A class of 2-colorable orthogonal double covers of complete graphs by hamiltonian paths

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Abstract

An orthogonal double cover (ODC) of the complete graph K_n by a graph G is a collection $\mathcal{G}$ of n spanning subgraphs of K_n, all isomorphic to G, such that any two members of $\mathcal{G}$ share exactly one edge and every edge of K_n is contained in exactly two members of $\mathcal{G}$. In the 1980's Hering posed the problem to decide the existence of an ODC for the case that G is an almost-hamiltonian cycle, i.e. a cycle of length $n-1$. It is known that the existence of an ODC of K_n by a hamiltonian path implies the existence of ODCs of K_{4n} and K_{16n}, respectively, by almost-hamiltonian cycles. Horton and Nonay introduced 2-colorable ODCs and showed: If for $n \geq 3$ and a prime power $q \geq 5$ there are an ODC of K_n by a hamiltonian path and a 2-colorable ODC of K_q by a hamiltonian path, then there is an ODC of K_{qn} by a hamiltonian path. We construct 2-colorable ODCs of K_n and K_{2n}, respectively, by hamiltonian paths for all odd square numbers $n \geq 9$.

Keywords: orthogonal double cover, graph decompositions, self-orthogonal decompositions, self-orthogonal 2-sequencings, Hering's problem

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1 Introduction

Motivated by questions from design theory [26] and from the theory of data bases [10], the intensive study of orthogonal double covers of complete graphs started in the 1980's. Since then there has been a steadily growing interest in the subject.

An *orthogonal double cover (ODC)* of the complete graph K_n is a collection $\mathcal{G}$ of n spanning subgraphs of K_n such that the following two conditions are satisfied:

(i) Every two distinct members of $\mathcal{G}$ share exactly one edge.

(ii) Every edge of K_n is contained in exactly two members of $\mathcal{G}$.

Every $G \in \mathcal{G}$ is called a *page* of $\mathcal{G}$. The above definition immediately implies $|E(G)| = n-1$ for all $G \in \mathcal{G}$. If all members of $\mathcal{G}$ are isomorphic to some graph G, then $\mathcal{G}$ is called an ODC of K_n by G.

The principle question concerning ODCs is: Given some graph G with $|E(G)| = n-1$, is there an ODC of K_n by G? This question has been investigated for several classes of graphs such as graphs consisting of short cycles [10, 13, 14], almost-hamiltonian cycles [27, 26, 2, 25, 24], graphs with maximum degree 2 [9, 18, 7], clique graphs [6, 12], and trees [16, 29, 30], in particular hamiltonian paths [24, 28, 3, 4, 31, 21]. Most of the constructions use group-actions or *PBD*-closure (see [5]). The notion of an ODC can be generalized in various directions, for instance the directed analogue of the problem [27, 2, 24, 28, 15, 20], suborthogonal double covers [19, 33, 22], and symmetric graph designs [17, 8, 11] have been considered.

Here we focus on ODCs by hamiltonian paths and almost-hamiltonian cycles.

2 Group-generated ODCs

In constructing ODCs it turned out to be fruitful to generate ODCs using the action of some group. Many of the known classes of ODCs are group-generated.

Let $\mathcal{A}$ be an additive abelian group of finite order n, and let G be a graph with $V(G) \subseteq \mathcal{A}$. Furthermore, for $x \in \mathcal{A}$ let $G+x$ denote the graph defined by

$$V(G+x) = \{y+x : y \in E(G)\}, \quad E(G+x) = \Big\{\{y+x, z+x\} : \{y,z\} \in E(G)\Big\}.$$

For every $x \in \mathcal{A}$ the graph $G + x$ is called a *translate* of G. If the set

$$G + \mathcal{A} := \{G + x : x \in \mathcal{A}\}$$

is an ODC of K_n, then we say that this ODC is *generated by* $\mathcal{A}$.

To formulate necessary and sufficient conditions for $G+\mathcal{A}$ to be an ODC, let us introduce two more notions. The *length* of an edge $e = \{x, y\} \in E(G)$ is defined by $\ell(e) := \{x - y, y - x\}$, and the *distance* of two edges $e = \{x, y\} \in E(G)$, $e' = \{x + z, y + z\} \in E(G)$ of the same length is the set $d(e, e') := \{z, -z\}$.

Lemma 1 (cf. [28]) *Let $\mathcal{A}$ be a finite abelian group, and let G be a graph on the vertex set $\mathcal{A}$. The set $G + \mathcal{A}$ is an ODC if and only if the following three conditions are satisfied:*

(1) For every $x \in \mathcal{A}$ of order 2 there is exactly one edge $e \in E(G)$ such that $\ell(e) = \{x\}$.

