THE ERDŐS-KO-RADO BOUND OF THE FUNCTION LATTICE

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Abstract. We answer the following question: When does a k-uniform family generated by some rank t element in the function lattice have maximum size among all k-uniform t-intersecting families?

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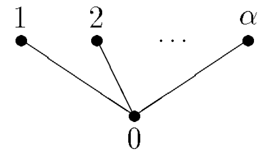
1. Introduction

Let F_α^n be the partially ordered set of all n-tuples over the alphabet $\{0,1,...,\alpha\}$, α a positive integer, with order relation given by $(x_1,...,x_n) \le (y_1,...,y_n)$ iff $x_i = 0$ or $x_i = y_i$ for all $i = 1,...,n$. Thus F_α^n is the product of n factors



We obtain the Boolean lattice B_n in the special case $\alpha = 1$, and the dual of the cubical lattice in the case $\alpha = 2$. In general, after adding a maximal element, F_α^n is called the function lattice.

The rank $r(X)$ of an element $X = (x_1,...,x_n)$ of F_α^n is given by the number of nonzero elements in X. A subset $\mathcal{F}$ (also called family) of F_α^n is called k-uniform t-intersecting if all elements of $\mathcal{F}$ have rank k and the infimum of every two elements of $\mathcal{F}$ has rank at least t.

Let us define

$$I_\alpha(n,k,t) := \{\mathcal{F} \subseteq F_\alpha^n : \mathcal{F} \text{ is } k\text{-uniform } t\text{-intersecting}\}$$
$$M_\alpha(n,k,t) := \max_{\mathcal{F} \in I_\alpha(n,k,t)} |\mathcal{F}|.$$

The following families $\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_{\lfloor \frac{n-t}{2} \rfloor}$ are candidates for achieving $M_\alpha(n,k,t)$:

$$\mathcal{F}_r := \{X = (x_1,...,x_n) \in F_\alpha^n : r(X) = k \text{ and } |B(X) \cap [1, t+2r]| \ge t + r\},$$

where $B(X) := \{i : x_i = \alpha\}$.

Ahlswede, Khachatrian proved in [1] the longstanding conjecture of Frankl [7] that in the Boolean lattice one of these families is always optimal:

Theorem 1.1. $M_1(n,k,t) = |\mathcal{F}_r|$ *for some* $r \in \{0\} \cup \mathbb{N}$.

This generalizes the previously known Erdős–Ko–Rado bound (see [6]) first established by Frankl [7] (for $t \ge 15$) and Wilson [12] (for all t):

Theorem 1.2. $M_1(n,k,t) = |\mathcal{F}_0|$ *if and only if* $n \ge (k-t+1)(t+1)$.

In a further paper [2] Ahlswede and Khachatrian extended their Theorem 1.1 to arbitrary α restricted to the n-th level of the function lattice:

Theorem 1.3. $M_\alpha(n,n,t) = |\mathcal{F}_r|$ *for some* $r \in \{0\} \cup \mathbb{N}$.

(The case $\alpha = 2$ is due to Kleitman [10] who showed that $M_2(n,n,t) = |\mathcal{F}_{\lfloor \frac{n-t}{2} \rfloor}|$.)

In particular it follows from [2] that

$$M_\alpha(n,n,t) = |\mathcal{F}_0| \text{ if and only if } \alpha \ge t+1.$$

It is known (due to Frankl [3]) that also for the function lattice the family $\mathcal{F}_0$ has maximum size among all k-uniform t-intersecting families provided n is sufficiently large:

Theorem 1.4. *$M_\alpha(n,k,t) = |\mathcal{F}_0|$ provided $n \geq n_\alpha(k,t)$ for suitable $n_\alpha(k,t)$.*

This paper demonstrates that the methods introduced by Ahlswede, Khachatrian in [1, 2] enables us to determine the least $n_\alpha(k,t)$ for which Theorem 1.4 remains true. We will prove the following generalization of Theorem 1.2.

Theorem 1.5. *Let $n > k$. Then $M_\alpha(n,k,t) = |\mathcal{F}_0|$ if and only if*

$$n \geq \left\lfloor \frac{(k-t+\alpha)(t+1)}{\alpha} \right\rfloor.$$

The case $t = 1$ was treated by Meyer [11], Frankl [3], and Engel [4].
A complete intersection theorem for the function lattice remains open.

