On simple and supersimple transversal designs

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Abstract

In this paper, the existence of a transversal design $TD_\lambda(4,g)$ is proved for all indices λ satisfying $2 \leq \lambda \leq g$ such that any two of its blocks intersect in at most two elements.

1 Introduction

For any positive integer g, let $\Omega_g = \{1, 2, \dots, g\}$. A set is called a g-set when it is of size g. A tuple $\underline{x}$ is called a k-tuple if it is of size k. Its entries shall be denoted by $x_1, \dots, x_k$.

Let $k \geq 2$, g and λ be positive integers. A transversal design $TD_\lambda(k,g)$ is a collection of subsets of size k (called blocks) chosen from a kg-set, where the kg-set is partitioned into disjoint subsets (called groups) of size g such that each block meets every group exactly once, and any two elements from disjoint groups occur together in precisely λ blocks. As usual, the index λ is suppressed when it equals one.

It is well-known that a $TD(k,g)$ is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order g. For a list of lower bounds on the number of MOLS for all orders up to 10,000 we refer to [6]. In 1960, Bose, Shrikhande and Parker [3] proved the existence of two MOLS, i.e. a $TD(4,g)$ for all $g \neq 2, 6$. As a consequence, it is easy to obtain a $TD_\lambda(4,g)$ for $\lambda > 1$: just take λ copies of a $TD(4,g)$ and join them. The remaining cases, namely $g = 2$ and 6 were settled by Hanani [9] which conclude the existence of a $TD_\lambda(4,g)$ iff the pair (λ, g) is neither $(1,2)$ nor $(1,6)$.

The transversal designs obtained in this way usually contain repeated blocks. This raises the question whether there exist transversal designs of larger index without multiple blocks. Given a positive integer r, we call a transversal design $TD_\lambda(k,g)$ r-simple iff any two distinct blocks have less than r elements in common. For brevity, k-simple and 3-simple designs are said to be simple and supersimple, respectively.

Supersimple transversal designs were introduced and first studied by Gronau and Mullin [8]. They conjectured the existence of a supersimple $TD_\lambda(4,g)$ iff $\lambda \leq g$ and (λ, g) is different from $(1,2)$ and $(1,6)$. Our objective is to prove the Gronau-Mullin conjecture.

2 Transversal designs and orthogonal arrays

In the sequel, a k-by-n matrix with entries from a g-set G shall be called an array of size n, degree k and order g (over G). An orthogonal array $OA_\lambda(t,k,g)$ is an array of size λg^t, degree k and order g (over a set G) having the property that every t-tuple with entries from G appears exactly λ times as a column in every t-by-λg^t submatrix. The parameters λ and t are the index and the strength of the orthogonal array, respectively. In this notation, if λ is omitted it is understood to be one, and if t is omitted it is understood to be two.

Orthogonal arrays and transversal designs have the following connection: A transversal design $TD_\lambda(k,g)$ exists iff an orthogonal array $OA_\lambda(k,g)$ exists. For more detailed information, we refer to [6]. As it is more convenient for our purposes here to work with orthogonal arrays, all results are stated in terms of arrays.

For further reference, we record some standard results on orthogonal arrays.

Theorem 1 (Bush). *If q is a prime power and $t \leq q$, then an $OA(t,q+1,q)$ exists. Moreover, if q is even, an $OA(3,q+2,q)$ exists, too.*

Theorem 2 (MacNeish, Bose). *If orthogonal arrays $OA_\lambda(t,k,g)$ and $OA_\mu(t,k,h)$ exist, then an $OA_{\lambda\mu}(t,k,gh)$ exists, too.*

The first theorem is due to Bush [5]. Deleting rows from an orthogonal array $OA_\lambda(t,k,g)$ yields again an orthogonal array $OA_\lambda(t,l,g)$ for any integer l with $t \leq l \leq k$. The second theorem is due to MacNeish [12] for $t=2$, and to Bose [2] (see also [4]).

