Testing bandwidth k for k–connected graphs

Konrad Engel
Fachbereich Mathematik
Universität Rostock
18051 Rostock, Germany
konrad.engel@mathematik.uni-rostock.de

Sven Guttmann
Fachbereich Informatik
Universität Rostock
18051 Rostock, Germany
guttmann@informatik.uni-rostock.de

January 1999

Abstract

We present a linear–time algorithm to decide whether a given k–connected graph has bandwidth k, where k is a fixed positive integer. This improves the general $O(n^{k+1})$–time–algorithm of Saxe for the recognition of bandwidth–k graphs on n vertices under the special supposition of connectivity k.

1 Introduction

In this paper we deal with simple graphs $G = (V, E)$ on n vertices without loops and multiple edges. A *(linear) layout* of G is a bijective mapping $f : V \to \{1, \dots, n\}$. The *bandwidth of a layout* f, denoted by $bw(f)$, is defined to be

$$bw(f) := \max\{|f(u) - f(v)| : \{u, v\} \in E\}.$$

The *bandwidth of the graph* G is

$$bw(G) := \min\{bw(f) : f \text{ is a layout of } G\}.$$

A graph G is *k–connected*, if k is the smallest number such that there exists a subset V' of V with $|V'| = k$ and $G \setminus V'$ is disconnected.

If $bw(f) = k$ (or $bw(G) = k$) we speak briefly of a *bw–k layout* (resp. a *bw–k graph*). A linear–time algorithm for the recognition of bw–2 graphs was given by Garey, Graham, Johnson, and Knuth [2]. Studying the structure of bw–2 graphs we presented in [1] a new algorithm for this problem. Saxe [4] developed an algorithm which recognizes bw–k graphs with complexity $O(n^{k+1})$. In [3] Makedon, Sheinwald, and Wolfsthal designed a simple linear-time algorithm for testing bandwidth 2 for 2–connected graphs.

In this paper we generalize [3] in order to test bandwidth k for k–connected graphs in linear time (k fixed). In the first part we present some general results for bw–k k–connected graphs. It will turn out, that we have to distinguish the cases k odd and even. For the case that k is even we have to work with a more structural characterization of G than in the odd case. Each bw–1 1–connected graph is obviously a chain. Thus let throughout $k \geq 2$.

2 On the structure of bw–k k–conneceted graphs

First we repeat some definitions from graph theory. A vertex v is a *neighbor* of some given vertex u if v is *adjacent* to u, i.e. $\{u, v\}$ is an edge. We denote edges briefly by uv. The *(first) neighborhood* $N^1(u)$ of a vertex $u \in V$ is the set of all neighbors of u. The *degree* $\deg(u)$ of a given vertex u is the number of all neighbors of u, i.e. $\deg(u) = |N^1(u)|$. The *second neighborhood* $N^2(u)$ of a vertex $u \in V$ is the set of all vertices $w \in V \setminus N^1(u)$ such that there is a vertex $v \in N^1(u)$ with $vw \in E$, i.e., the distance between u and w in G is two.

Let $A \subseteq V$. The *boundary of A* is defined to be the set

$$B(A) := \{v \in V \setminus A : \ v \text{ is adjacent to some } w \in A\}.$$

Lemma 1 *Let G be k–connected and $A \subseteq V$. Then*

$$|B(A)| \geq \min\{k, n - |A|\}.$$

Proof. If $|B(A)| < n - |A|$, the deletion of $B(A)$ disconnects the graph. Hence $|B(A)| \geq k$. ■

In particular, we have for a k–connected graph G:

$$\deg(v) \geq k \text{ for all } v \in V. \tag{1}$$

From (1) it follows that $bw(G) \geq k$ if G is a k–connected graph. Thus we have $bw(G) = k$ for such a graph if there is a bw–k layout of G. Let $\Delta(G)$ be the largest degree of a vertex of G. If G is a bw–k graph, then obviously

$$\Delta(G) \leq 2k. \tag{2}$$

The inequality (2) can be clearly tested in linear time. Thus we can throughout suppose this inequality.

Given a layout f of G, let for $j = 1, \ldots, n$,

$$A_j := \{v \in V : f(v) \leq j\}, \tag{3}$$
$$E_j := \{v \in V : f(v) \geq j\}. \tag{4}$$

Lemma 2 *Let G be k–connected and f a bw–k layout of G. Then, for $j = 1, \ldots, n$,*

$$B(A_j) = \{v \in V : f(v) \in \{j+1, \ldots, \min\{n, j+k\}\}\}, \tag{5}$$
$$B(E_j) = \{v \in V : f(v) \in \{\max\{1, j-k\}, \ldots, j-1\}\}. \tag{6}$$