(2) For every $x \in \mathcal{A}$ of order > 2 there are exactly two distinct edges $e, e' \in E(G)$ such that $\ell(e) = \ell(e') = \{x, -x\}$.

(3)

$$\bigcup_{\substack{e, e' \in E(G) \\ e \neq e' \\ \ell(e) = \ell(e')}} d(e, e') = \{x \in \mathcal{A} : \mathrm{ord}(x) > 2\}$$

3 ODCs by almost-hamiltonian cycles and by hamiltonian paths

The oldest question in the area of ODCs is due to Hering and Rosenfeld [27], it is also known as *Hering's problem* [26]: Is there an ODC of K_n by an almost-hamiltonian cycle, i.e. every page of the ODC consists of a cycle of length $n - 1$ and an isolated vertex? Hering [26, 25] showed that the answer is affirmative for all prime powers $n \geq 4$ and for $4 \leq n \leq 36$. The authors of [1] remark that it seems likely that the answer is affirmative for all $n \geq 4$.

According to the following theorem, ODCs by almost-hamiltonian cycles can be obtained from ODCs by hamiltonian paths. The K_{4n}-part is due to Heinrich and Nonay [24], the K_{16n}-part to Volker Leck [32]. The path with n vertices is denoted by P_n.

Theorem 2 *If for some n there is an ODC of K_n by P_n, then there are ODCs of K_{4n} and K_{16n}, respectively, by almost-hamiltonian cycles.*

Therefore, we are interested in ODCs of K_n by P_n. It is easy to check that there is no ODC of K_4 by P_4. It is conjectured that this is the only exception. More general, the authors of [16] conjecture that P_4 is the only tree which does not admit an ODC. The existence of an ODC of K_n by P_n was established for $2 \leq n \leq 20$, $n \neq 4$ in [24]. This result was extended to $2 \leq n \leq 101$, $n \neq 4$ in [31]. Anderson and Leonard [3, 4] constructed solutions for a certain (infinite) class of sufficiently large prime powers n.

4 2-colorable ODCs by paths

Other infinite classes of ODCs by hamiltonian paths can be obtained using the concept of 2-colorability introduced by Horton and Nonay [28]. Let $\mathcal{G}$ be an ODC of K_n by P_n. We say that $\mathcal{G}$ is 2-*colorable* if K_n has an edge-coloring using two colors such that the edges of each page of $\mathcal{G}$ are alternately colored. The following multiplication theorem generalizes an earlier result by Heinrich and Nonay [24] (see also [23]).

Theorem 3 (Horton and Nonay [28]) *Let $n \geq 3$ be an integer, and let $q \geq 5$ be a prime power. If there are an ODC of K_n by P_n and a 2-colorable ODC of K_q by P_q, then there is an ODC of K_{qn} by P_{qn}.*

This motivates interest in 2-colorable ODCs of K_n by P_n, in particular for prime powers n. Unfortunately, if $n \equiv 3 \pmod 4$, then such an ODC can not exist because every page should contain exactly $\frac{n-1}{2}$ edges from each color class but K_n has an odd number of edges. As first examples, 2-colorable ODCs of K_n by P_n were constructed for $n = 5, 13, 17$ in [24], for $n = 9, 10, 12, 29$ in [28], and for $10 \leq n \leq 101$, $n \not\equiv 3 \pmod 4$ in [31].

All these ODCs are group-generated, and the constructions make use of the following lemma which is easy to verify (see [28]). For a finite abelian group $\mathcal{A}$ and a hamiltonian path G on the vertex set $\mathcal{A}$, we say that G is 2-colorable if its edges can be colored alternately with two colors such that $e_1, e_2 \in E(G)$, $\ell(e_1) = \ell(e_2)$ implies that e_1 and e_2 receive the same color.

Lemma 4 *Let $\mathcal{A}$ be a finite abelian group, and let G be a hamiltonian path on the vertex set $\mathcal{A}$ such that $G + \mathcal{A}$ is an ODC. If G is 2-colorable, then $G + \mathcal{A}$ is 2-colorable.*

In particular, note that there are group-generated 2-colorable ODCs of K_n by P_n for $n = 2^4, 2^5$. As remarked in [28], there is no group-generated 2-colorable ODC of K_8 by P_8 but a 2-colorable ODC of K_8 by P_8 was given in [31].