We call an array r-simple iff any two different columns agree in less than r entries. If an array of degree k is k-simple, i.e. contains no repeated columns, it is said to be simple, for short. An orthogonal array $OA(t,k,g)$ of index one is obviously t-simple, but not r-simple for any $r < t$. Hence when considering the simplicity of orthogonal arrays we are usually interested in orthogonal arrays of index at least two. The problem is to find the smallest r $(t+1 \leq r \leq k+1)$ such that there is an r-simple $OA_\lambda(t,k,g)$. If, in particular, an orthogonal array $OA_\lambda(t,k,g)$ is $(t+1)$-simple, we call it supersimple.

Rosenberg [13] proved that an $OA_\lambda(t,k,g)$ exists whenever $\lambda \geq 2^{k-2}g^{2k-t}$ It is noteworthy that this result does not apply to r-simple orthogonal arrays when $r \leq k$: here the index is bounded from above by $\lambda \leq g^{r-t}$.

3 Supersimple orthogonal arrays from difference matrices

An array $D = (d_{ij})$ $(i \in \Omega_k,\, j \in \Omega_{\lambda g})$ of size λg, degree k and order g over a group G is said to be a (g, h, λ)-difference matrix iff each of the multisets $\{d_{ij} - d_{i'j} : j \in \Omega_{\lambda g}\}$ with $i \neq i'$ contains every element from G precisely λ times. Jungnickel [10] proved that every (g, k, λ)-difference matrix yields an orthogonal array $OA_\lambda(k + 1, g)$ of strength two. For further details, the reader may consult [1].

Note, that a difference matrix D is r-simple iff the multisets $\{d_{ij} - d_{ij'} : i \in \Omega_k\}$ with $j \neq j'$ contain every element from G at most $r - 1$ times.

Lemma 3. *If an r-simple (g, k, λ)-difference matrix exists, then an r-simple $OA_\lambda(k, g)$ exists, too.*

Proof. Let $D = (d_{ij})$ be an r-simple (g, k, λ)-difference matrix over a group G. Following Jungnickel [10], we obtain an orthogonal array $OA_\lambda(k, g)$ by joining all the matrices $D(x) = (d_{ij} + x)$ where $x \in G$. It remains to check the simplicity of the orthogonal array. Let $\underline{d}_j(x)$ and $\underline{d}_{j'}(x')$ be two distinct columns in submatrices $D(x)$ and $D(x')$, respectively. Whenever they agree in a position $i \in \Omega_k$, we obtain $d_{ij} - d_{ij'} = x' - x$. But, the multiset $\{d_{ij} - d_{ij'} : i \in \Omega_k\}$ contains every element from G at most $r - 1$ times. This implies that the constructed orthogonal array is r-simple. □

An array H of order λg, degree λg and order g is said to be a (g, λ)-generalized Hadamard matrix iff both H and H^T are $(g, \lambda g, \lambda)$-difference matrices. Obviously, every (g, λ)-generalized Hadamard matrix is $(\lambda + 1)$-simple. We now list two existence results for generalized Hadamard matrices to be used later on. The first theorem is due to Drake [7], the second one to Jungnickel [10].

Theorem 4 (Drake). *If p is a prime, $i \geq 1$ and j are non-negative integers, then a (p^i, p^j)-generalized Hadamard matrix exists.*

Theorem 5 (Jungnickel). *If q is an odd prime power, then a $(q, 2)$-generalized Hadamard matrix exists.*

We are now in a position to state a first result on supersimple orthogonal arrays.

Lemma 6. *If q is a prime power, then a supersimple $OA_2(2q, q)$ exists.*

Proof. By Jungnickel's theorem there is a $(q, 2)$-generalized Hadamard matrix for odd q, while a $(q, 2)$-generalized Hadamard matrix for even q exists by Drake's theorem. Clearly, both are 3-simple. By Lemma 3 this yields a 3-simple $OA_2(2q, q)$. □

As an immediate consequence of Lemma 6 and MacNeish's theorem, we obtain the following result.

Corollary 7. *Let g be a positive integer, and $g = q_1 \cdots q_m$ be a factorization of g into prime power factors. Moreover, let k be a positive integer satisfying $k - 1 \leq q_i$ for every $i \in \Omega_{m-1}$ and $k \leq 2q_m$. Then a supersimple $OA_2(k, g)$ exists.*

A corollary to the last result is as follows.

Corollary 8. *A supersimple $OA_2(4, g)$ exists for every integer $g \geq 2$.*

4 Orthogonal arrays of larger index

We start with the following important result.

