Thermomechanical coupled waves in a nonlinear medium

Witold Kosiński

Center of Mechanics and Information Technology
IPPT PAN, 00-049 Warszawa, Poland
Polish–Japanese Institute of Information Technology
ul. Koszykowa 86 , 02-008 Warszawa
Department of Mathematics, Technology and Natural Sciences
Pedagogical University of Bydgoszcz, 85-064 Bydgoszcz, Poland

Kurt Frischmuth

Fachbereich Mathematik, Universität Rostock
Universitätsplatz 1, D–18051 Rostock, Germany

Abstract

Several technological situations at moderate and high temperatures as well as many physical experiments at low temperatures show the necessity of taking into account the wave structure of heat transport. Motivated by experimental evidence a thermo-mechanical framework for a deformable heat-conducting continuum has been developed. In this framework a history effect is present, it is introduced via a proportionality law between the heat flux vector and the gradient of a scalar thermal variable. The theory leads to a modified model of thermoelasticity with an extra thermal stress effect and wave–type heat conduction. The model is governed by a system of quasi–linear hyperbolic equations. All essential material functions are examined, and the impact of their nonlinearity on the solution to initial-boundary value problems is studied.

A numerical scheme is developed to solve initial-boundary value problems on finite domains by a modification of LAX-WENDROFF's scheme. It turns out that the previously studied case of NaF crystals considered as *rigid heat conductors* was quantitatively quite acceptable: while it does not show any elastic waves, it still gives correct speeds and amplitudes of the pure thermal waves. However, this feature of *upward compatibility* of theories is lost for materials with higher thermal expansion coefficients. Hence, for new applications, it is of the essence to develop numerical methods for the fully coupled theory in 2D and 3D cases.

Preprint submitted to Elsevier Preprint *8 February 2000*

1 Introduction

Finite speed thermal waves, known collectively as *second sound*, distinguishing them from generally faster propagating mechanical waves, were first detected in ^{3}He, [1], and then in high purity dielectric crystals of sodium fluoride, NaF, [2], and bismuth, Bi, [3]. Heat pulses were initiated by the temperature rise at one end (side) of the crystal, while the temperature measurement on the other end (side) reported the arrival signal of thermomechanical waves, propagating with the sound speeds. No mechanical variables, like strain, stress or particle velocity were measured in the heat pulse experiments. Signals were recorded graphically. The arrival time of the leading edges of the pulses were measured. At low temperatures (i.e. at 12 K for NaF and 1.38 K for Bi) the fastest peak, denoted by L, travelled with the speed of a longitudinal sound wave. The peak which arrived next, denoted by T, travelled with the speed of a transversal sound wave. This behaviour is interpreted theoretically as the ballistic phonon effect. In some range of temperatures, an extra peak following those two already observed peaks was detected. That third peak has been interpreted as partially developed second sound. It was observed (cf. figures from [4] and the review article [5]), that in the range of low temperatures at which experiments have been performed, the time of arrival of the third heat pulse is approximately a linear function of the reference temperature.

Several technological situations at moderate and high temperatures (such as distribution of temperature around propagating cracks [6], and temperature distribution in solid materials due to laser pulse train of a very short duration [7]) and many physical experiments at low temperatures (such as experiments performed at low temperatures [1–3,8–10]), show the necessity of taking into account the wave structure of the heat transport.

The above cited experimental facts contradict quantitative and qualitative results that could be obtained using phenomenological approaches to the heat transfer based on a proportionality law between the heat flux and the temperature gradient, called the Fourier law. That law incorporated with the balance of energy leads to a parabolic equation for the temperature field, in solutions of which heat pulses are transmitted with infinite speed (not mentioning some nonlinear and singular parabolic initial, boundary–value problems that could lead to particular solutions with finite–speed wave structures). Only ballistic thermomechanical wave propagation could be described in such a model: no possibility exists for describing the second sound, i.e. thermal wave (i.e. the third peak in experimental results). However, the present approach can describe all three types of observed waves.