Proof. Let $v \in B(A_j)$. By definition of $B(A_j)$, $f(v) \geq j+1$. Moreover, there exists some $w \in A_j$ with $vw \in E$. Thus $k \geq f(v) - f(w) \geq f(v) - j$, i.e., $f(v) \leq j + k$. Consequently,

$$B(A_j) \subseteq \{v \in V : f(v) \in \{j+1, \ldots, \min\{n, j+k\}\}\}.$$

By Lemma 1, $|B(A_j)| \geq \min\{k, n-j\}$ and hence the inclusion is an equality. The sets E_j can be treated analogously. ■

Note that under the suppositions of Lemma 2 for $1 < j \leq n - k$

$$|B(A_j) \setminus B(A_{j-1})| = 1$$

and that for $v \in B(A_j) \setminus B(A_{j-1})$, $f(v) = j + k$. This vertex v is adjacent to only one vertex in A_j, namely the vertex w with $f(w) = j$.

Often we present layouts f as n–tuples $(u_1, \ldots, u_n)$. Here u_i is the vertex of G with $f(u_i) = i$, $i = 1, \ldots, n$. For a bw–k layout $(u_1 \ldots, u_n)$ the vertex u_1 (resp. u_n) is called *start vertex* (resp. *end vertex*). Note that we have $A_j = \{u_1, \ldots, u_j\}$ and $E_j = \{u_j, \ldots, u_n\}$. We say that u_i is *left* to u_j (and u_j is *right* to u_i) if $i < j$. A *partial layout* of G is a layout of an induced subgraph of G. We present also partial layouts as certain l–tuples $(u_1, \ldots, u_l)$ where $1 \leq l \leq n$. A partial layout

$(u_1, \ldots, u_l)$ is called *start sequence* if $l = k + 1$ and there exists a *bw–k* layout $(u_1, \ldots, u_l, u_{l+1}, \ldots, u_n)$.

The following lemma is a generalization of Lemma 1 in [3] where the case $k = 2$ was considered. It follows immediately from Lemma 2 and the subsequent remark.

Lemma 3 *Let G be a k–connected graph with $n \geq k + 1$.*

(a) Let $f = (u_1, \ldots, u_n)$ be a bw–k layout of G. Then $u_1u_2, \ldots, u_1u_k$, $u_iu_{i+k}, i = 1, \ldots, n - k$, and $u_{n-k+1}u_n, \ldots, u_{n-1}u_n$ are edges in G.

(b) A partial layout $(u_1, \ldots, u_{k+1})$ can be extended in at most one way to a bw–k layout $(u_1, \ldots, u_n)$, i.e., start sequences and bw–k layouts are in one–to–one correspondence.

Given a partial layout $f = (u_1, \ldots, u_l)$ we call the vertices $u_1, \ldots, u_l$ *labeled* and all other vertices *unlabeled*. Let $kToLast(f) := u_{l-k+1}$. For $u \notin f$ let $f||u := (u_1, \ldots, u_l, u)$. The extension of a partial layout for a k–connected graph with $\Delta(G) \leq 2k$ as mentioned in the preceding lemma can be carried out easily in the following way:

Procedure Start–sequence

> **Input:** The partial layout $f = (u_1, \ldots, u_{k+1})$;
>
> While $|f| < |V|$ do
>
> > $x := kToLast(f)$;
> > Let U be the set of unlabeled neighbors of x;
> > If $|U| > 1$ then STOP – $(u_1, \ldots, u_{k+1})$ is not a start sequence;
> > Let $U = \{u\}$;
> > $f := f||u$;
>
> **Output:** The *bw–k* layout f.

Note that we cannot have $U = \emptyset$ in the while loop since otherwise $|B(A)| < k$ for $A := \{u_1, \ldots, x\}$, a contradiction to Lemma 1. The correctness of the procedure follows immediately from Lemma 3. The linear–time complexity is obvious. Thus the main problem is the determination of a start sequence.

Lemma 4 *Let G be a k–connected graph with $n \geq 2k + 1$ and let $f = (u_1, \ldots, u_n)$ be a bw–k layout of G. Then*

$$|N^1(u_1)| = |N^2(u_1)| = k, \tag{7}$$

$$|N^1(u_n)| = |N^2(u_n)| = k. \tag{8}$$

Proof. Take in Lemma 2 (5) $j := 1$ and $j := k + 1$ (resp. in Lemma 2 (6) $j := n$ and $j := n - k$). ∎

