Corollary 3.4. *Let $i \geq t-1$, and let $\mathcal{F} \subseteq N_i(\mathcal{S})$ such that $C_J(\mathcal{F}) = s'_{ab}(\mathcal{F}) = \mathcal{F}$ for all $J \in \{A, B, N \setminus A, N \setminus B\}$ and $a \in A$, $b \in B$. If $\mathcal{F}$ contains the greatest element of $N_i(\mathcal{B}(a_s))$ w.r.t. $\prec$, then $N_i(\mathcal{B}(a_s)) \subseteq \mathcal{F}$ holds.*

Proof. Since $i \geq t-1$, the greatest element of $N_i(\mathcal{B}(a_s))$ is $E = (B \setminus \{b_1\}) \cup D$, where D is the set of the greatest $i-t+1$ elements of $(N \setminus B) \setminus \{a_s\}$. By definition, $N_i(\mathcal{B}(a_s))$ is the family of all i–subsets of $N \setminus \{a_s\}$ which do not contain B. Therefore, the assertion follows by Lemma 3.3. ∎

3.3. **Move of segments.** We will define now the operation *move of segments* that transforms a family $\mathcal{F} \subseteq N_i(\mathcal{S})$ into a family $M(\mathcal{F}) \subseteq N_i(\mathcal{S})$ preserving cardinality, where $M(\mathcal{F})$ is obtained from $\mathcal{F}$ by replacing a subset $\mathcal{H} \subseteq \mathcal{F}$ by a segment $\mathcal{G} \subseteq N_i(\mathcal{S})$ satisfying $\mathcal{G} \cap \mathcal{F} = \emptyset$ and $\mathcal{G} \prec \mathcal{H}$.

Let $\mathcal{F} \subseteq N_i(\mathcal{S})$ such that $N_i(\mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$ or $N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$. In this case simply define

$$M(\mathcal{F}) := \mathcal{F}.$$

Let now $\mathcal{F} \subseteq N_i(\mathcal{S})$ such that $N_i(\mathcal{B}(a_s, b_1)) \not\subseteq \mathcal{F}$ and $N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \not\subseteq \mathcal{F}$. Put $\mathcal{H}' := N_i(\mathcal{B}(a_s, b_1)) \cap \mathcal{F}$. Furthermore, let $\mathcal{G}' \subseteq N_i(\mathcal{B}(a_s, b_1)) \setminus \mathcal{F}$ be the segment such that $F \in \mathcal{F}$ for all $F \in N_i(\mathcal{S})$ which are smaller than the smallest element of $\mathcal{G}'$ w.r.t. $\prec$ and such that the element of $N_i(\mathcal{S})$ succeeding the greatest element of $\mathcal{G}'$ in the order $\prec$ is in $\mathcal{F} \cup N_i(\mathcal{B}(a_s, b_1))$. If $|\mathcal{G}'| \leq |\mathcal{H}'|$, then put $\mathcal{G} := \mathcal{G}'$ and let $\mathcal{H}$ be the set of the greatest $|\mathcal{G}|$ elements of $\mathcal{H}'$ w.r.t. $\prec$. If $|\mathcal{G}'| > |\mathcal{H}'|$, then put $\mathcal{H} := \mathcal{H}'$ and let $\mathcal{G}$ be the set of the smallest $|\mathcal{H}|$ elements of $\mathcal{G}'$ w.r.t. $\prec$. Finally, define

$$M(\mathcal{F}) := (\mathcal{F} \setminus \mathcal{H}) \cup \mathcal{G}.$$

Lemma 3.5. *If $\mathcal{F} \subseteq N_i(\mathcal{S})$ such that $C_J(\mathcal{F}) = s'_{ab}(\mathcal{F}) = \mathcal{F}$ for all $J \in \{A, B, N \setminus A, N \setminus B\}$ and $a \in A$, $b \in B$, then $|\Delta(M(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$ holds.*

Proof. Let $\mathcal{F} \subseteq N_i(\mathcal{S})$ like in the lemma. First note that without loss of generality we can assume that $i \geq t-1$ since otherwise $\mathcal{B}(a_s, b_1) = \emptyset$. Clearly, the only interesting case is when $M(\mathcal{F}) \neq \mathcal{F}$. So let $N_i(\mathcal{B}(a_s, b_1)) \not\subseteq \mathcal{F}$, $N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \not\subseteq \mathcal{F}$, and let $\mathcal{G}', \mathcal{H}'$ and $\mathcal{G}, \mathcal{H}$ be defined like above.