2. The candidate families

The aim of this section is to prove a monotonicity property of the sequence $(|\mathcal{F}_r| : r = 0, \dots, \lfloor \frac{n-t}{2} \rfloor)$. This is done in Lemma 2.7. Furthermore we determine the least natural number $n_\alpha(k,t)$ such that for all lattices F_α^n with $n \geq n_\alpha(k,t)$ the family $\mathcal{F}_0$ has maximum cardinality among all candidate families $\mathcal{F}_0, \dots, \mathcal{F}_{\lfloor \frac{n-t}{2} \rfloor}$.

Throughout the paper let $\alpha \geq 2$.

Lemma 2.1. *Let $n \geq t + 2(r+1)$. Then*

$$|\mathcal{F}_r \setminus \mathcal{F}_{r+1}| = \binom{t+2r}{r} \sum_{i=0}^{r+2} \binom{r+2}{i} (\alpha-1)^i \binom{n-t-2r-2}{k-t-r-i} \alpha^{k-t-r-i}$$

and

$$|\mathcal{F}_{r+1} \setminus \mathcal{F}_r| = \binom{t+2r}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (\alpha-1)^i \binom{n-t-2r-2}{k-t-r-i-1} \alpha^{k-t-r-i-1}.$$

Proof.

$$\mathcal{F}_r \setminus \mathcal{F}_{r+1} = \big\{X \in F_\alpha^n : r(X) = k, B(X) \cap [1, t+2r] = t + r \text{ and } \{t+2r+1, t+2r+2\} \cap B(X) = \emptyset\big\}$$

and

$$\mathcal{F}_{r+1} \setminus \mathcal{F}_r = \big\{X \in F_\alpha^n : r(X) = k, B(X) \cap [1, t+2r] = t + r - 1 \text{ and } \{t+2r+1, t+2r+2\} \subseteq B(X)\big\}.$$

□

Lemma 2.2. *Let $n \geq t + 2(r+1)$. Then*

$$\frac{|\mathcal{F}_r \setminus \mathcal{F}_{r+1}|}{\binom{t+2r}{r}\alpha^{k-t-2r-2}} = \sum_{i=0}^{r+2} \binom{r+2}{i}(\alpha-1)^{r+2-i}\binom{n-t-r-i}{k-t-r},$$

$$\frac{|\mathcal{F}_{r+1} \setminus \mathcal{F}_r|}{\binom{t+2r}{r+1}\alpha^{k-t-2r-2}} = \sum_{i=0}^{r+1} \binom{r+1}{i}(\alpha-1)^{r+1-i}\binom{n-t-r-i-1}{k-t-r-1}.$$

Proof. Starting with Lemma 2.1 it follows that

$$\begin{aligned}
\frac{|\mathcal{F}_r \setminus \mathcal{F}_{r+1}|}{\binom{t+2r}{r}\alpha^{k-t-2r-2}} &= \sum_{i=0}^{r+2} \binom{r+2}{i}(\alpha-1)^{i}\binom{n-t-2r-2}{k-t-r-i}(\alpha-1+1)^{r+2-i} \\
&= \sum_{i=0}^{r+2}\sum_{j=0}^{r+2-i} \binom{r+2}{i}\binom{r+2-i}{j}(\alpha-1)^{i}(\alpha-1)^{j}\binom{n-t-2r-2}{k-t-r-i} \\
&= \sum_{s=0}^{r+2}\sum_{i=0}^{s} \binom{r+2}{i}\binom{r+2-i}{s-i}(\alpha-1)^{s}\binom{n-t-2r-2}{k-t-r-i} \\
&= \sum_{s=0}^{r+2}\sum_{i=0}^{s} \binom{r+2}{s}\binom{s}{i}(\alpha-1)^{s}\binom{n-t-2r-2}{k-t-r-i} \\
&= \sum_{s=0}^{r+2} \binom{r+2}{s}(\alpha-1)^{s}\binom{n-t-2r-2+s}{k-t-r} \\
&= \sum_{i=0}^{r+2} \binom{r+2}{i}(\alpha-1)^{r+2-i}\binom{n-t-r-i}{k-t-r},
\end{aligned}$$

and similar manipulations show the second statement. □

Lemma 2.3. *Let $n \geq t + 2(r+1)$. Then $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}|$ if and only if*

$$\alpha \sum_{i=0}^{r+1} \binom{r+1}{i}(\alpha-1)^{r+1-i}\binom{n-t-r-i}{k-t-r} \geq$$

$$\left(2 + \frac{t-1}{r+1}\right)\sum_{i=0}^{r+1} \binom{r+1}{i}(\alpha-1)^{r+1-i}\binom{n-t-r-i-1}{k-t-r-1}.$$