Lemma 5 *Let G be a k-connected graph with $n \geq 2k + 1$ and let $f = (u_1, \ldots, u_n)$ be a bw-k layout of G. Then, for $i = 1, \ldots, k$,*

$$\deg(u_i) \leq k + i - 1, \tag{9}$$

$$|N^1(u_i) \cap N^1(u_1)| \geq k - i, \tag{10}$$

$$\deg(u_{n+1-i}) \leq k + i - 1, \tag{11}$$

$$|N^1(u_{n+1-i}) \cap N^1(u_n)| \geq k - i. \tag{12}$$

Proof. We prove only (9) and (10) since (11) and (12) can be treated analogously. Take in Lemma 2 (5) $j := i$. This yields,

$$\deg(u_i) \leq |A_{i-1}| + |B(A_i)| = i - 1 + k.$$

Obviously,

$$|N^1(u_i) \setminus N^1(u_1)| \leq |\{u_1, u_{k+2}, \ldots, u_{i+k}\}| = i.$$

Consequently,

$$|N^1(u_i) \cap N^1(u_1)| = |N^1(u_i)| - |N^1(u_i) \setminus N^1(u_1)| \geq k - i.$$

∎

Theorem 1 *Let $G = (V, E)$ be a k-connected graph with $n \geq 4k + 1$ and let $f = (u_1, \ldots, u_n)$ be a bw-k layout of G. Let $2k + 1 \leq i \leq n - 2k$ and $\deg(u_i) = k$. Let $u_{l_1}, \ldots, u_{l_\lambda}$ with $l_1 > \ldots > l_\lambda$ and $u_{r_1}, \ldots, u_{r_\rho}$ with $r_1 < \ldots < r_\rho$ be the left resp. right neighbors of u_i ($\lambda + \rho = k$). Let $t := k/2$.*

(a) We have $|N^2(u_i)| \geq k$.

(b) If $|N^2(u_i)| = k$ then $\lambda = \rho = t$ and for $j = 1, \ldots, t$,

$$N^1(u_{l_j}) = \{u_{l_j - k}, \ldots, u_{l_1 - k}, u_{l_t}, \ldots, u_{l_1}, u_i, u_{r_1}, \ldots, u_{r_{t-j}}\} \setminus \{u_{l_j}\},$$
$$N^1(u_{r_j}) = \{u_{r_j + k}, \ldots, u_{r_1 + k}, u_{r_t}, \ldots, u_{r_1}, u_i, u_{l_1}, \ldots, u_{l_{t-j}}\} \setminus \{u_{r_j}\}.$$

Proof. Let $M := \{u_{l_\lambda - k}, \ldots, u_{l_1 - k}, u_{r_1 + k}, \ldots, u_{r_\rho + k}\}$. By Lemma 3(a), $M \subseteq N^2(u_i)$, $l_\lambda = i - k$ and $r_\rho = i + k$. Consequently, $|N^2(u_i)| \geq |M| = k$ and (a) is proved.

Thus let $|N^2(u_i)| = k$, i.e. $N^2(u_i) = M$. By Lemma 3(a) $\{u_{l_{\lambda-1}+k}, \ldots, u_{l_1+k}\} \subseteq N^1(u_i) \cup N^2(u_i)$. Hence $N^2(u_i) = M$ implies

$$\{u_{l_{\lambda-1}+k}, \ldots, u_{l_1+k}\} \subseteq \{u_{r_1}, \ldots, u_{r_{\rho-1}}\}.$$

Analogously,

$$\{u_{r_1-k}, \dots, u_{r_{\rho-1}-k}\} \subseteq \{u_{l_{\lambda-1}}, \dots, u_{l_1}\}.$$

From both set inclusions it follows that $\lambda - 1 \leq \rho - 1$ and $\rho - 1 \leq \lambda - 1$, i.e. $\lambda = \rho = k/2 = t$, and that the set inclusions are in fact equalities. In particular, $l_j + k = r_{t-j}$, $j = 1, \dots, t-1$. It remains to study the neighborhood of a left neighbor u_{l_j} of u_i, $1 \leq j \leq t$ (right neighbors can be treated analogously). Obviously,

$$N^1(u_{l_j}) \cap M \subseteq \{u_{l_j-k}, \dots, u_{l_1-k}\}, \tag{13}$$

$$N^1(u_{l_j}) \cap N^1(u_i) \subseteq \{u_{l_t}, \dots, u_{l_1}, u_{r_1}, \dots, u_{r_{t-j}}\} \setminus \{u_{l_j}\}, \tag{14}$$

$$N^1(u_{l_j}) \cap \{u_i\} \subseteq \{u_i\}. \tag{15}$$

Since $N^1(u_{l_j}) \subseteq M \cup N^1(u_i) \cup \{u_i\}$ we have

$$|N^1(u_{l_j})| \leq j + (t + t - j - 1) + 1 = k.$$

Since also $|N^1(u_{l_j})| \geq k$ (recall (1)) we obtain that $|N^1(u_{l_j})| = k$ and all set inclusions in (13)–(15) are equalities, i.e.,

$$N^1(u_{l_j}) = \{u_{l_j-k}, \dots, u_{l_1-k}, u_{l_t}, \dots, u_{l_1}, u_i, u_{r_1}, \dots, u_{r_{t-j}}\} \setminus \{u_{l_j}\},$$

and (b) is proved. ■

From now on we assume that $n \geq 4k+1$. Note that "smaller" graphs can be analyzed by complete enumeration.