5 Supplementary half-starters

To construct more 2-colorable ODCs by hamiltonian paths, it turned out to be a reasonable approach to use supplementary half-starters and their translates. The above mentioned solutions in [31] were obtained by this method.

Consider some abelian group $\mathcal{A}$ of order $n \equiv 1 \pmod 4$. A *half-starter* with respect to $\mathcal{A}$ is a 1-factor (see [1] for definition) of the complete graph on the vertex set $\mathcal{A} \setminus \{0\}$ such that for every $x \in \mathcal{A} \setminus \{0\}$ the set $E(H)$ contains either no edge of length $\{x, -x\}$ or exactly two edges of length $\{x, -x\}$. Two half-starters H and H' are said to be *supplementary* if $G + \mathcal{A}$ is an ODC, where G denotes the graph on the vertex set $\mathcal{A}$ with $E(G) = E(H) \cup E(H')$.

By these definitions, Lemma 1, and the trivial fact that the occuring lengths and distances are invariant under changing from an half-starter to one of its translates, the following is obvious:

Lemma 5 *Let $\mathcal{A}$ be a finite abelian group. Furthermore, let H and H' be supplementary half-starters w.r.t. $\mathcal{A}$. Then $(H \cup (H' + t)) + \mathcal{A}$ is an ODC for all $t \in \mathcal{A} \setminus \{0\}$. Moreover, if $H \cup (H' + t)$ is a path, then it is 2-colorable.*

A construction over finite fields and Lemma 5 were used in [21] to show:

Lemma 6 *If $2000 > n \equiv 1 \pmod 4$ is a prime power, then there exists a group-generated 2-colorable ODC of K_n by P_n.*

6 New classes of 2-colorable solutions

Our main result is the following theorem.

Theorem 7 *Let $n = m^2$, where $m \geq 3$ is odd.*

(a) There is a 2-colorable ODC of K_n by P_n generated by $\mathbb{Z}_m \times \mathbb{Z}_m$.

(b) There is a 2-colorable ODC of K_{2n} by P_{2n} generated by $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$.

In the sequel, let $m \geq 3$ always be odd and let $n = m^2$. It will be convenient for us to represent $\mathbb{Z}_m \times \mathbb{Z}_m$ as the set of all $xi + y$ with $x, y \in \mathbb{Z}_m$ together with the addition defined by $(x_1 i + y_1) + (x_2 i + y_2) = (x_1 + x_2)i + (y_1 + y_2)$. Let $\mathcal{R}$ denote the ring obtained from this representation of $\mathbb{Z}_m \times \mathbb{Z}_m$ by introducing the multiplication which works like for complex numbers, i.e. $(x_1 i + y_1) \cdot (x_2 i + y_2) = (x_1 y_2 + y_1 x_2)i + (y_1 y_2 - x_1 x_2)$.

To formulate some lemmas needed in proving the main theorem, let us introduce the graph G by $V(G) = \mathcal{R}$ and

$$\{u, v\} \in E(G) \iff u = vi \ \text{ or } \ v = ui.$$

Clearly, G is the collection of the $\frac{n-1}{4}$ disjoint 4-cycles

$$C(x, y) := (xi + y,\ yi - x,\ -xi - y,\ -yi + x,\ xi + y)$$

with $x \in \{0, 1, \ldots, \frac{m-1}{2}\}$, $y \in \{1, 2, \ldots, \frac{m-1}{2}\}$, where 0 is an isolated vertex.

Lemma 8 *$G + (\mathbb{Z}_m \times \mathbb{Z}_m)$ is an ODC.*

Proof. Let $0 \neq u \in \mathcal{R}$. Obviously, the following edges are pairwise distinct and contained in $E(G)$:

$$e_1 := \left\{\tfrac{i+1}{2}u,\ \tfrac{i-1}{2}u\right\}, \qquad e_2 := \left\{\tfrac{i-1}{2}u,\ -\tfrac{i+1}{2}u\right\},$$
$$e_3 := \left\{-\tfrac{i+1}{2}u,\ -\tfrac{i-1}{2}u\right\}, \qquad e_4 := \left\{-\tfrac{i-1}{2}u,\ \tfrac{i+1}{2}u\right\}.$$