Proof. By Lemma 2.2 $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}|$ holds if and only if

$$\sum_{i=0}^{r+2} \binom{r+2}{i}(\alpha-1)^{r+2-i}\binom{n-t-r-i}{k-t-r} \geq$$

$$\frac{t+r}{r+1}\sum_{i=0}^{r+1} \binom{r+1}{i}(\alpha-1)^{r+1-i}\binom{n-t-r-i-1}{k-t-r-1}.$$

Addition of

$$\sum_{i=0}^{r+1} \binom{r+1}{i}(\alpha-1)^{r+1-i}\binom{n-t-r-i-1}{k-t-r-1}$$

and elementary manipulations prove the lemma. $\square$

Corollary 2.4. *If* $\alpha \geq 2 + \frac{t-1}{r+1}$, *then* $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}| \geq |\mathcal{F}_{r+2}| \geq \dots$. $\square$

Corollary 2.5. *Let* $n \geq t + 2(r+1)$.

a) *If* $n \leq (2 + \frac{t-1}{r+1})\frac{k-t-r}{\alpha} + t + r$, *then* $|\mathcal{F}_r| \leq |\mathcal{F}_{r+1}|$.

b) *If* $n \geq (2 + \frac{t-1}{r+1})\frac{k-t-r}{\alpha} + t + r + \frac{r+1}{2}$, *then* $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}|$. $\square$

Lemma 2.6. *Let* $n > k \geq t+1$. *Then* $|\mathcal{F}_0| \geq |\mathcal{F}_1|$ *if and only if*

$$n \geq \left\lfloor \frac{(k-t+\alpha)(t+1)}{\alpha} \right\rfloor .$$

Proof. By Lemma 2.3 we have $|\mathcal{F}_0| \geq |\mathcal{F}_1|$ if and only if

$$\alpha \left\{ (\alpha - 1)\binom{n-t}{k-t} + \binom{n-t-1}{k-t} \right\} \geq$$
$$(t+1)\left\{ (\alpha-1)\binom{n-t-1}{k-t-1} + \binom{n-t-2}{k-t-1} \right\}.$$

With $p := n - t$ and $q := k - t$ this inequality is equivalent to

$$\alpha p \big\{ (\alpha-1)(p-1) + \frac{p-1}{p}(p-q) \big\} \geq (t+1)q\big\{ (\alpha-1)(p-1) + (p-q) \big\}$$

If $\alpha p \leq (t+1)q$ then $|\mathcal{F}_0| < |\mathcal{F}_1|$ follows immediately. If $\alpha p \geq (t+1)q + 1$ then $|\mathcal{F}_0| \geq |\mathcal{F}_1|$ follows from $(\alpha-1)(p-1) + (p-q) \geq \alpha(p-q)$. $\square$

Lemma 2.7. $|\mathcal{F}_0| \geq |\mathcal{F}_1|$ *implies* $|\mathcal{F}_1| \geq |\mathcal{F}_2| \geq |\mathcal{F}_3| \geq \dots$.

Proof. We show $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}|$ for $r \geq 1$.

If $k < t + r + 1$ then $\mathcal{F}_{r+1} = \emptyset$ and the claim is trivial. Let $k \geq t + r + 1$. Furthermore, let $\alpha < 2 + \frac{t-1}{r+1}$ (in particular $t \geq 2$), because otherwise Corollary 2.4 yields the claim.

In view of Corollary 2.5 and Lemma 2.6 it is sufficient to show that

$$\text{(1)} \quad \frac{(t+1)(k-t)+1}{\alpha} + t \geq \left(2 + \frac{t-1}{r+1}\right)\left(\frac{k-t-r}{\alpha}\right) + t + r + \frac{r+1}{2},$$

or equivalently

$$(k-t)\frac{r}{r+1}(t-1) \geq \left(\alpha - 2 - \frac{t-1}{r+1}\right) r + \alpha\frac{r+1}{2} - 1.$$

Because of $k \geq t + r + 1$ and $\alpha < 2 + \frac{t-1}{r+1}$

$$r(t-1) \geq \left(2 + \frac{t-1}{r+1}\right)\left(\frac{r+1}{2}\right) - 1,$$

or equivalently

$$(2r-1)(t-1) \geq 2r,$$

is sufficient for (1). The last inequality holds if $t > 2$. If $t = 2$ and $\alpha = 2$ then (1) obviously holds, using $2 + \frac{1}{r+1} < 3$. $\square$

We summarize our results in the following Lemma.