Lemma 9. *Let $\lambda \geq 2, t, k, g$ and r be positive integers. An r-simple orthogonal array $OA_\lambda(t, k, g)$ exists only if*

$$\binom{k-t+u}{u} \leq \frac{\lambda g^u}{\lambda - 1} \binom{r-t+u-1}{u}$$

holds for all positive integers $u \leq t$.

Proof. Suppose, there exists an r-simple $OA_\lambda(t, k, g)$ which will be denoted by A, for short. There are exactly $\binom{k}{t}$ possible ways to choose a t-by-λg^t submatrix of A. Any two columns of A agree in all rows of at most $\binom{r-1}{t}$ of these submatrices.

Now, fix a column $\underline{a}$ of A. Let B be a t-by-λg^t submatrix of A. Here, $\underline{a}$ induces a column $\underline{b}$ which is of course a t-tuple. This t-tuple occurs in B $(\lambda - 1)$ more times. When considering all t-by-λg^t submatrices simultanously, we obtain

$$(\lambda - 1)\binom{k}{t} \leq (\lambda g^t - 1)\binom{r-1}{t}.$$

Finally, an r-simple $OA_\lambda(t, k, g)$ yields an $(r - s)$-simple $OA_\lambda(t - s, k - s, g)$ for every positive integer $s < t$. This gives us

$$(\lambda - 1)\binom{k-s}{t-s} \leq (\lambda g^{t-s} - 1)\binom{r-s-1}{t-s}$$

and thus the claimed inequality. □

Lemma 9 in particular implies

$$k \leq \frac{\lambda g}{\lambda - 1}(r - t) + t - 1$$

for the degree of an r-simple $OA_\lambda(t, k, g)$ with index $\lambda > 1$. Thus, the orthogonal arrays claimed by Lemma 6 have maximum possible degree.

An orthogonal array $OA_\lambda(t, k, g)$ is said to be completely reducible iff it is the union of λ orthogonal arrays $OA(t, k, g)$ of index one.

Lemma 10. *If an $OA(t, k, g)$ exists, then there is also a completely reducible t-simple $OA_{g^s}(t - s, k - s, g)$ for every non-negative integer $s < t$.*

Proof. Consider an $OA(t, k, g)$ over a g-set G. There are exactly g^s pairwise distinct s-tuples with entries from G. Fix one of them and consider all columns of the $OA(t, k, g)$ containing this fixed s-tuple in their first s rows. When deleting the first s rows these columns form an $OA(t - s, k - s, g)$. When repeating this procedure for each s-tuple and joining the resulting orthogonal arrays, we obtain a completely reducible $OA_{g^s}(t - s, k - s, g)$. Moreover, it is t-simple since deleting rows in an array does not affect its simplicity. □

Lemma 11. *If $\lambda \geq 2$, t, k, r are positive integers with $2 \leq t < r \leq k$, then there is a constant $q_0 = q_0(\lambda, t, k, r)$ such that an r-simple $OA_\lambda(t, k, q)$ exists for all prime powers $q \geq q_0$.*

Proof. Put $q_0 = \max\{\lambda^{1/(r-t)}, k + r - t - 1\}$. By Bush's theorem, we find an $OA(r, q{+}1, q)$ since $r \leq q$. Thus, Lemma 10 yields an r-simple $OA_\lambda(t, q{+}1{-}(r{-}t), q)$ for all $\lambda \leq q^{r-t}$. This concludes the proof since $k \leq q - r + t + 1$. □

To continue with, we shall present two generalizations of Bose's theorem. For any two k-tuples $\underline{a}$ and $\underline{b}$, let their product be the k-tuple $\underline{a} \otimes \underline{b}$ with entries $a_i b_i$, $i \in \Omega_k$. Let A and B be two arrays of degree k, sizes m and n, and orders g and h, respectively. Their product shall be the array $A \otimes B$ of degree k, size mn and order gh consisting of all the k-tuples $\underline{a} \otimes \underline{b}$ as columns, whereby $\underline{a}$ is any column in A, and $\underline{b}$ in B.

