ON THE EXISTENCE OF MANDATORY REPRESENTATION DESIGNS WITH BLOCK SIZES 3 AND k

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Abstract. A mandatory representation design $MR[v,K]$ is a pairwise balanced design on v points with block sizes from K in which for each $k \in K$ there is a block in the design of size k. Mendelsohn and Rees [4] investigated the existence of $MR[v,K]$'s, where $3 \in K$. In this report we consider additional necessary conditions, where $K = \{3,k\}$. These conditions are proved to be sufficient for $4 \le k \le 50$ with one genuine exception.

1. Introduction

Let K be a set of positive integers. Then a *pairwise balanced design* (*PBD*$[v,K]$) of order v with block sizes from K is a pair $(V,\mathcal{B})$, where V is a finite set (the *point set*) of cardinality v and $\mathcal{B}$ is a family of subsets (called *blocks*) of V which satisfy the following properties:

1. if $B \in \mathcal{B}$, then $|B| \in K$;
2. every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$.

A *mandatory representation design* is a *PBD* with at least one block of each size. The basic necessary conditions are due to Wilson [7]. Let $\alpha(K) = \gcd\{k-1 \mid k \in K\}$ and $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$. Then $v-1 \equiv 0 \pmod{\alpha(K)}$ and $v(v-1) \equiv 0 \pmod{\beta(K)}$.

Mendelsohn and Rees [4] investigated in 1988 the existence of $MR[v,K]$'s, particularly the case $K = \{3,k\}$. Before stating their result, we need to define some terminology.

A pair $(V,\mathcal{B})$ consisting of a point set V ($|V| = v$) and a block set $\mathcal{B}$ with block sizes from K, so that any pair of distinct points occurs in at most one block, is called a *partial design* of order v. When $K = \{k\}$, the notation (v,k) *packing* is in common use. The *packing number* $D(v,k)$ is the maximum number of blocks in any (v,k) packing. For

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m	$p(m)$	m	$p(m)$	m	$p(m)$	m	$p(m)$
0	1	8	8	16	11	24	13
1	3	9	9	17	11	25	13
2	5	10	9	18	12	26	13
3	6	11	9	19	12	27	14
4	6	12	9	20	12	28	14
5	7	13	10	21	13	29	15
6	7	14	11	22	13	30	15
7	7	15	11	23	13	31	15

TABLE 1. $p(m)$, $0 \leq m \leq 31$

$k = 3$, the packing number is well known (see Hanani [3])

$$D(v,3) = \begin{cases} \left\lfloor \frac{v}{3} \left\lfloor \frac{v-1}{2} \right\rfloor \right\rfloor - 1, & \text{if } v \equiv 5 \pmod 6; \\ \left\lfloor \frac{v}{3} \left\lfloor \frac{v-1}{2} \right\rfloor \right\rfloor, & \text{otherwise,} \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; and, $\lceil x \rceil$ is the least integer greater than or equal to x.

By $p(m)$ we denote the minimum number of points v such that there exists a $(v,3)$ packing, $p(m) = \min\{v > 0 \mid D(v,3) \geq m\}$.

Theorem 1. [4] *Suppose that there exists a mandatory representation design $MR[v, \{3, k\}]$. Then the following conditions hold*

(i) *If $k \equiv 1,3 \pmod 6$, then $v \equiv 1,3 \pmod 3$ and $v \geq 2k+1$.*
(ii) *If $k \equiv 5 \pmod 6$, then either $v \equiv 5 \pmod 6$ and $v \geq 2k+1$, or $v \equiv 1,3 \pmod 6$ and $v \geq 3k-2$.*
(iii) *If $k \equiv 0,4 \pmod 6$ and v is odd, then $v \equiv 1,3 \pmod 6$ and $v \geq \frac{1}{2}k(k+1)$. If $k \equiv 2 \pmod 6$ and v is odd, then either $v \equiv 1,3 \pmod 6$ and $v \geq \frac{1}{2}k(k+1)$, or $v \equiv 5 \pmod 6$ and $v \geq \frac{1}{2}k(k+2)+1$.*
(iv) *If $k \equiv 0,4 \pmod 6$ and v is even then $v \equiv 0,4 \pmod 6$, while if $k \equiv 2 \pmod 6$ and v is even there are no (further) congruential conditions on v. In each case here we have $v \geq k \cdot p(t) - 2t$ when $v \equiv -2t \pmod k$, $0 \leq t < k/2$.*

We remark here that the monotone non decreasing function $p(m)$ will be useful in Section 2. Therefore, we give Table 1 for the first few values of $p(m)$ and mention the following lemma which was already proved by Mendelsohn and Rees.