Let

$$S := \{v \in V : |N^1(v)| = |N^2(v)| = k\}.$$

From Lemma 4 we know that the start and end vertex of a *bw–k* layout of a k–connected graph belong to S. The set S plays an essential role. If $|S| \leq 1$ we may stop our investigations since the given graph is not a k–connected *bw–k* graph. Given a vertex $s \in S$, it is easy to test whether it is a start vertex:

Procedure Start–vertex

> **Input:** The vertex $s \in S$;
>
> Determine the neighbors $u_1, \dots, u_k$ of s;
>
> For each permutation π of $\{1, \dots, k\}$ do
>
> > If $\deg(u_{\pi(i)}) \leq k+i$ and $|N^1(u_{\pi(i)}) \cap N^1(s)| \geq k-(i+1)$, $i = 1, \dots, k-1$, then **Start–sequence** $(s, u_{\pi(1)}, \dots, u_{\pi(k)})$ and STOP if a *bw–k* layout is found;
>
> **Output:** A *bw–k* layout of G with start vertex s iff there exists one.

The correctness of the procedure follows from Lemma 5. Since there are only $k!$ (i.e. constant many) permutations of the neighbors also this procedure has linear–time complexity.

3 The case k is odd

Since $t = k/2$ is not an integer, we have by Theorem 1(b) $|S| \leq 4k$ if the given k–connected graph has a *bw–k* layout. This leads us to the following easy algorithm:

Algorithm bw–k(odd)

> **Input:** A positive odd integer $k \geq 3$, a k–connected graph $G = (V, E)$ with $|V| \geq 4k + 1$ and $\Delta(G) \leq 2k$;
>
> Determine $S := \{v \in V : |N^1(v)| = |N^2(v)| = k\}$;
>
> If $|S| \leq 1$ or $|S| \geq 4k + 1$ then STOP – G is not a *bw–k* graph;
>
> For all $s \in S$ do
>
> > **Start–vertex**(s) and STOP if a *bw–k* layout is found;
>
> **Output:** A *bw–k* layout of G iff G is a *bw–k* graph.

The linear–time complexity follows from the linear–time complexity of the procedure **Start–vertex**.

4 The case k is even

Let throughout $t := k/2$. In contrast to the preceding case, cycles (for $k = 2$) and their obvious generalizations show that the set S may contain vertices which are "far away" from a start vertex. But these vertices $s \in S$ must have the following property by Theorem 1:

> The elements of $N^1(s) \cup N^2(s)$ can be ordered in the form $(v_{-k}, \ldots, v_{-1}, v_1, \ldots, v_k)$ such that, with $v_0 := s$,

$$N^1(v_i) = \{v_{i-t}, \ldots, v_{i+t}\} \setminus \{v_i\}, \qquad -t \leq i \leq t. \tag{16}$$

We call vertices $s \in S$ with this property *suspicious vertices* and a corresponding sequence $(v_{-k}, \ldots, v_{-1}, v_0, v_1, \ldots .v_k)$ the *bw–k sequence of* s. Note that the inverse sequence $(v_k, \ldots, v_0, \ldots, v_{-k})$ is also a *bw–k* sequence. Suspicious vertices s can be recognized in constant time: After testing $|N^1(s)| = |N^2(s)| = k$ we may either look at all $(2k)!$ permutations of $N^1(s) \cup N^2(s)$ or proceed more elegantly as follows:

Procedure Suspicious-vertex

> **Input:** The vertex $v_0 := s$;

Let G' be the graph induced by $N^1(s) \cup \{s\}$ and consider degrees only in G';

Determine the set Q of vertices of G' of degree t;

If $|Q| \neq 2$ then STOP else denote one element of Q by v_{-t} and the other by v_t;

For $j = t$ to 2 do

Determine the set Q of vertices of G' of degree $k-(j-1)$;

Let Q_1 (resp. Q_2) be the set of vertices from Q which are adjacent to $v_{-t}, \dots, v_{-j}$ (resp. to $v_j, \dots, v_t$);

If $|Q_1| = |Q_2| = 1$ and $Q_1 \cup Q_2 = Q$ then denote the element of Q_1 by $v_{-(j-1)}$ and the element of Q_2 by v_{j-1} else STOP;

For $j = t-1$ to 1 do

If not $v_1, \dots, v_{t-j}$ are adjacent to v_{-j} then STOP;

Consider neighborhoods again in G (and not in G');

For $j = 1$ to t do

If there is some h with $-t-j+1 \leq h \leq -t-1$ such that v_{-j} is not adjacent to v_h then STOP;

If there is some h with $t+1 \leq h \leq t+j-1$ such that v_j is not adjacent to v_h then STOP;

Let $Q_1 := N^1(v_{-j}) \setminus \{v_{-t-j+1}, \dots, v_{t-j}\}$ and $Q_2 := N^1(v_j) \setminus \{v_{t+j-1}, \dots, v_{j-t}\}$;

If $|Q_1| = |Q_2| = 1$ and $|Q_1 \cup Q_2| = 2$ then denote the element of Q_1 by v_{-t-j} and the element of Q_2 by v_{t+j} else STOP;

Output: The *bw–k* sequence $(v_{-k}, \dots, v_{-1}, v_0, v_1, \dots, v_k)$ iff s is suspicious.