Lemma 12. *Let $A_1, \ldots, A_m$ be r-simple arrays whose union forms an $OA_\lambda(t, k, g)$, and let $B_1, \ldots, B_m$ be orthogonal arrays $OA_\mu(t, k, h)$, whose union forms an r-simple $OA_{m\mu}(t, k, h)$. Then there exists an r-simple $OA_{\lambda\mu}(t, k, gh)$.*

Proof. Consider the arrays $A_i \otimes B_i$ $(i \in \Omega_m)$ and denote their union by C. It is easily verified that C is an $OA_{\lambda\mu}(t, k, gh)$. It remains to prove that C is r-simple. Let $\underline{c} = \underline{a} \otimes \underline{b}$ and $\underline{c}' = \underline{a}' \otimes \underline{b}'$ be two distinct columns of C. Suppose $\underline{b}$ and $\underline{b}'$ are distinct columns of the orthogonal array given by the union of $B_1, \dots, B_m$. Then they agree in less than r entries, and so do $\underline{c}$ and $\underline{c}'$. Conversely, let $\underline{b}$ and $\underline{b}'$ denote the same column. Then $\underline{a}$ and $\underline{a}'$ lie in the same array A_i, $i \in \Omega_m$. Therefore they agree in less than r entries, which again holds for $\underline{c}$ and $\underline{c}'$, too. This proves the resulting array C to be r-simple. □

Lemma 13. *Let $A_1, \dots, A_m$ be r-simple orthogonal arrays $OA_{\lambda_i}(t, k, g)$, $i \in \Omega_m$, and let $B_1, \dots, B_m$ be orthogonal arrays $OA_{\mu_i}(t,k,h)$, $i \in \Omega_m$, whose union forms an r-simple $OA_\mu(t, k, h)$ with $\mu = \mu_1 + \ldots + \mu_m$. Then there exists an r-simple $OA_\sigma(t, k, gh)$ with $\sigma = \lambda_1\mu_1 + \ldots + \lambda_m\mu_m$.*

Proof. Let us consider the orthogonal arrays $A_i \otimes B_i$, $i \in \Omega_m$. By Lemma 12, each of them is an r-simple $OA_{\lambda_i\mu_i}(t, k, gh)$. Denote their union by C which is of course an $OA_\sigma(t, k, gh)$. Let $\underline{c} = \underline{a} \otimes \underline{b}$ and $\underline{c}' = \underline{a}' \otimes \underline{b}'$ be two distinct columns in C. If $\underline{c}$ and $\underline{c}'$ lie in the same subarray $A_i \otimes B_i$ $(i \in \Omega_m)$, then they have less than r entries in common. Otherwise, $\underline{c}$ and $\underline{c}'$ belong to different subarrays of C. But then $\underline{b}$ and $\underline{b}'$ are distinct, and by assumption agree in less than r entries. Thus, C is r-simple, too. □

Let G be a group, $\underline{a}$ be a k-tuple over G and $\underline{b}$ be an l-tuple over G. Their sum will be the tuple $\underline{c} = \underline{a} + \underline{b}$ over G with entries

$$c_i = \begin{cases} a_i + b_i & \text{if } i \in \Omega_k \cap \Omega_l, \\ a_i & \text{if } i \in \Omega_k \backslash \Omega_l, \\ b_i & \text{otherwise.} \end{cases}$$

Let A and B be two arrays of degrees k and l, sizes m and n over G, respectively. Their sum shall be the array $A + B$ of degree $\max\{k, l\}$ and size mn consisting of all the tuples $\underline{a} + \underline{b}$ as columns, whereby $\underline{a}$ is any column in A, and $\underline{b}$ in B.

Lemma 14. *If an $OA(t, k, g)$ and an u-simple array B of size $\lambda \geq 2$, degree $l \leq k$ and order g exist, then a completely reducible r-simple $OA_\lambda(t, k, g)$ exists for $r = \max\{t + l, u + k - l\}$.*

Proof. Let A be an $OA(t, k, g)$ and C denote the sum $A + B$. It is easy to check that C is a completely reducible $OA_\lambda(t, k, g)$. Let $\underline{c} = \underline{a} + \underline{b}$ and $\underline{c}' = \underline{a}' + \underline{b}'$ be two distinct columns in C. If $\underline{a} = \underline{a}'$, then $\underline{c}$ and $\underline{c}'$ agree in less than $u + (k - l)$ entries. Conversely, if $\underline{a} \neq \underline{a}'$, then $\underline{c}$ and $\underline{c}'$ have less than $l + t$ entries in common. This concludes the proof. □

As immediate consequences we state the following results.