By Corollary 3.4, the greatest element of $N_i(\mathcal{B}(a_s, b_1))$ w.r.t. $\prec$ cannot be in $\mathcal{F}$. Consequently, $C_{N \setminus B}(\mathcal{F}) = \mathcal{F}$ yields that $\mathcal{F} \cap \mathcal{B}(a_s, b_1)$ is an initial segment of $N_i(\mathcal{B}(a_s, b_1))$ and that $\mathcal{F} \cap \mathcal{B}(a_h, b_1) = \emptyset$ for $h = 2, \dots, s$. Therefore, the only members of $\mathcal{F}$ containing $B \setminus \{b_1\}$ are the elements of $\mathcal{H}'$. Now Proposition 2.2 implies

$$|\Delta(\mathcal{F})| = |\Delta(\mathcal{F} \setminus \mathcal{H})| + |\Delta_{new}(\mathcal{H})|,$$

and it suffices to show

$$|\Delta_{new}(\mathcal{G})| \leq |\Delta_{new}(\mathcal{H})|. \tag{3.3}$$

Next we partition $N_i(\mathcal{B}(a_s))$ into consecutive segments. For $F \in N_i(\mathcal{B}(a_s))$ let $L(F) \subseteq F$ be the set of the greatest $t-1$ elements of F, and for $L \in \binom{N \setminus \{a_s\}}{t-1}$ define

$$\mathcal{E}(L) := \{F \in N_i(\mathcal{B}(a_s)) : L(F) = L\}.$$

By the definition of the reverse–lexicographic order, we have $\mathcal{E}(L_1) \prec \mathcal{E}(L_2)$ iff $L_1 \prec_{rl} L_2$. Moreover, for every L the first element of $\mathcal{E}(L)$ w.r.t. $\prec$ is the set $L \cup K$, where K is the set of the smallest $i-(t-1)$ elements of $N \setminus \{a_s\}$. Clearly, $\mathcal{E}(B \setminus \{b_1\}) = N_i(\mathcal{B}(a_s, b_1))$ holds and $B \setminus \{b_1\}$ is the reverse–lexicographically greatest element of $\binom{N\setminus\{a_s\}}{t-1}$. Therefore, $(B \setminus \{b_1\}) \cup K \in \mathcal{F}$ holds. Now Lemma 3.3 implies $L \cup K \in \mathcal{F}$ for all $L \in \binom{N\setminus\{a_s\}}{t-1}$.

Consequently, $\mathcal{G}' \subset \mathcal{E}(G)$ for some $G \in \binom{N\setminus\{a_s\}}{t-1}$, $G \neq B\setminus\{b_1\}$. Obviously, $F \cap B = G \cap B$ holds for every $F \in \mathcal{E}(G)$. Therefore, $C_{N\setminus B}(\mathcal{F}) = \mathcal{F}$ implies that $\mathcal{G}'$ is a final segment of $\mathcal{E}(G)$.

Let $E \subseteq N \setminus \{a_s\}$ be the set of all elements of $N \setminus \{a_s\}$ which are smaller than the smallest element of G. Clearly,

$$\mathcal{E}(G) = \{D \cup G : D \in \tbinom{E}{i-(t-1)}\}$$

holds. Moreover, the family

$$\mathcal{H}'' := \{D \cup (B \setminus \{b_1\}) : D \in \tbinom{E}{i-(t-1)}\}$$

is an initial segment of $N_i(\mathcal{B}(a_s, b_1))$. If $\mathcal{H}'' \subseteq \mathcal{H}'$ would hold, then Lemma 3.3 would yield $\mathcal{E}(G) \subseteq \mathcal{F}$, a contradiction to the choice of $\mathcal{G}'$. Consequently, $\mathcal{H}'$ is an initial segment of $\mathcal{H}''$.