It is easy to verify that in each STOP–situation the vertex s is not a suspicious vertex and that (16) holds if there is no STOP–situation. From the procedure it follows that the *bw–k* sequence of a suspicious vertex s is unique up to inversion.

By Theorem 1, each vertex $s \in S$ which is not supicious must be a start vertex or belong to the first or second neighborhood of a start vertex. Hence, if we find a vertex $s \in S$ which is not suspicious, we may proceed as in the case k is odd: We test all vertices from $\{s\} \cup N^1(s) \cup N^2(s)$ for being a start vertex.

Let $P = (v_l, v_{l+1}, \dots, v_r)$ be a path in G. It is said to be a *bw–k path* if $r - l \geq 2k$ and

$$N^1(v_i) = \{v_{i-t}, \dots, v_{i+t}\} \setminus \{v_i\}, \qquad l+t \leq i \leq r-t.$$

Note that the bw–k sequence of a suspicious vertex is a bw–k path. For a bw–k path $P = (v_l, \dots, v_r)$ let

$$L^1(P) := (v_{l+t}, \dots, v_{l+2t-1}),\ L^2(P) := (v_l, \dots, v_{l+t-1}),$$
$$R^1(P) := (v_{r-2t+1}, \dots, v_{r-t}), R^2(P) := (v_{r-t+1}, \dots, v_r).$$

Further let

$$L(P) := L^1(P) \cup L^2(P), \quad R(P) := R^1(P) \cup R^2(P), \quad M(P) := L(P) \cup R(P).$$

Clearly, the vertices $v_{l+2t}, \dots, v_{r-2t}$ of P are suspicious vertices. Let $tToLeft(P) := v_{l+t-1}$ and $tToRight(P) := v_{r-t+1}$. For $u \notin P$ let $u||P := (u, v_l, \dots, v_r)$ and $P||u := (v_l, \dots, v_r, u)$. Given a bw–k path P it is easy to test whether it can be extended:

Procedure Path–extension

> **Input:** The bw–k path P;
>
> $StopLeft := false$;
>
> Repeat
>
> > $x := tToLeft(P)$;
> > If x is not adjacent to all vertices of $L^2(P) \setminus \{x\}$ then $StopLeft := true$
> >
> > else
> >
> > > $LeftU := N^1(x) \setminus L(P)$;
> > > If $LeftU \cap R^2(P) = \emptyset$ and $|LeftU| = 1$ with $U = \{u\}$ then $P := u||P$
> > > else $StopLeft := true$
> >
> > until $StopLeft$
>
> $StopRight := false$;
>
> Repeat
>
> > $x := tToRight(P)$;
> > If x is not adjacent to all vertices of $R^2(P) \setminus \{x\}$ then $StopRight := true$
> >
> > else
> >
> > > $RightU := N^1(x) \setminus R(P)$;
> > > If $RightU \cap L^2(P) = \emptyset$ and $|RightU| = 1$ with $U = \{u\}$ then $P := P||u$ else $StopRight := true$
> >
> > until $StopRight$;
>
> **Output:** The not extendable bw–k path P.

Note that, under the general supposition that G is k–connected, e.g. for $LeftU \cap R^2(P) = \emptyset$ and $|LeftU| = 1$ the new path $u||P$ is indeed a *bw–k* path: Since $\deg(\mathrm{tToLeft(P)}) \geq k$ this vertex $tToLeft(P)$ must be adjacent to all other vertices of $L^2(P)$. Obviously, the procedure **Path–extension** can be carried out in linear time. Let, as in the procedure, for a *bw–k* path $P = (v_l, \ldots, v_r)$,

$$LeftU := N^1(v_{l+t-1}) \setminus L(P),$$
$$RightU := N^1(v_{r-t+1}) \setminus R(P).$$

Lemma 6 *Let G be k–connected and $P = (v_l, \ldots, v_r)$ be a bw–k path. Then $LeftU \neq \emptyset$ and $RightU \neq \emptyset$. If $LeftU \subseteq R^2(P)$ or $RightU \subseteq L^2(P)$ then $V = \{v_l, \ldots, v_r\}$.*