2 Internal variable as a thermal history

There are several phenomenological approaches aimed at a new heat conduction equation, that could describe that phenomena. Some of them are reviewed in the papers [11–13]. The present approach keeps a proportionality law between the heat flux $\boldsymbol{q}$ and the spatial gradient of a thermal internal state variable β in the form

$$\boldsymbol{q} = -\alpha \nabla \beta. \tag{1}$$

The classical Fourier law, on the other hand, is

$$\boldsymbol{q} = -k(\vartheta) \nabla \vartheta, \tag{2}$$

where the heat conduction coefficient $k(\vartheta)$ can depend, in general on the absolute temperature ϑ. The scalar internal state variable β will represent the history of the temperature in the sense that its value at the present time will be a solution to an initial-value problem for a suitable ODE in which the right-hand side is a function of (at least) ϑ and β. Let us notice that in the linear case such an ODE can have the form:

$$\dot{\beta} = \frac{1}{\tau}(\vartheta - \beta), \tag{3}$$

where the coefficient τ is called the *thermal relaxation time*. If we perform spatial differentiation in (3) and make use of (1) then we get

$$\tau \dot{\boldsymbol{q}} + \boldsymbol{q} = -\alpha \nabla \vartheta. \tag{4}$$

The last equation can be regarded as a modification of the Fourier law, if the heat conductivity coefficient $k(\vartheta)$ is identified with α and one has added to (2) a term in which the time derivative of the heat flux $\boldsymbol{q}$ is multiplied by the coefficient τ. If one omits introducing the thermal internal state variable β the last equation (4) becomes the governing equation of the so-called rate type approach modelling heat conduction, and is called the Maxwell-Cattaneo-Vernotte-Kaliski equation ([14–18]). It is interesting to add that the approach based on the last equation can be obtained as a particular case of the summed temperature–gradient history approach proposed by the authors of [19].

The thermoelastic developments of the modification (4) one can find in [20–23] (see the review paper [24]). There are some drawbacks in applying the equation (4) to the thermodynamic theory of thermomechanics, especially because of the internal dissipation inequality. This was pointed out in [25–27].

The present author (W. K.) with Perzyna in [28] (and later in [29] in a more general setup) calculated for the first time finite speeds of thermal (and thermomechanical) waves in the framework of a thermodynamic theory with internal state variables. Later on Coleman *et al* [30] repeated the first results of Kosiński and Perzyna from 1972 in a slightly different setup, without referring to the internal state variable approach and to the original paper [28]. The model proposed in [31] is based on a different procedure for the derivation of a wavetype heat propagation equation; in fact it is somehow a next generation of [28,29]. In that model a scalar internal state variable β has been introduced to represent at the same time a history of the thermodynamic temperature and the potential for its gradient. In the course of obtaining the consequences of the laws of thermodynamics, a modified Fourier-type law was found leading to finite speeds of propagation of thermal and thermomechanical waves. The model has been mostly applied to heat conductors in 1D and 3D cases and to thermoelastic solids (cf. [32–36]). Its numerical implementations are discussed in [37–40]. A well–posedness result is published in [41].

3 Thermodynamic foundations

From this point it is not difficult to arrive at a more general evolution equation than the linear one (3), and to introduce a new dependent variable (a function of t and X) β that satisfies the following ODE

$$\dot{\beta} = F(\vartheta, \beta), \tag{5}$$

which is accompanied by a more general heat conduction constitutive equation than (1), namely

$$\boldsymbol{q} = -\alpha^*(\vartheta)\nabla\beta \tag{6}$$

which together with (5) are regarded as modified Fourier laws. This extra thermal variable β (called in [33] a new semi-empirical temperature) in this general setup still represents a *thermal history* of the material and can be interpreted as an internal state variable. In some applications ([35,36]) we have used the linear evolution equation (3). However, in more advanced problems related to the temperature–dependence of the speed of thermal waves we had to look for a non-linear one ([42]). Before the development of the thermodynamic foundation let us make the following
Assumption:
Fourier law is necessary to obtain from (5) in the limit transition case (called a quasi-stationary case) when the left-hand side of (5) tends to zero. In other

words, we suppose that there exists an invertible *quasi-equilibrated* or *quasi-stationary* relation between ϑ and β, a differentiable homeomorphism $B(\vartheta)$ such that $F(\vartheta, B(\vartheta)) \equiv 0$.