To conclude the proof, note that $\mathcal{E}(G)$ and $\mathcal{H}''$ are both isomorphic to $N_{i-t+1}(B_{|E|})$, where $B_{|E|}$ denotes the Boolean lattice of order $|E|$. Furthermore, $\Delta_{new}(\mathcal{E}(G))$ and $\Delta_{new}(\mathcal{H}'')$ are both isomorphic to $N_{i-t}(B_{|E|})$, the families $\mathcal{G}'$ and $\mathcal{H}''$ are isomorphic to a final segment and an initial segment of $N_{i-t+1}(B_{|E|})$, respectively, w.r.t. the reverse–lexicographic order. Since Boolean lattices are additive w.r.t. the order $\prec_{rl}$ (3.3) holds in both possible cases: $\mathcal{G}$ is a final segment of $\mathcal{E}(G)$ and $\mathcal{H}$ is a segment of $\mathcal{H}''$ or $\mathcal{G}$ is a segment of $\mathcal{E}(G)$ and $\mathcal{H}$ is an initial segment of $\mathcal{H}''$. ∎

3.4. **Stable families.** Consider some operation transforming a subset $\mathcal{F} \subseteq N_i(\mathcal{S})$ into another subset $\sigma(\mathcal{F}) \subseteq N_i(\mathcal{S})$ such that $|\mathcal{F}| = |\sigma(\mathcal{F})|$. We say that the operation *works from right to left* if there is a bijection $\varphi : \mathcal{F} \longrightarrow \sigma(\mathcal{F})$ such that $\varphi(F) = F$ or $\varphi(F) \prec F$ for all $F \in \mathcal{F}$. Partial compression works from right to left (for any $\mathcal{F}$) by Proposition 2.1(e) and its definition. Partial shifting works from left to right since if an element F is replaced by $(F \setminus \{b\}) \cup \{a\}$, then this new element is in the same block but reverse–lexicographically smaller. Finally, the move of segment works from right to left by its definition.

We call a family $\mathcal{F} \subseteq N_i(\mathcal{S})$ *stable* if $C_J(\mathcal{F}) = s'_{ab}(\mathcal{F}) = M(\mathcal{F}) = \mathcal{F}$ for all $J \in \{A, B, N \setminus A, N \setminus B\}$ and $a \in A$, $b \in B$. Lemma 3.1, Lemma 3.2, Lemma 3.5 and the fact that all our operations work from right to left yield the following lemma.

Lemma 3.6. *For every $\mathcal{F} \subseteq N_i(\mathcal{S})$ there exists a stable $\mathcal{F}' \subseteq N_i(\mathcal{S})$ such that $|\mathcal{F}| = |\mathcal{F}'|$ and $|\Delta(\mathcal{F}')| \leq |\Delta(\mathcal{F})|$.* ∎

The next lemma provides that we have three essential classes of stable families.

Lemma 3.7. *If $\mathcal{F} \subseteq N_i(\mathcal{S})$ is stable, then*

(a) *$N_i(\mathcal{B}(a_s)) \subseteq \mathcal{F}$ or*

(b) *$\mathcal{F} \cap \mathcal{B}(a, b_1) = \emptyset$ for all $a \in A$ or*

(c) *$N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$ and $\mathcal{F} \cap \mathcal{B}(a, b_1) = \emptyset$ for all $a \in A \setminus \{a_s\}$*

holds.

Proof. Let $\mathcal{F}$ be like in the lemma. By $M(\mathcal{F}) = \mathcal{F}$, we just have the following two cases.

Case 1. Assume that $N_i(\mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$. If $N_i(\mathcal{B}(a_s, b_1))$ is empty, then $N_i(\mathcal{B}(a, b_1))$ is empty for all $a \in A$, and we have (b). Otherwise, $i \geq t-1$ must hold, and we obtain (a) by Corollary 3.4.