Proof. We consider only $LeftU$. We have $LeftU \neq \emptyset$ because otherwise $\deg(v_{l+t-1}) \leq (t-1) + t = k - 1$, in contrast to (1). Let $LeftU \subseteq R^2(P)$. Let

$$A := \{v_{l+t-1}, \ldots, v_{r-t}\}.$$

Obviously,

$$B(A) := \{v_l, \ldots, v_{l+t-2}, v_{r-t+1}, \ldots, v_r\}.$$

Since $|B(A)| = k - 1$ we have, by Lemma 1, $|A| + |B(A)| = n$, i.e. $A \cup B(A) = V$. ■

Lemma 7 *Let $P = (v_l, \ldots, v_r)$ be a bw–k path in G. If $V = \{v_l, \ldots, v_r\}$, then $f = (v_l, v_r, v_{l+1}, v_{r-1}, v_{l+2}, \ldots)$ is a bw–k layout.*

Proof. Let v_iv_j be an edge $(l \leq i < j \leq r)$. If $\{i, j\} \subseteq \{l, \ldots, l+t-1, r-t+1, \ldots, r\}$, clearly $|f(v_j) - f(v_i)| \leq 2t - 1 < k$. Otherwise, obviously $|j - i| \leq t$. Since $|f(v_{h+1}) - f(v_h)| \leq 2$ for $h = l, \ldots, r-1$, we have

$$|f(v_j) - f(v_i)| \leq \sum_{h=i}^{j-1} |f(v_{h+1}) - f(v_h)| \leq 2t = k.$$

■

Lemma 8 *Let $P = (v_l, \ldots, v_r)$ be a not extendable bw–k path in G such that $LeftU \not\subseteq R^2(P)$ and $RightU \not\subseteq L^2(P)$. Then the vertices $v_l, \ldots, v_{l+2t-1}$ and $v_{r-2t+1}, \ldots, v_r$ are not suspicious.*

Proof. Assume the contrary and let v_i be suspicious where w.l.o.g. $l \leq i \leq l+2t-1$. The neighborhood structure implies that $(v_l, \ldots, v_i, \ldots, v_{i+2t})$ is a subsequence (the right part) of the *bw–k* sequence of v_i (up to inversion). In particular,

$$N^1(v_{l+t-1}) = \{u, v_l, \ldots, v_{l+2t-1}\} \setminus \{v_{l+t-1}\}$$

with some $u \notin \{v_l, \ldots, v_{2l+t-1}\}$. By the supposition, $u \notin R^2(P)$, hence $u \notin P$. Consequently, $u||P$ is also a *bw–k* path, a contradiction to the supposition that P cannot be extended. ■

Recall the sets A_j from (3).

Lemma 9 *Let G be k–connected and $P = (v_l, \ldots, v_r)$ be a bw–k path. Let f be a bw–k layout of G and let $f(v_i) = 1$ for some i with $l + k \le i \le r - k$. Then, for each positive integer p with $l \le i - pt$ and $i + pt \le r$,*

$$A_{1+pk} = \{v_{i-pt}, v_{i-pt+1}, \ldots, v_i, \ldots, v_{i+pt-1}, v_{i+pt}\}, \tag{17}$$

$$\begin{aligned}\{f(v_{i-pt}), \ldots, f(v_{i-(p-1)t-1}), f(v_{i+(p-1)t+1}), \ldots, f(v_{i+pt})\} \\ = \{2 + (p-1)k, \ldots, 1 + pk\},\end{aligned} \tag{18}$$

$$f(v_{i-pt}) > \cdots > f(v_{i-(p-1)t-1}) \text{ and } f(v_{i+(p-1)t+1}) < \cdots < f(v_{i+pt}). \tag{19}$$

Proof. We proceed by induction on p. Let $p = 1$. Then (17) and (18) are satisfied since $v_{i-t}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+t}$ are the neighbors of v_i. Assume, e.g., that for some g with $i - t \le g < g + 1 \le i - 1$, $f(v_g) < f(v_{g+1})$. Let $j := f(v_g)$. Then $B(A_j)$ contains t elements v_h with $h < g$, t elements v_h with $h > i$ and the element v_{g+1}, i.e. at least $k + 1$ elements. This is a contradiction to Lemma 2. Thus also (19) is proved for $p = 1$. Now look at the step $p - 1 \to p$. Since by supposition $2pt \le r - l$, we have $2(p-1)t + 1 \le n - k$ and hence $f(v) \le n - k$ for all $v \in A_{1+(p-1)k}$. The structure of P and Lemma 3 imply that

$$f(v_h) = \begin{cases} f(v_{h+t}) + k & \text{if } i - pt \le h \le i - (p-1)t - 1, \\ f(v_{h-t}) + k & \text{if } i + (p-1)t + 1 \le h \le i + pt, \end{cases}$$

and (17)–(19) follow immediately from the induction hypothesis. ■

Lemma 10 *Let G be k–connected and $P = (v_l, \ldots, v_r)$ be a bw–k path. If $|LeftU| \ge 2$ (resp. $|RightU| \ge 2$) then the vertices v_i with $l + k \le i \le \frac{l+r}{2} - t$ (resp. $\frac{l+r}{2} + t \le i \le r - k$) are not start vertices.*