Moreover, w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_m$ we have $\ell(e_2) = \ell(e_4) = \{ui, -ui\}$ and

$$\{u, -u\} = \ell(e_1) = \ell(e_3) = d(e_2, e_4).$$

Now the claim follows by Lemma 1. ∎

Since any two vertex-disjoint edges from $E(G)$ which lie in the same component have the same length, in an obvious way we can consider G as the union of two edge-disjoint half-starters. Namely, let H be a subgraph of G such that $V(H) = \mathcal{R} \setminus \{0\}$ and H shares exactly two edges with every non-trivial component of G. Furthermore, let iH denote the graph with $V(iH) = \mathcal{R}$ and

$$\{u, v\} \in E(H) \iff \{ui, vi\} \in E(iH).$$

Then H and iH are edge-disjoint half-starters w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_m$. Moreover, $E(H) \cup E(iH) = E(G)$ together with Lemma 8 implies:

Corollary 9 *H and iH are supplementary half-starters w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_m$.*

Consider $H \cup (iH + t)$ for some $t \in \mathcal{R} \setminus \{0\}$. Clearly, one component of this graph is a path joining the vertices 0 and t, and all other components (if there are any) are cycles of even length. To prove Theorem 7a we will need a choice of H and t such that $H \cup (iH + t)$ is a path, and to prove Theorem 7b we will need a choice of H and t such that $H \cup (H + t)$ is a path.

To give such appropriate choices, we introduce the half-starters (w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_m$) H_1, H_2, H_3 each of which, by definition, has the property that it shares exactly two vertex-disjoint edges with every non-trivial component of G. For each of the 4-cycles $C(x, y)$ ($x = 0, 1, \ldots, \frac{m-1}{2}$; $y = 1, 2, \ldots, \frac{m-1}{2}$) and each of the half-starters H_1, H_2, H_3 these two edges are chosen according to:

$$\{xi + y,\ yi - x\},\ \{-xi - y,\ -yi + x\} \in E(H_1) \iff x \text{ is even},$$
$$\{xi + y,\ -yi + x\},\ \{-xi - y,\ yi - x\} \in E(H_1) \iff x \text{ is odd},$$

$$\{xi + y,\ yi - x\},\ \{-xi - y,\ -yi + x\} \in E(H_2) \iff x \text{ is even or } y = \tfrac{m-1}{2},$$
$$\{xi + y,\ -yi + x\},\ \{-xi - y,\ yi - x\} \in E(H_2) \iff x \text{ is odd and } y \neq \tfrac{m-1}{2},$$

$$\{xi + y,\ yi - x\},\ \{-xi - y,\ -yi + x\} \in E(H_3) \iff x \text{ is even or } y = 1,$$
$$\{xi + y,\ -yi + x\},\ \{-xi - y,\ yi - x\} \in E(H_3) \iff x \text{ is odd and } y \neq 1.$$

As an example, for $n = 49$ the partition of G into H_1 and iH_1 is illustrated in Figure 1.

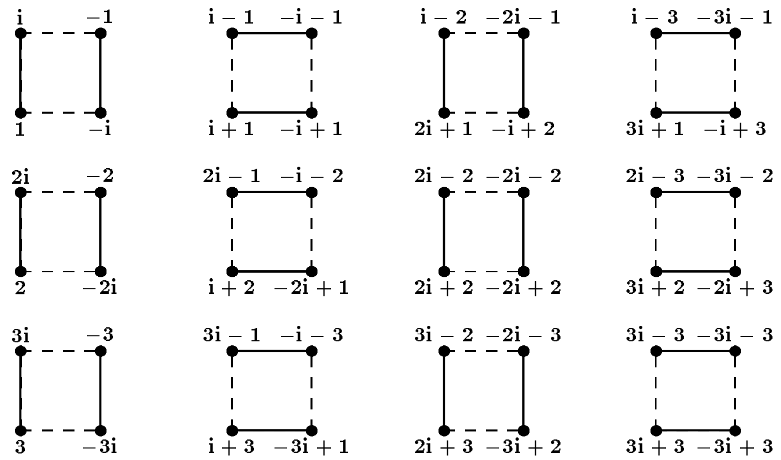

Figure 1: H_1 (solid lines) and iH_1 (dashed) for $n = 49$

Lemma 10 *(a) If $m \equiv 3 \pmod 4$, then $H_1 \cup (iH_1 - 1)$ is a path.*

(b) If $m \equiv 1 \pmod 4$, then $H_2 \cup (iH_2 - 1)$ is a path.

(c) If $m \equiv 3 \pmod 4$, then $iH_1 \cup (iH_1 - 1)$ is a path.

(d) If $m \equiv 1 \pmod 4$, then $H_3 \cup (H_3 - 1)$ is a path.

Proof. The proofs of the four parts of the theorem are very similar. Therefore, we give the proof of (a) in detail, and just outline the proofs of (b), (c), (d). Having the proof of (a) at hand, it is an easy exercise to work out the details for (b), (c), (d).