Lemma 2.8. *Let $n > k$. Equivalent are*

a) $|\mathcal{F}_0| = \max_r |\mathcal{F}_r|$
b) $|\mathcal{F}_0| \geq |\mathcal{F}_1| \geq |\mathcal{F}_2| \geq \dots$
c) $n \geq \left\lfloor \frac{(k-t+\alpha)(t+1)}{\alpha} \right\rfloor$. $\square$

We will show in section 4 that $\mathcal{F}_0$ has indeed maximum cardinality among all k-uniform t-intersecting families if the conditions of Lemma 2.8 hold.

3. Pushing up, shifting and generating sets

This section supplies the necessary tools from extremal set theory which are needed in the next section. While pushing up and shifting are known techniques for a long time (see e.g. [9], [8], or [5]), the method of generating sets introduced by Ahlswede, Khachatrian ([1, 2]) is new. Most of this section is up to minor changes taken from their paper [2]. We omit attributions (see [2]) and the easy proofs.

Definition 3.1 (Pushing up and canonical families). *For $\mathcal{A} \subseteq F_\alpha^n$, $A = (a_1, a_2, \dots, a_n) \in \mathcal{A}$ and $1 \leq i \leq n$, $0 < j \leq \alpha$ we define*

$$P_{ij}(A) = \begin{cases} (a_1, \dots, a_{i-1}, \alpha, a_{i+1}, \dots, a_n), & \text{if this is not an element of } \mathcal{A} \text{ and} \\ & a_i = j \\ A & \text{otherwise} \end{cases}$$

and

$$P_{ij}(\mathcal{A}) = \{P_{ij}(A) : A \in \mathcal{A}\}.$$

Furthermore, we call a family $\mathcal{A} \subseteq F_\alpha^n$ canonical *if*

$$P_{ij}(\mathcal{A}) = \mathcal{A} \text{ for all } 1 \leq i \leq n,\ 0 < j \leq \alpha.$$

$CI_\alpha(n, k, t)$ denotes the set of all canonical families in $I_\alpha(n, k, t)$.

Lemma 3.2. *Repeated application of P_{ij} to $\mathcal{A} \in I_\alpha(n, k, t)$ yields, after finitely many steps, a family $\mathcal{A}' \in CI_\alpha(n, k, t)$ with $|\mathcal{A}| = |\mathcal{A}'|$.*

Corollary 3.3.

$$M_\alpha(n, k, t) = \max_{\mathcal{A} \in CI_\alpha(n,k,t)} |\mathcal{A}|$$

Definition 3.4 (α-support). *For $A = (a_1, \dots, a_n) \in F_\alpha^n$, $\mathcal{A} \subseteq F_\alpha^n$ we define*

$$B(A) := \{i : a_i = \alpha\},$$

and

$$\mathcal{B}(\mathcal{A}) := \{B(A) : A \in \mathcal{A}\}.$$

$\mathcal{B}(\mathcal{A})$ is called α-support *of $\mathcal{A}$.*

Lemma 3.5. *If $\mathcal{A} \in CI_\alpha(n,k,t)$ then*

$$\mathcal{B}(\mathcal{A}) \in I(n,t),$$

where $I(n,t)$ denotes the set of all t-intersecting families in the Boolean lattice B_n.