Proof. We consider only $LeftU$. Assume the contrary and let, for some i with $l + k \le i \le \frac{l+r}{2} - t$ the vertex v_i be a start vertex with a corresponding *bw–k* layout f. Let p be the largest integer such that $l + t - 1 < i - (p-1)t$. Then $l \le i - pt$ which implies $i + pt \le 2i - l \le r - 2t$. Let $j := f(v_{l+t-1})$. From Lemma 9 it follows that

$$A_j = \{v_{l+t-1}, v_{l+t}, \ldots, v_{l+t+j-2}\}$$

and that

$$l + t + j - 2 \le i + pt \le r - 2t.$$

Consequently,

$$B(A_j) = \{v_l, \dots, v_{l+t-2}\} \cup LeftU \cup \{v_{l+t+j-1}, \dots, v_{l+2t+j-2}\}.$$

Since $LeftU \cap \{v_l, \dots, v_{r-t}\} = \emptyset$, it follows

$$|B(A_j)| \geq (t-1) + 2 + t = k + 1,$$

a contradiction to Lemma 2. ■

Recall, that for a *bw–k* path $(v_l, \dots, v_r)$, $M(P) = \{v_l, \dots, v_{l+k-1}, v_{r-k+1}, \dots, v_r\}$.

Lemma 11 *Let G be a k–connected bw–k graph and let $P = (v_l, \dots, v_r)$ be a not extendable bw–k path. Suppose that no vertex of $M(P)$ is a start vertex. Let $LeftU \not\subseteq R^2(P)$ and $RightU \not\subseteq L^2(P)$. Further let $LeftU \cap R^2(P) \neq \emptyset$ or $RightU \cap L^2(P) \neq \emptyset$. Then one of the vertices v_i with $\frac{l+r}{2} - t < i < \frac{l+r}{2} + t$ is a start vertex.*

Proof. Assume the contrary. By the supposition and Lemma 10, no vertex of P is a start vertex (note that e.g. $LeftU \cap R^2(P) \neq \emptyset$ implies $|LeftU| \geq 2$ and that e.g. for $RightU \cap L^2(P) = \emptyset$, $|RightU| \geq 2$ since P cannot be extended). Let f be a *bw–k* layout of G and let w.l.o.g. $v_h \in LeftU \cap R^2(P)$, $r - t + 1 \leq h \leq r$. Let $j := f(v_{l+k})$. By Lemma 3, v_{l+k} is adjacent to vertices with f–values $j-k$ and $j+k$. It is easy to see that by the structure of P these vertices are v_{l+t} and v_{l+k+t}. Let w.l.o.g. $f(v_{l+t}) = j-k$ and $f(v_{l+k+t}) = j+k$. Again the structure of P implies that

$$f(v_i) \begin{cases} < j & \text{if } i \in \{l+t, \dots, l+k-1\}, \\ > j & \text{if } i \in \{l+t+1, \dots, l+k+t\}. \end{cases}$$

By Lemma 3,

$$f(v_{l+t-1}) = f(v_{l+k-1}) - k < j - k.$$

Let ρ be that integer from $\{1, \dots, t\}$ for which $l + t + \rho \equiv h \mod t$. Obviously, $h = l + t + \rho + pt$ where $p \geq 1$. By Lemma 3 (and the fact that $f(v_i) \neq 1$ for $i \in \{l, \dots, r\}$),

$$f(v_h) = f(v_{l+t+\rho}) + pk > j + k.$$

Since v_{l+t-1} and v_h are adjacent we have a contradiction: $f(v_h) - f(v_{l+t-1}) > (j+k) - (j-k) = 2k$. ■

Lemma 12 *Let G be k–connected and $P = (v_l, \dots, v_r)$ be a bw–k path. If $|LeftU \setminus R^2(P)| \geq 2$ and $|RightU \setminus L^2(P)| \geq 2$ then the vertices v_i with $l + k \leq i \leq r - k$ are not start vertices.*

Proof. The proof is similar to the proof of Lemma 10. Assume that for some i with $l + k \leq i \leq r - k$ the vertex v_i is a start vertex with a corresponding *bw–k* layout f. From Lemma 9 it follows that there is some j such that

$$A_j = \{v_{l+t-1}, \dots, v_{l+t+j-2}\} \text{ and } l + t + j - 2 \leq r - t + 1$$

or

$$A_j = \{v_{r-t-j+2}, \ldots, v_{r-t+1}\} \text{ and } r-t-j+2 \geq l+t-1.$$

It is easy to see that $|B(A_j)| \geq k+1$, a contradiction to Lemma 2. ■

Now we may present the algorithm. For a *bw–k* path $P = (v_l, \ldots, v_r)$ let

$$\nu(P) := v_{\lfloor (l+r)/2 \rfloor}.$$

Algorithm bw–k(even)