Proof of (a). Let $m \equiv 3 \pmod 4$. We define two subsets of $\mathcal{R}$ by

$$S_1 := \Big\{ si + t : \quad s \in \{-\tfrac{m-1}{2}, -\tfrac{m-3}{2}, \ldots, \tfrac{m-1}{2}\},\ t \in \{-\tfrac{m-1}{2}, -\tfrac{m-3}{2}, \ldots, \tfrac{m-5}{2}\}, \\ s \equiv t \equiv 1 \pmod 2,\ \operatorname{sgn}(s) \neq \operatorname{sgn}(t) \Big\}$$

and $S_2 := \{-u - 1 : u \in S_1\}$. Obviously, $|S_1| = |S_2| = \frac{n-1}{8}$ and $S_1 \cap S_2 = \emptyset$ hold. For brevity, put $S := S_1 \cup S_2$.

Next, we introduce a linear order on S_1. For any two distinct elements $u = s_1 i + t_1 \in S_1$ and $v = s_2 i + t_2 \in S_1$ with $s_1, s_2 \in \{-\frac{m-1}{2}, \ldots, \frac{m-1}{2}\}$ and $t_1, t_2 \in \{-\frac{m-1}{2}, \ldots, \frac{m-5}{2}\}$ put $u \prec v$ if

(1) $|t_1| < |t_2|$, or

(2) $t_1 = -t_2 < 0$, or

(3) $t_1 = t_2$ and $s_1 < s_2$.

For two distinct elements $u, v \in S_1$ we write $u \prec\!\cdot\, v$ if $u \prec v$ and there is no $w \in S_1 \setminus \{u, v\}$ such that $u \prec w \prec v$.

Let $u, v \in S_1$ such that $u \prec\!\cdot\, v$. We will show that $H_1 \cup (iH_1 - 1)$ contains two paths of the form $(u,\ p_1,\ q_1,\ r_1,\ v)$ resp. $(-u - 1,\ p_2,\ q_2,\ r_2,\ -v - 1)$ with $p_1, q_1, r_1, p_2, q_2, r_2 \notin S$. This is done by studying the following three cases separately:

Case 1: $u = si + t$ with $s \in \{-\frac{m-1}{2}, \ldots, \frac{m-5}{2}\} \setminus \{-1\}$ and $t \in \{-\frac{m-1}{2}, \ldots, \frac{m-5}{2}\}$. This implies $v = (s + 2)i + t$, and we have:

$$\begin{array}{rclcrcl} p_1 &=& (t+1)i - (s+1), & \qquad & p_2 &=& ti - (s+1), \\ q_1 &=& -(s+1)i - (t+1), & & q_2 &=& (s+1)i + t, \\ r_1 &=& ti - (s+2), & & r_2 &=& (t+1)i - (s+2). \end{array}$$

Case 2: $u = \frac{m-1}{2}i + t = -\frac{1}{2}i + t$ with $t \in \{-\frac{m-1}{2}, \ldots, -1\}$. This implies $v = \frac{1}{2}i - t$, and we have:

$$
\begin{aligned}
p_1 &= (t+1)i - \tfrac{1}{2}, & p_2 &= ti - \tfrac{1}{2},\\
q_1 &= -\tfrac{1}{2}i - (t+1), & q_2 &= -\tfrac{1}{2}i - t,\\
r_1 &= -ti - \tfrac{1}{2}, & r_2 &= -(t-1)i - \tfrac{1}{2}.
\end{aligned}
$$

Case 3: $u = -i + t$ with $t \in \{1, \ldots, \frac{m-5}{2}\}$. This implies $v = i - (t+2)$, and we have:

$$
\begin{aligned}
p_1 &= (t+1)i, & p_2 &= ti,\\
q_1 &= t+1, & q_2 &= t,\\
r_1 &= -(t+2)i - 1, & r_2 &= -(t+1)i - 1.
\end{aligned}
$$

By the definitions of H_1 and S, in all three cases we have $p_1, q_1, r_1, p_2, q_2, r_2 \notin S$ and

$$
\begin{aligned}
\{u, p_1\},\ \{q_1, r_1\},\ \{-u-1, p_2\},\ \{q_2, r_2\} &\in E(iH_1 - 1),\\
\{p_1, q_1\},\ \{r_1, v\},\ \{p_2, q_2\},\ \{r_2, -v-1\} &\in E(H_1).
\end{aligned}
$$

Altogether, we established the existence of two disjoint subpaths of $H_1 \cup (iH_1 - 1)$, each containing exactly $\frac{n-9}{2}$ edges. The endpoints of one of these paths are the first element and the last element of S_1 w.r.t. the linear order given above, i.e. the endpoints are $i-1$ and $\frac{m-1}{2}(i-1) = -\frac{1}{2}(i-1)$. The endpoints of the other path are $-i$ and $\frac{1}{2}(i-1) - 1$.