Hence in terms of the function B, we get

$$\text{if} \quad \dot{\beta} = 0 \quad \text{then} \quad \beta = B(\vartheta), \quad \text{with} \quad F(\vartheta, B(\vartheta)) \equiv 0. \tag{7}$$

This function will appear in the representation of the steady-state conductivity coefficient. Let us present this for the case of the quasi-linear proportionality constitutive law (6). Namely, admitting that in the general dynamic case, the heat flux vector $\boldsymbol{q}$ is proportional to the gradient of β (cf. (6)), then under the quasi-stationarity condition (steady-state) (7), the heat flux turns out to be proportional to $\nabla\vartheta$ itself. This is as is in the Fourier law (2), however, with a different heat conduction coefficient κ, called now steady-state heat conductivity coefficient, and given by

$$\boldsymbol{q} = -\kappa(\vartheta)\nabla\vartheta \qquad \text{with} \qquad \kappa(\vartheta) := \alpha^*(\vartheta)\frac{dB(\vartheta)}{d\vartheta}. \tag{8}$$

One has to underline once more that the equation (8) is valid only in the limit transition case when the time derivative of β can be assumed zero. The above assumption is important in the course of fitting material functions from experimental results related to classical heat conduction measurements.

3.1 Notations

In this paper, we denote by X the space variables in reference configuration identified with particles, and by x the spatial variable in the actual configuration. Gradients with respect to the former will be denoted by $Grad$, with respect to the latter by ∇.

The symbol ∂ will be used for partial derivatives. If derivatives with respect to tensor quantities are taken, the result is a tensor (of a higher order).

We emphasize the difference between the tensor $\boldsymbol{F}$ (independent variable) and the scalar F (dependent variable). They are not related to each other.

Whenever convenient, we use a comma for partial derivatives, there is no difference between a superposed dot and a suffix t after a comma. In particular, in the 1D case we use the comma notation for both gradient and divergence. A dot $\cdot$ between two vectors or tensors denotes a scalar product (a full contraction).

Now we quote the foundations of thermodynamics with the modified Fourier law presented in [43,44] and repeat most important consequences of that presentation. The case of thermomechanics at finite strains requires the introduction of the variables $\rho_0(X)$ – the reference mass density, $\boldsymbol{F}$ – the deformation gradient tensor, that leads to the relation of $\rho_0(X)$ to the actual mass density ρ by the law $\rho_0 = J\rho$, with $J = \det\boldsymbol{F}$. If $\boldsymbol{T}$ is the symmetric Cauchy stress tensor then the first Piola-Kirchhoff stress tensor $\boldsymbol{S}$ is given by the identity $\boldsymbol{S} = J\ \boldsymbol{T}\boldsymbol{F}^{-T}$. Further, $\boldsymbol{v}$ is the particle velocity, $\boldsymbol{q}$ – the heat flux vector related to the reference heat flux $\boldsymbol{q}_\kappa$ by the identity $\boldsymbol{q}_\kappa = J\boldsymbol{q}\boldsymbol{F}^{-T}$, ϵ – the specific internal energy per unit mass, η – the specific entropy, ϑ – the absolute temperature, r – the body heat supply. We assume that the spatial gradient $\operatorname{grad}\beta = \nabla\beta$ appears in the constitutive equations for free energy ψ, entropy η, stress $\boldsymbol{S}$ and heat flux $\boldsymbol{q}_\kappa$:

$$\begin{aligned} \psi &= \psi^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta), & \eta &= \eta^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta), \\ \boldsymbol{S} &= \boldsymbol{S}^*(\boldsymbol{F}, \vartheta, \nabla\vartheta, \nabla\beta), & \boldsymbol{q}_\kappa &= \boldsymbol{Q}^*(\boldsymbol{F}, \vartheta, \nabla\beta). \end{aligned} \tag{9}$$

Notice that the independence of the heat flux of the actual value of the temperature gradient is crucial for the development of the hyperbolic model.