Case 2. Assume that $N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$. If $N_i(\mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$, then we have (a). Otherwise we obtain (c) by $C_{N \setminus B}(\mathcal{F}) = \mathcal{F}$. ∎

4. The main result

Theorem 4.1. *If $\mathcal{F} \subseteq N_i(\mathcal{S})$ with $1 \leq i \leq n-2$, then $\Delta(C(\mathcal{F})) \subseteq C(\Delta(\mathcal{F}))$ holds.*

Proof. Let i and $\mathcal{F}$ be like in the theorem. According to the introduction, it suffices to show (1.2) and (1.3) for $X = \mathcal{F}$ and $N_i = N_i(\mathcal{S})$. The relation (1.3) is implied by Proposition 2.2, it remains to show

$$|\Delta(C(\mathcal{F}))| \leq |\Delta(\mathcal{F})|. \tag{4.1}$$

By Lemma 3.6, we can assume that $\mathcal{F}$ is stable. Furthermore, without loss of generality, we can assume that $i \geq t$. Otherwise, $C(\mathcal{F})$ is isomorphic to the set of the $|\mathcal{F}|$ reverse–lexicographically smallest i–subsets of $\{1, 2, \ldots, n\}$, and (4.1) follows from the Kruskal–Katona theorem [6, 8].

Since the first block $\mathcal{B}(a_s)$ consists of the reverse–lexicographically smallest subsets of N, the Kruskal–Katona theorem implies

$$|\Delta(C(\mathcal{G}))| \leq |\Delta(\mathcal{G})| \text{ for } \mathcal{G} \subseteq N_i(\mathcal{S}),\ |\mathcal{G}| \leq |N_i(\mathcal{B}(a_s))|. \tag{4.2}$$

We proceed by induction on s. If $s = 1$, then (4.1) follows by Proposition 2.1(c) and the Kruskal–Katona theorem. Let $s \geq 2$, and suppose that the theorem is true for all $SM(N'; A', B')$ with $|A'| < s$. By Proposition 2.1(f) and our induction hypothesis we obtain

$$|\Delta(C(\mathcal{G}))| \leq |\Delta(\mathcal{G})| \text{ for } \mathcal{G} \subseteq N_i(\mathcal{S}),\ \mathcal{G} \cap N_i(\mathcal{B}(a_1)) = \emptyset. \tag{4.3}$$

Furthermore, the induction hypothesis together with Proposition 2.1(g) and Proposition 2.2 yields

$$|\Delta(C(\mathcal{G}))| \leq |\Delta(\mathcal{G})| \text{ for } \mathcal{G} \subseteq N_i(\mathcal{S}),\ N_i(\mathcal{B}(a_s)) \subseteq \mathcal{G}. \tag{4.4}$$

By Lemma 3.7, we can distinguish the following four cases.

Case 1. Assume that $N_i(\mathcal{B}(a_s)) \subseteq \mathcal{F}$. In this case the assertion is implied by (4.4).

Case 2. Assume that $s = t$ and that $\mathcal{F} \cap \mathcal{B}(a, b_1) = \emptyset$ for all $a \in A$. In this case just exchange the roles of A and B. Clearly, this does not affect $|\Delta(\mathcal{F})|$ and (4.1) follows by (4.3).

Case 3. Assume that $s = t$, $N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}$ and $\mathcal{F} \cap \mathcal{B}(a, b_1) = \emptyset$ for all $a \in A \setminus \{a_s\}$. In this case $\mathcal{F}$ is of the form $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where

$$\mathcal{F}_1 = N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)),$$

$\mathcal{F}_2$ is empty or an initial segment of $\mathcal{B}(a_s, b_1)$ and

$$\mathcal{F}_3 \subseteq \bigcup_{h=1}^{s-1} N_i(\mathcal{B}(a_h) \setminus \mathcal{B}(a_h, b_1)).$$

Clearly,

$$|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{F}_1 \cup \mathcal{F}_2)| + |\Delta_{new}(\mathcal{F}_3)|. \tag{4.5}$$

Let $\mathcal{S}' := SM(N \setminus \{a_s\}; A \setminus \{a_s\}, B \setminus \{b_1\})$ and

$$\mathcal{F}'_3 := \{F \setminus \{a_s\} : F \in \mathcal{F}_3\}.$$

Now $\mathcal{F'}_3 \subseteq N_{i-1}(\mathcal{S}')$ holds. Moreover, we have $\Delta(\mathcal{F}_1) = N_{i-1}(\mathcal{B}(a_s)\backslash\mathcal{B}(a_s, b_1)) = N_{i-1}(\mathcal{S}')$ by Proposition 2.2. Consequently,

$$|\Delta_{new}(\mathcal{F}_3)| = |\Delta(\mathcal{F'}_3)| \tag{4.6}$$

holds.