It follows that $\mathcal{A} \in CI_\alpha(n,k,t)$ with $|\mathcal{A}| = M_\alpha(n,k,t)$ is uniquely determined by its α-support. Families with this property are called *α-upsets*:

Definition 3.6 (α-upsets and generating sets). *For $G \in B_n$ and $\mathcal{G} \subseteq B_n$ we define*

$${}_\alpha\nabla(G) := \{A \in F_\alpha^n : r(A) = k \quad and \quad B(A) \supseteq G\}$$

and

$${}_\alpha\nabla(\mathcal{G}) := \bigcup_{G \in \mathcal{G}} {}_\alpha\nabla(G).$$

$\mathcal{A} \subseteq F_\alpha^n$ is called α-upset *if there is some $\mathcal{G} \subseteq B_n$ with $\mathcal{A} = {}_\alpha\nabla(\mathcal{G})$, $\mathcal{G}$ is then called* generating family *for $\mathcal{A}$.*

For a family $\mathcal{A} \subseteq F_\alpha^n$ we denote by $\mathcal{M}(\mathcal{A})$ the set of all minimal elements of $\mathcal{B}(\mathcal{A})$. Then obviously $\mathcal{G} \subseteq B_n$ is a generating family for the α-upset $\mathcal{A}$ if and only if $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{G} \subseteq \mathcal{B}(\mathcal{A})$.

Definition 3.7 (Shifting). *For $G \in B_n$, $\mathcal{G} \subseteq B_n$ and $1 \leq i < j \leq n$ we define*

$$S_{ij}(G) = \begin{cases} G \setminus \{j\} \cup \{i\}, & \text{if this is not an element of } \mathcal{G} \text{ and } G \cap \{i,j\} = \{j\} \\ G & \text{otherwise} \end{cases}$$

and

$$S_{ij}(\mathcal{G}) = \{S_{ij}(G) : G \in \mathcal{G}\}.$$

$\mathcal{G} \subseteq B_n$ is called left-shifted *if*

$$S_{ij}(\mathcal{G}) = \mathcal{G} \quad for\ all \quad 1 \leq i < j \leq n.$$

$LI(n,t)$ denotes the set of all left-shifted families in $I(n,t)$. $LCI_\alpha(n,k,t)$ denotes the set of all families $\mathcal{A} \in CI_\alpha(n,k,t)$ with $\mathcal{B}(\mathcal{A}) \in LI(n,t)$.

Lemma 3.8. *Repeated application of S_{ij} to $\mathcal{G} \in I(n,t)$ yields, after finitely many steps, a family $\mathcal{G}' \in LI(n,t)$ with $|\mathcal{G}| = |\mathcal{G}'|$ and $|\mathcal{G} \cap \binom{[n]}{i}| = |\mathcal{G}' \cap \binom{[n]}{i}|$.*

Corollary 3.9.

$$M_\alpha(n,k,t) = \max_{\mathcal{A} \in LCI_\alpha(n,k,t)} |\mathcal{A}|$$

Definition 3.10. *For $G \in B_n$ and $\mathcal{G} \subseteq B_n$ let*

$$s^+(G) := \max_{i \in G} i$$

and

$$s^+(\mathcal{G}) := \max_{G \in \mathcal{G}} s^+(G).$$

Definition 3.11. *For $G \in B_n$, $\mathcal{G} \subseteq B_n$ and $1 \leq \ell \leq n$ let*

$${}_\alpha\nabla_\ell(G) := \{A = (a_1, ..., a_n) \in F_\alpha^n : r(A) = k \text{ and } B(A) \cap [1, \ell] = G\},$$

$${}_\alpha\nabla_\ell(\mathcal{G}) := \bigcup_{G \in \mathcal{G}} {}_\alpha\nabla_\ell(G).$$

Furthermore, we define for $1 \leq i, \ell \leq n$

$$\nabla(\ell, i) := |\,{}_\alpha\nabla_\ell([1, i])|.$$

Lemma 3.12. *Let $n \geq t + 2(r+1)$. Then*

$$|\mathcal{F}_r \setminus \mathcal{F}_{r+1}| = \binom{t+2r}{r} \nabla(t + 2r + 2, t + r)$$

and

$$|\mathcal{F}_{r+1} \setminus \mathcal{F}_r| = \binom{t+2r}{r+1} \nabla(t + 2r + 2, t + r + 1).$$

Lemma 3.13. *Let $1 < i \leq \ell \leq n$. Then*

a) $\nabla(\ell - 1, i - 1) = \nabla(\ell, i - 1) + \nabla(\ell, i)$
b) $\nabla(\ell, i - 1) > \nabla(\ell, i)$ *provided* $(\alpha, k) \neq (2, n)$