Again by the definition of H_1, the graph $H_1 \cup (iH_1 - 1)$ contains the path:

$$
\Big(-\tfrac{1}{2}(i-1),\ \tfrac{1}{2}(i-1) + i,\ -\tfrac{1}{2}(i+1) - 1,\ -\tfrac{1}{2}(i+1),\ \tfrac{1}{2}(i-1),\ \tfrac{1}{2}(i-1) - 1\Big),
$$

where the second edge and the fourth edge are in $E(H_1)$, the remaining are in $E(iH_1 - 1)$. Consequently, $H_1 \cup (iH_1 - 1)$ contains a path with exactly $n-4$ edges joining $i-1$ and $-i$.

Finally, observe that $\{-i, -1\}, \{i-1, -i-1\} \in E(H_1)$ and $\{-i-1, 0\} \in E(iH_1 - 1)$, which concludes the proof.

Proof of (b). Let $m \equiv 1 \pmod 4$, and define

$$
S_1 := \Big\{ si + t : \ s \in \{-\tfrac{m-3}{2}, -\tfrac{m-5}{2}, \ldots, \tfrac{m-3}{2}\},\ t \in \{-\tfrac{m+1}{2}, -\tfrac{m-1}{2}, \ldots, \tfrac{m-3}{2}\},\\
s \equiv t \equiv 1 \pmod 2,\ \operatorname{sgn}(s) \neq \operatorname{sgn}(t) \Big\},
$$

$S_2 := \{-u-1 : u \in S_1\}$ and $S := S_1 \cup S_2$. Then $|S_1| = |S_2| = \frac{n-1}{8}$ and $S_1 \cap S_2 = \emptyset$ hold.

For any two distinct elements $u = s_1 i + t_1 \in S_1$ and $v = s_2 i + t_2 \in S_1$ with $s_1, s_2 \in \{-\frac{m-3}{2}, \ldots, \frac{m-3}{2}\}$ and $t_1, t_2 \in \{-\frac{m+1}{2}, \ldots, \frac{m-3}{2}\}$ put $u \prec v$ if one of the conditions (1), (2), (3) given in the proof of (a) is satisfied.

Analogously to (a), for any two consecutive elements $u, v \in S_1$ the graph $H_2 \cup (iH_2 - 1)$ contains a path with four edges joining u and v and another path with four edges joining $-u-1$ and $-v-1$, where only the ends of these two paths are members of S. Consequently, $H_2 \cup (iH_2 - 1)$ contains path with $\frac{n-9}{2}$ edges joining the smallest and the greatest element of S_1, i.e. $i-1$ and $-\frac{3}{2}i - \frac{1}{2}$, and $H_2 \cup (iH_2 - 1)$ contains a disjoint path joining $-i$ and $\frac{3}{2}i - \frac{1}{2}$.

Finally, it is easily seen that in $H_2 \cup (iH_2 - 1)$ there are two edges joining 0 to $i-1$, another 5 edges joining $-\frac{3}{2}i - \frac{1}{2}$ to $\frac{3}{2}i - \frac{1}{2}$, and an edge joining $-i$ to -1.

Proof of (c). Let $m \equiv 3 \pmod 4$, and define

$$\begin{aligned} S_1 := \Big\{ si + t : \ & s \in \{-\tfrac{m-3}{2}, -\tfrac{m-5}{2}, \ldots, \tfrac{m+1}{2}\}, \\ & t \in \{-\tfrac{m-3}{2}, -\tfrac{m-5}{2}, \ldots, \tfrac{m-3}{2}\} \setminus \{0\}, \\ & s \equiv t \equiv 0 \pmod 2,\ \operatorname{sgn}(s) = \operatorname{sgn}(t) \text{ or } s = 0 \Big\} \cup \Big\{0, -\tfrac{1}{2}\Big\}, \end{aligned}$$

$S_2 := \{-u-1 : u \in S_1\}$ and $S := S_1 \cup S_2$. Then $|S_1| = |S_2| = \frac{n+7}{8}$ and $S_1 \cap S_2 = \{-\frac{1}{2}\}$ hold.