Lemma 1.1. *For each $m \geq 0$,*

$$\frac{(p(m)-1)(p(m)-2)}{2} - \frac{p(m)+1}{2} < 3m.$$

The necessary conditions in Theorem 1 are sufficient if k is odd [4]; but, in case (iv) where $k \equiv 2 \pmod 6$ they are not sufficient. For example, Gronau, Mullin and Pietsch [1] proved that there exist no PBD's on $v = 30, 36$ or 38 points with $K = \{3, 8\}$. There are no (further) congruential conditions on v, but on the number of blocks of size k. In this paper, we will show the following additional nonexistence criteria in the case (iv).

Theorem 2. *Suppose that there exists a mandatory representation design $MR[v, \{3, k\}]$. If $k \equiv 2 \pmod 6$ and v is even, then there are no (further) congruential conditions on v. Let $v \equiv -2t \pmod k$, $0 \leq t < k/2$; and let $d \equiv -\lceil \frac{v}{k} \rceil \pmod 3$ if $v \equiv 0, 4 \pmod 6$ and $d \equiv -\lceil \frac{v}{k} \rceil + 1 \pmod 3$ if $v \equiv 2 \pmod 6$, $0 \leq d < 3$. Then*

$$v \geq k \cdot p(t + \frac{1}{2}dk) - 2(t + \frac{1}{2}dk).$$

In Section 3, we give a brief description how the existence results are established.

2. An additional necessary condition

In this section, we will expand on the work done by Mendelsohn and Rees by using the information on the number of blocks of size k. In order to establish Theorem 2, we need some of the results stated in their paper.

Proposition 1. *Let $(V, \mathcal{B})$ be a partial design of order v, with block size k, in which there are b blocks. Then letting $r = 1 + \lfloor (b-1)/k \rfloor$*

$$v \geq b \cdot \frac{2rk - b + 1}{r^2 + r}.$$

Proof. See [4]. □

Corollary 2.1. *Let $b > 2k + 1$. Then $v > \frac{1}{3}k(2k + 1)$.*

The next three statements examine the number of blocks of size k ($k \equiv 2 \pmod 6$) which an $MR[v, \{3, k\}]$ can have and give a lower bound for the number of points. The results obtained are used to prove the main theorem. First, the minimum number of blocks is determined.

We remark that in any $MR[v, \{3, k\}]$, with v and k both even, every point lies on an odd number of blocks of size k. This follows from counting pairs of points in all blocks which contain an arbitrary, fixed point.

Lemma 2.2. *Let v be even and $k \equiv 2 \pmod 6$. Suppose that there exists an $MR[v, \{3, k\}]$. Let $d \equiv -\lceil \frac{v}{k} \rceil \pmod 3$ if $v \equiv 0, 4 \pmod 6$ and $d \equiv -\lceil \frac{v}{k} \rceil + 1 \pmod 3$ if $v \equiv 2 \pmod 6$, $0 \leq d < 3$. Then for the number b of blocks of size k, the following holds*

$$b \geq \left\lceil \frac{v}{k} \right\rceil + d.$$

Proof. Every point lies on at least one block of size k. Hence, we need at least $\left\lceil \frac{v}{k} \right\rceil$ blocks. Observe that $b \equiv 0 \pmod 3$ if $v \equiv 0, 4 \pmod 6$, and $b \equiv 1 \pmod 3$ if $v \equiv 2 \pmod 6$. This is by counting the pairs of elements in V in two ways and by taking the equation obtained

$$k(k-1)b + 6b_3 = v(v-1)$$

modulo 3, where b_3 denotes the number of blocks of size 3. The definition of d implies that $\left\lceil \frac{v}{k} \right\rceil + d$ is the least integer that is not less than $\left\lceil \frac{v}{k} \right\rceil$ and satisfies the congruential conditions on b. □