Lemma 3.14. *Let $\mathcal{A} \in LCI_\alpha(n, k, t)$ be an α-upset, $\mathcal{M} := \mathcal{M}(\mathcal{A})$ the set of all minimal elements of $\mathcal{B}(\mathcal{A})$, and $\ell := s^+(\mathcal{M})$.*
Let furthermore $\mathcal{M}_0 := \{G \in \mathcal{M} : s^+(G) = \ell\}$, $\mathcal{G} \subseteq \mathcal{M}_0$ and $\mathcal{G}' := \{G \setminus \{\ell\} : G \in \mathcal{G}\}$. Then

$$\begin{aligned} {}_\alpha\nabla(\mathcal{G}) \setminus {}_\alpha\nabla(\mathcal{M} \setminus \mathcal{G}) &= {}_\alpha\nabla_\ell(\mathcal{G}), \\ {}_\alpha\nabla(\mathcal{G}') \setminus {}_\alpha\nabla(\mathcal{M}) &= {}_\alpha\nabla_\ell(\mathcal{G}'), \\ {}_\alpha\nabla(\mathcal{G}') \setminus {}_\alpha\nabla(\mathcal{M} \setminus \mathcal{M}_0) &= {}_\alpha\nabla_{\ell-1}(\mathcal{G}'). \end{aligned}$$

Lemma 3.15. *Let $\mathcal{A} \in LCI_\alpha(n, k, t)$ be an α-upset and let $G_1, G_2 \in \mathcal{M}(\mathcal{A})$ have the properties $i \notin G_1 \cup G_2$, $j \in G_1 \cap G_2$ for some $i, j \in [n]$ with $i < j$. Then*

$$|G_1 \cap G_2| \geq t + 1.$$

4. Left-generated maximum $I_\alpha(n, k, t)$-families

In order to prove that the family $\mathcal{F}_0$ is optimal we will show that among all k-uniform t-intersecting families of maximum size there is one generated by $[1, t]$. This will follow from the next lemma, whose proof is a close analogue of the arguments given in [1, 2].

Lemma 4.1. *Let $(\alpha, k) \neq (2, n)$. Assume that $|\mathcal{F}_r| \geq |\mathcal{F}_{r+1}| \geq |\mathcal{F}_{r+2}| \geq ...$ for some $r \in \{0\} \cup \mathbb{N}$. Then there exists a family $\mathcal{A} \in LCI_\alpha(n, k, t)$ with $|\mathcal{A}| = M_\alpha(n, k, t)$ and*

$$s^+(\mathcal{M}(\mathcal{A})) = t + 2\bar{r} \leq t + 2r \text{ for some } \bar{r} \in \{0\} \cup \mathbb{N}.$$

Proof. First we deal with an arbitrary $\mathcal{A} \in LCI_\alpha(n,k,t)$ with $|\mathcal{A}| = M_\alpha(n,k,t)$.
Let $\ell := s^+(\mathcal{M}(\mathcal{A}))$ and consider the partitions

$$\begin{aligned} &\mathcal{M}(\mathcal{A}) = \mathcal{M}_0(\mathcal{A}) \,\dot{\cup}\, \mathcal{M}_1(\mathcal{A}), \text{ where} \\ &\mathcal{M}_0(\mathcal{A}) = \{G \in \mathcal{M}(\mathcal{A}) : s^+(G) = \ell\} \text{ and } \mathcal{M}_1(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \setminus \mathcal{M}_0(\mathcal{A}), \end{aligned}$$

and

$$\mathcal{M}_0(\mathcal{A}) = \bigcup_{t \le i \le \ell} \mathcal{R}_i \,, \text{ where } \mathcal{R}_i = \mathcal{M}_0(\mathcal{A}) \cap \binom{[n]}{i}.$$

Let furthermore

$$\mathcal{R}_i' = \{G \setminus \{\ell\} : G \in \mathcal{R}_i\}.$$

If $\mathcal{R}_t \neq \emptyset$ then $\mathcal{M}(\mathcal{A}) = \{[1,t]\}$. If $\mathcal{R}_\ell \neq \emptyset$ then $\mathcal{M}(\mathcal{A}) = \{[1,\ell]\}$ and necessarily $\ell = t$. In both cases the Lemma holds.