Input: A positive even integer $k \geq 2$, a k–connected graph $G = (V, E)$ with $|V| \geq 4k+1$ and $\Delta(G) \leq 2k$;

Determine $S := \{v \in V : |N^1(v)| = |N^2(v)| = k\}$; Let $T := S$;

If $|S| \leq 1$ then STOP – G is not a *bw–k* graph;

repeat

Pick a vertex $s_0 \in S$;
Test with **Suspicious-vertex** whether s_0 is suspicious;
If s_0 is not suspicious then
for all $s \in (\{s_0\} \cup N^1(s_0) \cup N^2(s_0)) \cap S$ do **Start–vertex**(s) and STOP
else
Let P be the *bw–k* sequence of s_0;
Extend P with **Path–extension** to a not extendable *bw–k* path P;
If P contains all vertices of G then Output of f according to Lemma 7 and STOP
else
If $LeftU \cap R^2(P) \neq \emptyset$ or $RightU \cap L^2(P) \neq \emptyset$ then
If $T \cap M(P) \neq \emptyset$ then
Pick some element s_1 from $T \cap M(P)$;
for all $s \in (\{s_1\} \cup N^1(s_1) \cup N^2(s_1)) \cap S$ do **Start–vertex**(s) and STOP
else
for all $s \in \{\nu(P)\} \cup N^1(\nu(P))$ do **Start–vertex**(s) and STOP
else delete the vertices of P not belonging to $M(P)$ from S

until $S = \emptyset$.

Output: A *bw–k* layout of G iff G is a *bw–k* graph.

Theorem 2 *The Algorithm* **bw–k(even)** *decides in linear time for even k whether the k–connected graph G with $\Delta(G) \leq 2k$ is a bw–k graph and provides in the positive case a bw–k layout.*

Proof. The first STOP follows from Lemma 4. We mentioned already after the procedure **Suspicious–vertex** that, for a non suspicious vertex s_0, by Theorem 1, a vertex s from $\{s_0\} \cup N^1(s_0) \cup N^2(s_0)$ must be a start vertex if G is a *bw–k* graph. This explains the second STOP. The third STOP follows from Lemma 7. Thus suppose that P does not contain all vertices of G. If $LeftU \cap R^2(P) = \emptyset$ and $RightU \cap L^2(P) = \emptyset$ we have $|LeftU| \geq 2$ and $|RightU| \geq 2$ since P cannot be extended and $LeftU \neq \emptyset$, $RightU \neq \emptyset$ by Lemma 6. By Lemma 12 the vertices from P not belonging to $M(P)$ cannot be start vertices and we may delete them from the set of all possible start vertices. Thus let $LeftU \cap R^2(P) \neq \emptyset$ or $RightU \cap L^2(P) \neq \emptyset$. According to Lemma 6, we have $LeftU \not\subseteq R^2(P)$ and $RightU \not\subseteq L^2(P)$. In view of Lemma 8 the vertices of $M(P)$ are not suspicious. Hence, if one of them, say s_1, belongs to the original set S (here T), then again by Theorem 1, a vertex from $\{s_1\} \cup N^1(s_1) \cup N^2(s_1)$ must be a start vertex if G is a *bw–k* graph and the fourth STOP follows. Otherwise Lemma 11 can be applied to verify the fifth STOP.

The deletion of m $(= r - l + 1 - 2k)$ suspicious vertices needs obviously only $O(m)$ steps. Thus, if there is not an earlier STOP, the deletion of all suspicious vertices is accomplished not later than after $O(|S|) = O(n)$ steps. Then the algorithm stops after $O(n)$ steps since **Start–vertex** has linear–time complexity. Consequently, the whole algorithm has only linear–time complexity. ∎

References

[1] K. Engel and S. Guttmann, *A structural characterization and a new linear–time algorithm for the recognition of bandwith–2 graphs*, Preprint 98/2, Universität Rostock, Fachbereich Mathematik, 1998.

[2] M.R. Garey, R.L. Graham, D.S. Johnson, and D.E. Knuth, *Complexity results for bandwidth minimization*, SIAM J. Appl. Math., 34 (1978), pp. 477–495.

[3] F. Makedon, D. Sheinwald, and Y. Wolfsthal, *A simple linear–time algorithm for the recognition of bandwidth–2 biconnected graphs*, Inform. Process. Lett., 46 (1993), pp. 103–107.

[4] J. B. Saxe, *Dynamic–programming algorithms for recognizing small–bandwidth graphs in polynomial time*, SIAM J. Alg. Disc. Meth., 1 (1980), pp. 363–369.