A linear order on S_1 is established by putting $u \prec v$ for $u = s_1 i + t_1 \in S_1$ and $v = s_2 i + t_2 \in S_1$ with $s_1, s_2 \in \{-\frac{m-3}{2}, \ldots, \frac{m+1}{2}\}$ and $t_1, t_2 \in \{-\frac{m+1}{2}, -\frac{m-1}{2} \ldots, \frac{m-3}{2}\}$ if

(4) $|t_1| < |t_2|$, or

(5) $t_1 = -t_2 < 0$, or

(6) $t_1 = t_2$ and $s_1 > s_2$.

Note that according to this definition the smallest element and the greatest element of S_1 in the linear order are 0 and $-\frac{1}{2}$, respectively.

Let $u, v \in S_1$ be consecutive w.r.t. the linear order $\prec$. By the definition of H_1, the graph $iH_1 \cup (iH_1 - 1)$ contains a path with four edges joining u and v and another path with four edges joining $-u-1$ and $-v-1$, and only the ends of these two paths are elements of S. Therefore, in $iH_1 \cup (iH_1 - 1)$ the vertices 0 and $-\frac{1}{2}$ are joined by a path with $\frac{n-1}{2}$ edges as well as -1 and $-\frac{1}{2}$ are joined by a path with $\frac{n-1}{2}$ edges.

Proof of (d). Let $m \equiv 1 \pmod 4$, and define Let $m \equiv 3 \pmod 4$, and define

$$\begin{aligned} S_1 := \quad & \Big\{ si+t : \; s \in \{-\tfrac{m-1}{2}, -\tfrac{m-3}{2}, \dots, \tfrac{m-1}{2}\}, \\ & \qquad t \in \{-\tfrac{m-1}{2}, -\tfrac{m-3}{2}, \dots, \tfrac{m-5}{2}\} \setminus \{0,-2\}, \\ & \qquad s \equiv t \equiv 0 \pmod 2, \; \operatorname{sgn}(s) = \operatorname{sgn}(t) \text{ or } s \in \{0,-2\} \Big\} \\ \cup \; & \Big\{ 0, \; -2, \; -2i - \tfrac{1}{2}, \; -\tfrac{1}{2} \Big\}, \end{aligned}$$

$S_2 := \{-u-1 : u \in S_1\}$ and $S := S_1 \cup S_2$. Then $|S_1| = |S_2| = \frac{n+7}{8}$ and $S_1 \cap S_2 = \{-\frac{1}{2}\}$ hold.

A linear order on S_1 is established by putting $u \prec v$ for $u = s_1 i + t_1 \in S_1$ and $v = s_2 i + t_2 \in S_1$ with $s_1, s_2, t_1, t_2 \in \{-\frac{m-1}{2}, \dots, \frac{m-1}{2}\}$ if one of the conditions (1), (2), (3) given in the proof of (a) is satisfied. Note that according to this definition the smallest element and the greatest element of S_1 in the linear order are 0 and $-\frac{1}{2}$, respectively.

The remaining argumentation is identical to the one following the definition of $\prec$ in the proof of (c), where $iH_1 \cup (iH_1 - 1)$ must be replaced by $H_3 \cup (H_3 - 1)$. ■

Proof of Theorem 7. To prove (a), we define

$$H := \begin{cases} H_1, & \text{if } m \equiv 3 \pmod 4, \\ H_2, & \text{if } m \equiv 1 \pmod 4). \end{cases}$$

The assertion follows immediately from the definition of H together with Corollary 9, Lemma 10 (a) and (b), Lemma 5, and Lemma 4.

To prove (b), we construct a 2-colorable hamiltonian path P on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$ such that $P + (\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2)$ is an ODC. By Lemma 4, the existence of such a P implies the claim. $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$ will be considered as the set of all pairs (u, v) with $u \in \mathcal{R}$ and $v \in \mathbb{Z}_2$.

Let

$$H := \begin{cases} iH_1, & \text{if } m \equiv 3 \pmod 4, \\ H_3, & \text{if } m \equiv 1 \pmod 4). \end{cases}$$

By Lemma 10 (c) and (d), $P_1 := H \cup (H-1)$ is a hamiltonian path on the vertex set $\mathcal{R}$. Now P is obtained from P_1 by the following procedure:

1. Replace every vertex u of P_1 by $(u, 1)$, and let P_2 denote the resulting path.

2. For $\{u, v\} \in E(H)$ replace in P_2 the edge $\{(u,1),(v,1)\}$ by the path $\Big((u,1),\, (ui,0),\, (vi,0),\, (v,1)\Big)$ (i.e. every second edge of P_2 is replaced), and let P_3 denote the resulting path.