Suppose there is an i with $t < i < \ell$, $i \neq \frac{\ell+t}{2}$ and $\mathcal{R}_i \neq \emptyset$. Consider

$$\begin{aligned} \mathcal{G}_1 &:= \big(\mathcal{M}(\mathcal{A}) \setminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_i' \quad \text{and} \\ \mathcal{G}_2 &:= \big(\mathcal{M}(\mathcal{A}) \setminus (\mathcal{R}_i \cup \mathcal{R}_{\ell+t-i})\big) \cup \mathcal{R}_{\ell+t-i}'. \end{aligned}$$

By Lemma 3.15 we have for $G' \in \mathcal{R}_i'$, $G \in \mathcal{R}_j$ with $i + j \neq \ell + t$

$$|G' \cap G| \ge t.$$

This shows that $\mathcal{G}_1, \mathcal{G}_2 \in I(n,t)$ and therefore

$$\mathcal{A}_i := {}_\alpha\nabla(\mathcal{G}_i) \in I_\alpha(n,k,t) \text{ for } i = 1,2.$$

Notice that $\mathcal{R}_{\ell+t-i} \neq \emptyset$ (otherwise $|\mathcal{A}_1| > |\mathcal{A}|$).

$\mathcal{A} \setminus \mathcal{A}_1$ contains exactly those elements of $\mathcal{A}$ that are generated only by $\mathcal{R}_{\ell+t-i}$. On the other hand, $\mathcal{A}_1 \setminus \mathcal{A}$ contains exactly those elements that are generated by $\mathcal{R}_i'$ but not by any element of $\mathcal{M}(\mathcal{A})$. Hence, Lemma 3.14 gives us

$$\begin{aligned} |\mathcal{A} \setminus \mathcal{A}_1| &= |{}_\alpha\nabla_\ell(\mathcal{R}_{\ell+t-i})| = |\mathcal{R}_{\ell+t-i}|\, \nabla(\ell, \ell+t-i), \\ |\mathcal{A}_1 \setminus \mathcal{A}| &= |{}_\alpha\nabla_\ell(\mathcal{R}_i')| = |\mathcal{R}_i|\, \nabla(\ell, i-1). \end{aligned}$$

Analogously, we have

$$\begin{aligned} |\mathcal{A} \setminus \mathcal{A}_2| &= |{}_\alpha\nabla_\ell(\mathcal{R}_i)| = |\mathcal{R}_i|\, \nabla(\ell, i), \\ |\mathcal{A}_2 \setminus \mathcal{A}| &= |{}_\alpha\nabla_\ell(\mathcal{R}_{\ell+t-i}')| = |\mathcal{R}_{\ell+t-i}|\, \nabla(\ell, \ell+t-i-1). \end{aligned}$$

Combining these equalities with Lemma 3.13 gives

$$|\mathcal{A}_1 \setminus \mathcal{A}| \cdot |\mathcal{A}_2 \setminus \mathcal{A}| > |\mathcal{A} \setminus \mathcal{A}_1| \cdot |\mathcal{A} \setminus \mathcal{A}_2|$$

in contradiction to

$$|\mathcal{A}_i| \le |\mathcal{A}| \text{ for } i = 1,2.$$

We have shown that $\mathcal{R}_i = \emptyset$ for all $i \neq \frac{\ell+t}{2}$. In particular, $2|(\ell - t)$.

Now choose a family $\mathcal{A} \in LCI_\alpha(n,k,t)$ with $|\mathcal{A}| = M_\alpha(n,k,t)$ for which

$$\ell = s^+(\mathcal{M}(\mathcal{A})) \text{ is minimal.}$$

Assume that $\ell > t + 2r$, hence

$$\ell = t + 2(\bar{r}+1) \text{ for some } \bar{r} \geq r. \tag{2}$$

Let us use the notation

$$\mathcal{T}_j := \{G \in \mathcal{R}'_{\frac{\ell+t}{2}} : j \notin G\} \text{ for all } j \in [1, \ell-1].$$

Double counting yields a family $\mathcal{T}_j$ satisfying

$$|\mathcal{T}_j| \geq \frac{\ell - t}{2(\ell-1)} \left|\mathcal{R}_{\frac{\ell+t}{2}}\right|. \tag{3}$$