Corollary 15. *If an $OA(t,k,g)$ as well as an $OA(u,l,g)$ with $l \leq k$ exist, then there is also a completely reducible r-simple $OA_{g^u}(t,k,g)$ for $r = \max\{t+l, u+k-l\}$.*

Corollary 16. *If an $OA(t,k,g)$ exists, then a completely reducible simple $OA_{g^{k-t}}(t,k,g)$ as well as a completely reducible $\lceil \frac{k-t+1}{2} \rceil$-simple $OA_g(t,k,g)$ exist, too.*

Proof. Both claims follow from Corollary 15. Note that both, an $OA(1,l,g)$ and an $OA(l,l,g)$ exist for all positive integers l and g. □

5 Main results

We are now in the position to establish our results on the existence of simple and supersimple orthogonal arrays of degree 4.

Theorem 17. *A supersimple orthogonal array $OA_\lambda(4,g)$ exists iff $\lambda \leq g$ and (λ, g) is neither $(1,2)$ nor $(1,6)$.*

Proof. For $\lambda = 1$, the claim follows from the well-known result of Bose, Shrikhande and Parker [3]. Hence it suffices to consider all λ with $2 \leq \lambda \leq g$. We shall proceed by induction on g. By Lemma 8, there exist supersimple $OA_2(4,2)$ and $OA_2(4,3)$. In addition, Bush's theorem yields an $OA(3,4,3)$ which is a supersimple $OA_3(4,3)$, too.

Suppose the statement is true for all integers less than $g \geq 4$. So we shall verify it for g, now. If g is a prime power, Bush's theorem gives us an $OA(3,5,g)$, and by Lemma 3 this yields a supersimple $OA_\lambda(4,g)$ for all $\lambda \leq g$. Otherwise, let $q < g$ denote the largest prime power dividing g. Hence, $g = qh$ holds for some integer $h \geq 2$.

First, assume $g = 6$. If λ is even, we have both, a supersimple $OA_2(4,2)$ and a supersimple $OA_{\lambda/2}(4,3)$. By Lemma 13 their product is a supersimple $OA_\lambda(4,6)$. For $\lambda = 3$ or 5, appropriate orthogonal arrays are to be found in the Appendix.

Next, assume $q \geq 4$ and $h = 2$. If λ is even, we again have a supersimple $OA_2(4,2)$ and a supersimple $OA_{\lambda/2}(4,q)$ whose product is a supersimple $OA_\lambda(4,g)$. Conversely, let λ be odd. Let A_1 and A_2 denote two supersimple arrays of degree 4 and order two whose union forms an $OA_3(4,2)$.

$$\begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \\ 1101001 \end{pmatrix} \qquad \begin{pmatrix} 00011 \\ 00101 \\ 01001 \\ 10001 \end{pmatrix}$$

Figure 1. An example for the supersimple arrays A_1 and A_2.

If we join them with $(\lambda - 3)/2$ copies of a supersimple $OA_2(4, 2)$ we obtain an $OA_\lambda(4, 2)$. On the other hand, we have a completely reducible supersimple $OA_q(4, q)$ by Lemma 3. Applying Lemma 12 we obtain a supersimple $OA_\lambda(4, g)$.

Finally, assume $q \geq 4$ and $h \geq 3$. Let $\lambda = mh+\mu$ for some integers $0 \leq m \leq q-1$ and $\mu \in \Omega_h$. Choose m copies of a supersimple $OA_h(4, h)$ and a supersimple $OA_\mu(4, h)$ which exist by the induction hypothesis. Moreover, we find a completely reducible supersimple $OA_q(4, q)$. By Lemma 13, this yields a supersimple $OA_\lambda(4, g)$ and concludes our proof. □

The next two results are about simple orthogonal arrays. Let C be the array consisting of all pairwise distinct k-tuples over a g-set G. Obviously C is an $OA_{g^{k-t}}(t, k, g)$ for every positive integer $t \leq k$. Suppose C is partioned into two submatrices A and B. If A is an $OA_\lambda(t, k, g)$, then B is an $OA_{g^{k-t}-\lambda}(t, k, g)$. Hence the existence of a simple $OA_\lambda(t, k, g)$ always yields a simple $OA_{g^{k-t}-\lambda}(t, k, g)$, too.