Let $\mathcal{F'}_4$ be a compressed subset of $N_{i-1}(\mathcal{S}')$ such that $|\mathcal{F'}_4| = |\mathcal{F'}_3|$. Further define $\mathcal{F}_4 := \{F \cup \{a_s\} : F \in \mathcal{F'}_4\}$ and $\mathcal{F}^* := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4$. Clearly, we also have

$$|\Delta_{new}(\mathcal{F}_4)| = |\Delta(\mathcal{F'}_4)|. \tag{4.7}$$

Applying the induction hypothesis on $\mathcal{S}'$ yields

$$|\Delta(\mathcal{F'}_4)| \leq |\Delta(\mathcal{F'}_3)|. \tag{4.8}$$

By the definitions of $\prec$ and $\mathcal{F}_4$, the family $\mathcal{F}_4$ consists of the smallest $|\mathcal{F}_3|$ elements of $\bigcup_{h=1}^{s-1} N_i(\mathcal{B}(a_h) \setminus \mathcal{B}(a_h, b_1))$ w.r.t. $\prec$, and by Proposition 2.2 we obtain

$$|\Delta(\mathcal{F}^*)| = |\Delta(\mathcal{F}_1 \cup \mathcal{F}_2)| + |\Delta_{new}(\mathcal{F}_4)|. \tag{4.9}$$

Putting together (4.5),(4.6),(4.7),(4.8),(4.9) gives

$$|\Delta(\mathcal{F}^*)| \leq |\Delta(\mathcal{F})|. \tag{4.10}$$

If $\mathcal{F}^* \cap \mathcal{B}(a_1) = \emptyset$, then (4.1) follows by (4.3) and (4.10). Therefore, let us assume that $\mathcal{H}' := \mathcal{F}^* \cap \mathcal{B}(a_1)$ is not empty. Further define $\mathcal{G}' := (\bigcup_{h=2}^{s} N_i(\mathcal{B}(a_h, b_1)) \setminus \mathcal{F}_2$.

If $|\mathcal{H}'| \leq |\mathcal{G}'|$, then put $\mathcal{H} := \mathcal{H}'$ and let $\mathcal{G}$ be the set of smallest $|\mathcal{H}|$ elements of $\mathcal{G}'$ w.r.t. $\prec$. By Corollary 2.4, for $\mathcal{F}^{**} := (\mathcal{F}^* \setminus \mathcal{H}) \cup \mathcal{G}$ we obtain

$$|\Delta(\mathcal{F}^{**})| \leq |\Delta(\mathcal{F}^*)|. \tag{4.11}$$

Obviously, we have $\mathcal{F}^{**} \cap \mathcal{B}(a_1) = \emptyset$. Now (4.1) follows by (4.3),(4.10),(4.11).

If $|\mathcal{H}'| > |\mathcal{G}'|$, then put $\mathcal{G} := \mathcal{G}'$ and let $\mathcal{H}$ be the set of largest $|\mathcal{G}|$ elements of $\mathcal{H}'$ w.r.t. $\prec$. By Corollary 2.4, for $\mathcal{F}^{**} := (\mathcal{F}^* \backslash \mathcal{H}) \cup \mathcal{G}$ (4.11) holds. Since we have $N_i(\mathcal{B}(a_s)) \subseteq \mathcal{F}^{**}$, the assertion follows by (4.4),(4.10),(4.11).

Case 4. Assume that $s < t$ and $\mathcal{F} \cap \mathcal{B}(a, b_1) = \emptyset$ for all $a \in A \setminus \{a_s\}$. We apply induction on t, using $t = s$ as the induction basis. So let the assertion be true for $t' < t$.

$\mathcal{F}$ is of the form $\mathcal{F} = \mathcal{F}_2 \cup \mathcal{F}_3$, where $\mathcal{F}_2$ is empty or an initial segment of $\mathcal{B}(a_s, b_1)$ and

$$\mathcal{F}_3 \subseteq \bigcup_{h=1}^{s} N_i(\mathcal{B}(a_h) \setminus \mathcal{B}(a_h, b_1)).$$

Obviously,

$$|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{F}_3)| + |\Delta_{new}(\mathcal{F}_2)| \tag{4.12}$$

holds.