Example: Let $v = 30$ and $k = 8$. It is possible to cover all points by $\left\lceil \frac{v}{k} \right\rceil = 4$ blocks of size 8, but $b \equiv 0 \pmod 3$. Hence, $b \geq 4 + 2$.
Let $v = 38$ and $k = 8$. At least $\left\lceil \frac{v}{k} \right\rceil = 5$ blocks are necessary to cover all points. Thus, $b \equiv 1 \pmod 3$ implies $b \geq 5 + 2$.

The next proposition, which gives an upper bound on the block number, is also due to Mendelsohn and Rees.

Proposition 2. *Let v be even and $k \equiv 2 \pmod 6$. Furthermore, let $v \equiv -2t \pmod k$ where $0 \leq t < \frac{k}{2}$. Let $(V, \mathcal{B})$ be a partial design of order v with b blocks ($b \leq 2k+1$) of size k in which each point lies on an odd number of blocks. Put $c = b - \left\lceil \frac{v}{k} \right\rceil$. Then it follows*

$$c \leq \frac{2}{3}(2k+1).$$

Proof. Let r_x denote the number of blocks containing $x \in V$. By double counting we have

$$\sum_{x \in V} r_x = kb$$

and, therefore,

$$v = kb - \sum_{x \in V}(r_x - 1) = kb - (2t + ck).$$

We may assume that each point is contained in either one or three blocks. Otherwise by deleting points of degree $r_x = 1$ and splitting up points of degree $r_x = 2m+1$ into m points of degree 3, we can deduce a partial design with the same number of points and blocks which has this property. For more details see [4]. Let $U \subseteq V$ consists of those points contained in three blocks. Since $\sum_{x \in V}(r_x - 1) = 2t + ck$, it follows $|U| = t + \frac{1}{2}ck$. By counting the blocks spanned by U, we obtain

$$3|U| = 3t + \frac{3}{2}ck \leq kb \leq k(2k+1).$$

This implies $c \leq \frac{2}{3}(2k+1)$. □

Corollary 2.3. *Let v be even and $k \equiv 2 \pmod 6$. Furthermore, let $v \equiv -2t \pmod k$, $0 \leq t < \frac{k}{2}$; $d \equiv -\left\lceil \frac{v}{k} \right\rceil \pmod 3$ if $v \equiv 0, 4 \pmod 6$*

and $d \equiv -\lceil \frac{v}{k} \rceil + 1 \pmod 3$ *if* $v \equiv 2 \pmod 6$, $0 \le d < 3$. *Assume that* $(V, \mathcal{B})$ *is an* $MR[v, \{3, k\}]$ *with* b *blocks* $(b \le 2k+1)$ *of size* k. *Then*

$$v \ge \min\{k \cdot p(t + ck/2) - (2t + ck) \mid d \le c \le \frac{2}{3}(2k+1)\}.$$

Proof. Let $c = b - \lceil \frac{v}{k} \rceil$ be defined as in Proposition 2. This proposition and Lemma 2.2 imply $d \le c \le \frac{2}{3}(2k+1)$. From the last proof one knows that there exists a partial design on the same point set V with b blocks of size k which has the property that precisely $u = t + \frac{1}{2}ck$ points lie on exactly 3 blocks of size k. The smallest number of blocks required to construct such a partial design is $p(u)$. These blocks cover $k \cdot p(u) - 2u$ distinct points. Hence, $v \ge k \cdot p(t + ck/2) - (2t + ck)$. □

We have now finished proving Corollary 2.3 and are about ready to prove Theorem 2. For convenience, we restate this theorem, which allows us (in addition to Theorem 1) to prove nonexistence of an *MR* just by doing a few simple calculations.