3. P is obtained from P_3 by introducing a new edge joining the end-vertex $(0,1)$ to $(0,0)$.

Obviously, P is a hamiltonian path on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$.

As an example, for $n = 9$ the paths P_1 and P are shown in Figure 2.

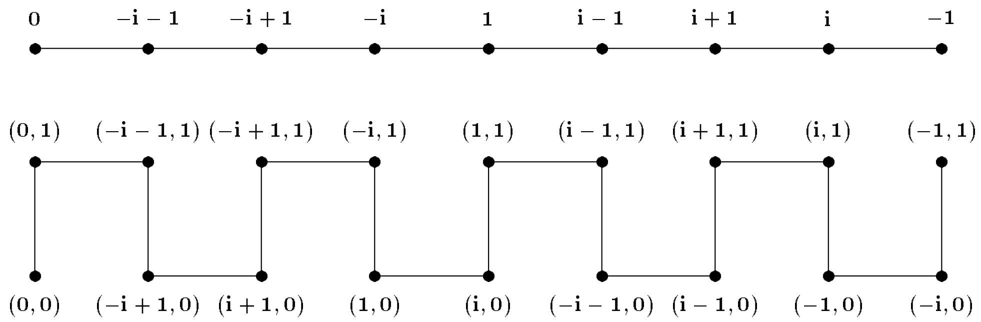

Figure 2: P_1 (top) and P (bottom) for $n = 9$

Clearly, P contains exactly one edge whose length w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2$ is $\{(0,1)\}$, namely the edge $\{(0,1),(0,0)\}$. Moreover, for $\{u,v\} \in E(iH)$ we have $\{(u,0),(v,0)\} \in E(P)$, and for $\{u,v\} \in E(H-1)$ we have $\{(u,1),(v,1)\} \in E(P)$. Now for every $0 \neq w \in \mathcal{R}$ it follows from Corollary 9, Lemma 5, and Lemma 1 that $\{(w,0),(-w,0)\}$ is the length of two edges in P and the distance of two edges of the same length in P. Furthermore, for $0 \neq w \in \mathcal{R}$ the following edges are in $E(P)$:

$$e_1 := \left\{ \left(\tfrac{i+1}{2}w, 1\right),\ \left(\tfrac{i-1}{2}w, 0\right) \right\}, \qquad e_2 := \left\{ \left(\tfrac{i-1}{2}w, 1\right),\ \left(-\tfrac{i+1}{2}w, 0\right) \right\},$$
$$e_3 := \left\{ \left(-\tfrac{i+1}{2}w, 1\right),\ \left(-\tfrac{i-1}{2}w, 0\right) \right\}, \qquad e_4 := \left\{ \left(-\tfrac{i-1}{2}w, 1\right),\ \left(\tfrac{i+1}{2}w, 0\right) \right\},$$

and we have $\{(w,1),(-w,1)\} = \ell(e_1) = \ell(e_3) = d(e_2,e_4)$. Hence, by Lemma 1, the set $P + (\mathbb{Z}_m \times \mathbb{Z}_m \times \mathbb{Z}_2)$ is an ODC.

Finally, the 2-colorability of P follows directly from its definition. ■

7 Corollaries

Together with the results mentioned in the preceding sections, our main theorem gives new large classes of ODCs by hamiltonian paths and almost-hamiltonian cycles. Summerizing Theorems 2, 3, 7, Lemma 6, and the fact that there are 2-colorable ODCs by P_8, P_{16} and P_{32}, respectively (see section 4), we obtain:

Corollary 11 *Let $k \geq 5$ be an integer, and let $k = 2^\alpha p_1^{\alpha_1} \ldots p_\ell^{\alpha_\ell}$ (with $\alpha \geq 0$, $\alpha_i \geq 1$) be its prime factor decomposition. Furthermore, suppose that α_i is even for all i with $p_i \equiv 3 \pmod 4$ or $2000 < p_i \equiv 1 \pmod 4$. Let $n \geq 3$ be another integer such that there exists an ODC of K_n by P_n.*

(a) If $\alpha \notin \{1, 2\}$, then there are an ODC of K_k by P_k and an ODC of K_{kn} by P_{kn}.

(b) If $\alpha = 1$ and $\alpha_i \geq 2$ for some i, then there is an ODC of K_k by P_k.

(c) If $2 \leq \alpha \neq 3$, then there are ODCs of K_k and K_{kn}, respectively, by almost-hamiltonian cycles.

(d) If $\alpha = 3$ and $\alpha_i \geq 2$ for some i, then there is an ODC of K_k by an almost-hamiltonian cycle.

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