By its definition, $\mathcal{F}_3$ is a subset of $N_i(\mathcal{S}')$, where $\mathcal{S}' = SM(N; A, B \setminus \{b_1\})$. Let $\mathcal{F}_5$ be the set of the smallest $|\mathcal{F}_3|$ elements of $N_i(\mathcal{S}')$ w.r.t. $\prec$. Further define $\mathcal{F}^* := (\mathcal{F} \backslash \mathcal{F}_3) \cup \mathcal{F}_5$. The induction hypothesis (induction on t) implies

$$|\Delta(\mathcal{F}_5)| \leq |\Delta(\mathcal{F}_3)|. \tag{4.13}$$

By the definition of $\prec$, the family $\mathcal{F}_5$ consists of the smallest $|\mathcal{F}_3|$ elements of $\bigcup_{h=1}^{s} N_i(\mathcal{B}(a_h) \backslash \mathcal{B}(a_h, b_1))$. If $N_i(\mathcal{B}(a_s) \backslash \mathcal{B}(a_s, b_1)) \not\subseteq \mathcal{F}_5$, then $|\mathcal{F}| < |N_i(\mathcal{B}(a_s))|$, and (4.1) follows by (4.2). Consequently, we can assume

$$\mathcal{F}_1 := N_i(\mathcal{B}(a_s) \setminus \mathcal{B}(a_s, b_1)) \subseteq \mathcal{F}^*.$$

This implies

$$|\Delta(\mathcal{F}^*)| = |\Delta(\mathcal{F}_5)| + |\Delta_{new}(\mathcal{F}_2)|. \tag{4.14}$$

Now by (4.12),(4.13),(4.14), we obtain

$$|\Delta(\mathcal{F}^*)| \leq |\Delta(\mathcal{F})|.$$

Put $\mathcal{F}_4 := \mathcal{F}_5 \setminus \mathcal{F}_1$. Finally, applying exactly the argumentation following (4.10) in Case 3. completes the proof of the theorem. ∎

5. Ideals of maximum weight

Let P be a Macaulay poset with the partial order $\leq$, the rank–function r and the associated linear order $\prec$. A subset $X \subseteq P$ is an *ideal* if $y \in X$ for all $x, y \in P$ with $x \in X$, $y \leq x$. We introduce a *weight function* $w : P \Longrightarrow \mathbb{R}$ such that $r(x) = r(y)$ implies $w(x) = w(y)$ and such that $r(x) < r(y)$ iimplies $w(x) \leq w(y)$, i.e. w is constant on the levels and increasing with the levels of P. The *weight* of a subset $X \subseteq P$ is the number $W(X) := \sum_{x \in X} w(x)$.

P is called *rank–greedy* (w.r.t. $\prec$) if $x \leq y$, $x \neq y$ imply $x \prec y$ (i.e. $\prec$ is a linear extension of the partial order) and if $r(x) > r(y)$, $z \prec y \ \forall z \in \Delta(x)$ imply $x \prec y$. Clearly, in a rank–greedy Macaulay poset the set of the smallest m elements w.r.t. $\prec$ is an ideal for every $m \leq |P|$. For $X \subseteq P$ let $c(X)$ denote the set of the smallest $|X|$ elements of P.

For rank–greedy Macaulay posets the following theorem gives a solution to the problem of finding an ideal of given size and maximum weight.

Theorem 5.1. (Engel [2]) *Let P be a rank–greedy Macaulay poset. If $X \subseteq P$ is an ideal, then $W(X) \leq W(c(X))$ holds.* ∎

To solve the above problem for $\mathcal{S}$ and its dual $\mathcal{S}^*$ it just remains to show the following lemma.

Lemma 5.2. *The posets $\mathcal{S}$ and $\mathcal{S}^*$ are rank–greedy with respect to the order $\prec$ introduced in Section 2.*

Proof. If P is a rank–greedy Macaulay poset, then also P^* is rank–greedy (see [2] for a proof). Therefore, it suffices to show that $\mathcal{S}$ is rank–greedy. This follows from Proposition 2.1(d), Proposition 2.2 and the fact that Boolean lattices are rank–greedy w.r.t. $\prec_{rl}$. ∎

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