Theorem 2 . *Suppose that there exists a mandatory representation design* $MR[v, \{3, k\}]$. *If* $k \equiv 2 \pmod 6$ *and* v *is even, then there are no (further) congruential conditions on* v. *Let* $v \equiv -2t \pmod k$, $0 \le t < k/2$; *and let* $d \equiv -\lceil \frac{v}{k} \rceil \pmod 3$ *if* $v \equiv 0, 4 \pmod 6$ *and* $d \equiv -\lceil \frac{v}{k} \rceil + 1 \pmod 3$ *if* $v \equiv 2 \pmod 6$, $0 \le d < 3$. *Then*

$$v \ge k \cdot p(t + \frac{1}{2}dk) - 2(t + \frac{1}{2}dk).$$

Proof. We examine the partial design consisting of all of the blocks of size k. Let b be the number of blocks. For the sake of brevity, we write t^* instead of $t + \frac{1}{2}dk$. Since the case $t^* = t$ is already solved in Theorem 1, we may assume that $\frac{1}{2}k \le t^* < \frac{3}{2}k$.

First, consider the case $b > 2k+1$. By Corollary 2.1 one knows that $v > \frac{1}{3}k(2k+1)$. If $k = 8, 14$ or 20, then it is easy to check that $v \ge k \cdot p(t^*) - 2t^*$ for $\frac{1}{3}k(2k+1) < v < k \cdot p(\frac{3}{2}k - 1) - k$. This is explicitly done in Table 2.

If $k \ge 26$, then $p(\frac{3}{2}k - 1) - 1 \le \frac{1}{3}(2k+1)$. Therefore, it follows

$$v > \frac{1}{3}k(2k+1) \ge k \cdot p(\frac{3}{2}k - 1) - k \ge k \cdot p(t^*) - 2t^*.$$

Now, concentrate on $b \le 2k+1$. By Corollary 2.3 we will be done if we can show that for each $d \le c \le \frac{2}{3}(2k+1)$ ($d = 1$ or 2),

$$k \cdot p(t + \frac{1}{2}ck) - (2t + ck) \ge k \cdot p(t^*) - 2t^*$$

or, equivalently,

$$p(t + \frac{1}{2}ck) \ge p(t^*) + c - d. \tag{1}$$

$k=8, 46 \le v < 64$			$k=14, 136 \le v < 154$			$k=20, 274 \le v < 280$		
v	t^*	$kp(t^*)-2t^*$	v	t^*	$kp(t^*)-2t^*$	v	t^*	$kp(t^*)-2t^*$
52	10	52	136	16	122	274	13	174
54	9	54	138	15	124	276	12	156
58	7	42	142	13	114	278	21	218
60	6	44	144	12	102			
62	9	54	146	18	132			
			148	10	106			
			150	9	108			
			152	15	124			

TABLE 2. $k = 8, 14, 20$; $\frac{1}{3}k(2k+1) < v < k \cdot p(\frac{3}{2}k-1) - k$; $\frac{1}{2}k \le t^* < \frac{3}{2}k$

This is true for $c = d$, so we may assume $c > d$. It is obvious from the definition that $p(m) > n$ whenever $3m > \binom{n}{2}$. Thus, to prove inequality (1) it is sufficient to show that

$$3(t + \frac{1}{2}ck) > \frac{(p(t^*)+c-d-1)(p(t^*)+c-d-2)}{2}$$

$$= \frac{p(t^*)(p(t^*)-4)}{2} + \frac{(2(c-d)+1)p(t^*)+(c-d-1)(c-d-2)}{2}.$$

By Lemma 1.1 the followings holds

$$\frac{p(t^*)(p(t^*)-4)}{2} < 3t^*$$

so that we need only show

$$3(c-d)k \ge (2(c-d)+1)p(t^*) + c^2 - (2d+3)c + d^2 + 3d + 2$$

where $d + 1 \le c \le \frac{2}{3}(2k+1)$.