By Lemma 3.15 we have

$$\mathcal{G}_j := \left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}_{\frac{\ell+t}{2}}\right) \cup \mathcal{T}_j \in I(n,t)$$

and therefore

$${}_\alpha\nabla(\mathcal{G}_j) \in I_\alpha(n,k,t).$$

Claim: We have $|{}_\alpha\nabla(\mathcal{G}_j)| \geq |\mathcal{A}|$.
Let us write

$$\begin{aligned}
\mathcal{A} = \mathcal{A}_1 \,\dot{\cup}\, \mathcal{A}_2, &\text{ where} \\
\mathcal{A}_1 &= {}_\alpha\nabla\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}_{\frac{\ell+t}{2}}\right), \\
\mathcal{A}_2 &= {}_\alpha\nabla\left(\mathcal{R}_{\frac{\ell+t}{2}}\right) \setminus {}_\alpha\nabla\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}_{\frac{\ell+t}{2}}\right), \\
{}_\alpha\nabla(\mathcal{G}_j) = \mathcal{A}_1 \,\dot{\cup}\, \mathcal{A}_3, &\text{ where} \\
\mathcal{A}_3 &= {}_\alpha\nabla\left(\mathcal{T}_j\right) \setminus {}_\alpha\nabla\left(\mathcal{M}(\mathcal{A}) \setminus \mathcal{R}_{\frac{\ell+t}{2}}\right).
\end{aligned}$$

We will show that $|\mathcal{A}_3| \geq |\mathcal{A}_2|$. By Lemma 3.14 we have

$$\begin{aligned}
|\mathcal{A}_2| &= \left|{}_\alpha\nabla_\ell(\mathcal{R}_{\frac{\ell+t}{2}})\right| = \left|\mathcal{R}_{\frac{\ell+t}{2}}\right| \nabla\left(\ell, \frac{\ell+t}{2}\right), \\
|\mathcal{A}_3| &= |{}_\alpha\nabla_{\ell-1}\left(\mathcal{T}_j\right)| = |\mathcal{T}_j| \nabla\left(\ell-1, \frac{\ell+t}{2} - 1\right).
\end{aligned}$$

Thus, in view of (3) and $\left|\mathcal{R}_{\frac{\ell+t}{2}}\right| \neq 0$

$$\frac{\ell-t}{2(\ell-1)} \nabla\left(\ell-1, \frac{\ell+t}{2} - 1\right) \geq \nabla\left(\ell, \frac{\ell+t}{2}\right)$$

will prove the claim. Together with our assumption (2) this reads as

$$\nabla(t+2\bar{r}+1, t+\bar{r}) \geq \left(1 + \frac{t+\bar{r}}{\bar{r}+1}\right) \nabla(t+2\bar{r}+2, t+\bar{r}+1).$$

In view of Lemma 3.12 and Lemma 3.13 the last inequality is equivalent to

$$\frac{|\mathcal{F}_{\bar{r}+1} \setminus \mathcal{F}_{\bar{r}}|}{\binom{t+2\bar{r}}{\bar{r}+1}} + \frac{|\mathcal{F}_{\bar{r}} \setminus \mathcal{F}_{\bar{r}+1}|}{\binom{t+2\bar{r}}{\bar{r}}} \geq \left(1 + \frac{t+\bar{r}}{\bar{r}+1}\right) \frac{|\mathcal{F}_{\bar{r}+1} \setminus \mathcal{F}_{\bar{r}}|}{\binom{t+2\bar{r}}{\bar{r}+1}}$$

resp. to

$$|\mathcal{F}_{\bar{r}}| \geq |\mathcal{F}_{\bar{r}+1}|.$$

This proves the claim.

Note that a strict inequality in (3) gives us the contradiction $|{}_{\alpha}\nabla(\mathcal{G}_j)| > |\mathcal{A}|$. Hence, we have equality in (3) for all families $\mathcal{T}_j$, $j \in [1, \ell-1]$. But it is easily seen that

$${}_{\alpha}\nabla(\mathcal{G}_{\ell-1}) \in LCI_{\alpha}(n,k,t),$$

in contradiction to our choice of $\mathcal{A}$. □

Theorem 1.5 now follows from Lemma 2.8 and Lemma 4.1.

Remark. Lemma 4.1 also holds in the case $(\alpha, k) = (2, n)$. This can be seen if one starts the proof with an optimal family $\mathcal{A} \in LCI_{\alpha}(n,k,t)$ satisfying first $s^{+}(\mathcal{M}(\mathcal{A}))$ minimal and second $|\mathcal{M}_0(\mathcal{A})|$ minimal. We have included the condition $(\alpha, k) \neq (2, n)$ in order to show the somewhat stronger statement in the case $i \neq \frac{\ell+t}{2}$. □

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