Theorem 18. *A simple orthogonal array $OA_\lambda(4, g)$ exists iff $\lambda \leq g^2$ and (λ, g) is non of the following: $(1, 2)$, $(1, 6)$, $(g^2 - 1, 2)$, $(g^2 - 1, 6)$.*

Proof. Let g be different from 2 and 6. Then there exists an $OA(4, g)$, and by Corollary 16, a completely reducible simple $OA_{g^2}(4, g)$.

As mentioned above, it remains to consider only indices $2 \leq \lambda \leq g^2/2$. If $g = 2$, Theorem 17 yields a simple $OA_2(4, 2)$. Finally, let $g = 6$. If $\lambda \leq 18$ is even, we have both, a simple $OA_2(4, 2)$ and a simple $OA_{\lambda/2}(4, 3)$. Their product is a simple $OA_\lambda(4, 6)$ according to Lemma 13. Otherwise, assume λ is odd. Let A_1 and A_2 be two simple arrays whose union forms an orthogonal array $OA_3(4, 2)$. If we join them with an appropriate number of simple $OA_4(4, 2)$ and $OA_2(4, 2)$, we obtain an $OA_\lambda(4, 2)$. However we also have a completely reducible simple $OA_9(4, 3)$. By virtue of Lemma 12 we find a simple $OA_\lambda(4, 6)$. This concludes the proof. □

Theorem 19. *A simple orthogonal array $OA_\lambda(3, 4, g)$ exists iff $\lambda \leq g$.*

Proof. Using Bush's theorem and Bose's theorem we obtain an orthogonal array $OA(3, 4, g)$ for every integer g. Now Corollary 16 yields a completely reducible simple $OA_g(3, 4, g)$ which implies the claim. □

6 Conclusion

In [8] Gronau and Mullin study supersimple $2-(v, 4, 2)$-designs. They construct these designs for all integers $v \geq 7$ with $v \equiv 1 \mod 3$ using supersimple transversal designs of degree 4 and index 2. Unfortunately, as Khodkar [11] pointed out, Theorem 3.1 of [8] is not correct as it stands. This gap in the proof of Theorem A of that paper is now closed by our Corollary 8. However, it should be mentioned that Khodkar [11] gave an alternative proof of the Gronau-Mullin result on supersimple $2-(v,4,2)$-designs.

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Appendix

The following figures show supersimple orthogonal arrays used in the proof of Theorem 17.

$$\left(\begin{array}{l}
111111111111111111222222222222222222333333333333333333\\
111222333444555666111222333444555666111222333444555666\\
124134356256146235356256124134235146124134356256146235\\
564265143321136245312134652456425316612316524435541326
\end{array}\right.\cdots$$

$$\cdots\left.\begin{array}{l}
444444444444444444555555555555555555666666666666666666\\
111222333444555666111222333444555666111222333444555666\\
356256124134235146124134356256146235356256124134235146\\
453542163261236451153451632246625431264623514315341265
\end{array}\right)$$

Figure 2. A supersimple orthogonal array $OA_3(4,6)$.

$$\left(\begin{array}{l}
111111111111111111111111111111222222222222222222222222222222\\
111112222233333444445555566666111112222233333444445555566666\\
123451234612356124561345623456123561234512456123462345613456\\
123452345634561456125612361234214633256143526546315123462145
\end{array}\right.\cdots$$

$$\cdots\begin{array}{l}
333333333333333333333333333333444444444444444444444444444444\\
111112222233333444445555566666111112222233333444445555566666\\
124561345612345234561235612346123461235623456134561234512456\\
653424123512465651233426151346532161635416243214356245145362
\end{array}\cdots$$

$$\cdots\left.\begin{array}{l}
555555555555555555555555555555666666666666666666666666666666\\
111112222233333444445555566666111112222233333444445555566666\\
234561245613456123561234612345134562345612346123451245612356\\
456216134252134324561634523651365145246126135613244365214523
\end{array}\right)$$

Figure 3. A supersimple orthogonal array $OA_5(4,6)$.