First, we prove that this is true for $k = 8$ and 14. Since $t^* < \frac{3}{2}k$, we have $p(t^*) \le k+1$; the roots of the quadratic $c^2 - (k+2d+1)c + (d+1)k + d^2 + d + 3$ are $\frac{1}{2}\left(k + 2d + 1 \pm \sqrt{k^2 - 2k - 11}\right)$. Since $k \ge 8$, we have $k^2 - 2k - 11 > (k-2)^2$; whence (1) is true for all c such that $d + 3/2 \le c \le k + d - 1/2$. For $c = d+1$ and $k + d \le c \le \frac{2}{3}(2k+1)$ see Table 3.

If $k \ge 20$, then $p(t^*) \le p(\frac{3}{2}k-1) < \frac{2}{3}k+2$. Now, the roots of $c^2-(\frac{5}{3}k+2d-1)c+(\frac{5}{3}d+\frac{2}{3})k+d^2-d+4$ are $\frac{1}{2}\left(\frac{5}{3}k + 2d - 1 \pm \sqrt{(\frac{5}{3}k)^2 - 6k - 15}\right)$. Since $k \ge 20$, we have $(\frac{5}{3}k)^2 - 6k - 15 \ge (\frac{5}{3}k - 3)^2$; whence

$$0 \ge c^2 - (\frac{5}{3}k + 2d - 1)c + (\frac{5}{3}d + \frac{2}{3})k + d^2 - d + 4$$

whenever $d + 1 \le c \le \frac{2}{3}(2k+1) \le \frac{5}{3}k + d - 2$. Thus, inequality (1) is valid for all $k \equiv 2 \pmod 6$ and all $d \le c \le \frac{2}{3}(2k+1)$, and the proof is complete. $\square$

$k=8$				$k=14$			
d	c	$p(m)$	$p(m^*)+c-d$	d	c	$p(m)$	$p(m^*)+c-d$
1	2	9	8	1	2	12	11
	9	17	15		15	27	24
	10	17	16		16	28	25
	11	18	17		17	29	26
					18	29	27
					19	30	28
2	3	11	10	2	3	14	13
	10	17	17		16	28	27
	11	17	18		17	29	28
					18	29	29
					19	30	30

TABLE 3. $k=8,14$; $m=\frac{c+1}{2}k-1$, $m^*=\frac{d+1}{2}k-1$

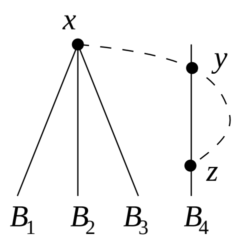

FIGURE 1. Point pair x, y covered twice

The usefulness of the main theorem should be illustrated in a short example.

Example: There exists no $MR[30, \{3, 8\}]$, since $t = 1, d = 2, t^* = 9$ and $p(t^*) = 9$ imply $30 < 54 = k \cdot p(t^*) - 2t^*$.

It is a simple observation that for a particular class of parameter sets the necessary conditions in Theorem 1 case (iv) and Theorem 2 are not sufficient.

Lemma 2.4. *If $k > 2$ is an even integer, then there exists no $MR[4k-2, \{3, k\}]$.*

Proof. Assume for the contrary that there exists an $MR[4k-2, \{3, k\}]$ with b blocks of size k. Clearly, $4 \leq b$ since every point must be covered. On the other hand, $b \leq 4$ since otherwise, the blocks of size k cover more than v points when $k > 4$; and, if $k = 4$, then $v = 4k-2$ does not satisfy Wilson's necessary conditions. Hence, $b = 4$ and there exists a point $x \in V$ that lies on exactly 3 blocks of length k, say B_1, B_2, B_3, and on exactly $k/2$ blocks of length 3. Let $\{x, y, z\}$ be such a block of size 3; then y and z do not lie on B_1, B_2 or B_3. Thus, $\{y, z\} \subset B_4$ the 4-th block of length k (see Figure 1), contradicting the fact that every pair of points occurs in exactly one block. $\square$

3. Sufficiency

The necessary conditions stated in Theorem 1, Theorem 2 and Lemma 2.4 are sufficient for $4 \leq k \leq 50$, with one exception. This exceptional case (Proposition 3) was proved in a recent paper of Rees [6]. He also showed that the conditions in Theorem 1 (iii) (k is even and v odd) are sufficient for $v \geq \frac{1}{2}k^2 + 6k$.

Proposition 3. *There exists no* $MR[21, \{3, 6\}]$.

The construction part for $4 \leq k \leq 50$ was done by using various recursive constructions and existence results from the *CRC Handbook of Combinatorial Designs* [5]. In particular, recursive constructions are obtained from Chapter *III.2 PBDs: Recursive Constructions*, (pp. 193-202) and we also used the known existence results from Section *III.1.3 Some Families of Group Divisible Designs*, (pp. 189-191). All necessary definitions, methods and examples are given there.

The ingredients for recursive constructions are obtained by a non exhaustive search technique, called hill-climbing. Starting with the partial design consisting of all the blocks of size k, the hill-climbing algorithm adds randomly the missing blocks of size 3. For more details see [2], for example.

4. Conclusions

The accompanying table gives a summary of the results obtained for $4 \leq k \leq 20$. A table with complete results ($k \leq 50$) is located at ftp://ftp.math.uni-rostock.de/pub/members/mgruttm/closure.htm . We point out that no references are included, see e.g. [1] or [5].

Notation: N is used when the closure (the set of all integers v for which an $MR[v, \{3, k\}]$ exists) contains all sufficiently large integers and $x_1, x_2, \ldots, x_n \mod y$ denotes the set of all nonnegative integers which are congruent x_1, or $x_2, \ldots$, or x_n modulo y.

3 k	Closure	Genuine Exceptions
3 4	0 1 mod 3	6 7 9
3 5	1 mod 2	7 9
3 6	0 1 mod 3	4 7 9 10 12 13 15 19 21 22
3 7	1 3 mod 6	9 13
3 8	N	4 5 6 7 9 10 11 12 13 14 15 16 17 18 19 20 21 23 25 26 27 28 29 30 31 33 34 35 36 38
3 9	1 3 mod 6	7 13 15
3 10	0 1 mod 3	4 6 7 9 12 13 15 16 18 19 21 22 24 25 27 31 33 34 36 37 39 42 43 45 49 51
3 11	1 mod 2	5 7 9 13 15 17 19 21 25 27
3 12	0 1 mod 3	4 6 7 9 10 13 15 16 18 19 21 22 24

		25 27 28 30 31 33 37 39 40 42 43 45 46 49 51 52 54 55 57 61 63 67 69 73 75
3 13	1 3 mod 6	7 9 15 19 21 25
3 14	N	4 5 6 7 8 9 10 11 12 13 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 41 43 44 45 46 47 48 49 50 51 52 53 54 55 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 77 79 80 81 83 85 87 88 89 90 91 93 94 95 96 97 99 100 101 103 104 107 110
3 15	1 3 mod 6	7 9 13 19 21 25 27
3 16	0 1 mod 3	4 6 7 9 10 12 13 15 18 19 21 22 24 25 27 28 30 31 33 34 36 37 39 40 42 43 45 49 51 52 54 55 57 58 60 61 63 66 67 69 70 72 73 75 79 81 82 84 85 87 91 93 97 99 103 105 109 111 115 117 121 123 127 129 133 135
3 17	1 mod 2	5 7 9 11 13 15 19 21 23 25 27 29 31 33 37 39 43 45
3 18	0 1 mod 3	4 6 7 9 10 12 13 15 16 19 21 22 24 25 27 28 30 31 33 34 36 37 39 40 42 43 45 46 48 49 51 55 57 58 60 61 63 64 66 67 69 70 73 75 76 78 79 81 82 84 85 87 91 93 94 96 97 99 103 105 109 111 115 117 121 123 127 129 133 135 139 141 145 147 151 153 157 159 163 165 169
3 19	1 3 mod 6	7 9 13 15 21 25 27 31 33 37
3 20	N	4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 59 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 113 115 116 117 119 121 122 123 124 125 126 127 129 130 131 132 133 135 136 137 138 139 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 157 158 159 161 163 164 165 167 169 170 171 173 175 177 179 181 183 184 185 186 187 189 190 191 192 193 195 196 197 198 199 201 203 205 206 207 209 215

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