Now the second law of thermodynamics

$$\rho_0(\dot\eta\vartheta - \dot\epsilon) + \boldsymbol{S}\cdot\dot{\boldsymbol{F}} - \vartheta^{-1}\boldsymbol{q}_\kappa \cdot \mathrm{Grad}\vartheta \geq 0, \tag{10}$$

is used to get the following potential–type consequences :

$$\begin{aligned} \mathbf{0} &= \partial\psi^*/\partial\nabla\vartheta, & \boldsymbol{S} &= \rho_0\partial\psi^*/\partial\boldsymbol{F} + (\vartheta\partial F/\partial\vartheta)^{-1}(\nabla\beta\otimes\boldsymbol{q}_\kappa), \\ \eta &= -\partial\psi^*/\partial\vartheta, & \boldsymbol{q}_\kappa &= -\rho_0(\vartheta\partial F/\partial\vartheta)(\partial\psi^*/\partial\nabla\beta)\boldsymbol{F}^{-T}, \end{aligned} \tag{11}$$

and a reduced inequality, with F substituted from (5),

$$(\vartheta\partial F/\partial\vartheta)^{-1}(\partial F/\partial\beta)\boldsymbol{q}_\kappa\boldsymbol{F}^T\cdot\nabla\beta \geq 0. \tag{12}$$

The first three identities of (11) are well known, the last one is rather not typical (cf. [29]). The stress potential relation in (11) contains two components: the first component is rather classical, while the second one represents the direct coupling between mechanical and thermal fields in which an extra stress term appears. Eliminating $\boldsymbol{q}_\kappa$ from the stress relation in (11) and (12) we get a new potential relation for the Piola-Kirchhoff stress and a residual inequality.

However, here the final potential relations for the Cauchy stress and the actual heat flux vector in a general anisotropic case are written

$$\begin{aligned} \boldsymbol{T} &= \rho\partial\psi^*/\partial\boldsymbol{F}\boldsymbol{F}^T - \rho\nabla\beta\otimes\partial\psi^*/\partial\nabla\beta, \\ \boldsymbol{q} &= -\rho\vartheta(\partial\psi^*/\partial\nabla\beta)(\partial F/\partial\vartheta). \end{aligned} \tag{13}$$

Let us assume in $(13)_2$ a linear relation with respect to $\nabla\beta$. Then due to the principle of material frame indifference this will be true iff the free energy function is of the form

$$\psi^* = \psi_1^*(\boldsymbol{B},\vartheta) + 0.5\psi_2^*(\boldsymbol{B},\vartheta)\nabla\beta\cdot\nabla\beta, \tag{14}$$

where the factor 0.5 has been assumed for convenience only, with $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$; then Eq. $(13)_2$ turns into

$$\boldsymbol{q} = -\alpha^*(\boldsymbol{B},\vartheta)\nabla\beta \quad \text{with} \quad \alpha^*(\boldsymbol{B},\vartheta) = \rho\vartheta\partial F/\partial\vartheta\ \psi_2^*(\boldsymbol{B},\vartheta). \tag{15}$$

In the present thermomechanical case, the independence of the free energy function of β leads to splitting of the function $F(\vartheta,\beta)$ into two independent terms

$$F(\vartheta,\beta) = F_1(\vartheta) + F_2(\beta). \tag{16}$$

Moreover, in the linear case (15) the proportionality material coefficient α^* has to be independent of strain, i.e.

$$\alpha^*(\boldsymbol{B},\vartheta) = \alpha^*(\vartheta), \quad \psi_2^*(\boldsymbol{B},\vartheta) = \psi_{21}^*(\vartheta)J \quad \text{and} \quad \boldsymbol{q} = -\alpha^*(\vartheta)\ \nabla\beta, \tag{17}$$

with $\psi_{21}^* \geq 0$. If the internal energy $\epsilon^* = \psi^* + \vartheta\eta^*$ is independent of $\nabla\beta$ then the function ψ_{21}^* is linear in ϑ, i.e. $\psi_{21}^*(\vartheta) = \psi_{20}\vartheta$. In this case Eq. $(13)_1$ due to (14) will take the form

$$\boldsymbol{T} = 2\rho(\partial\psi_1^*/\partial\boldsymbol{B})\boldsymbol{B} + \rho\psi_{20}\vartheta J(0.5|\nabla\beta|^2\boldsymbol{I} - \nabla\beta\otimes\nabla\beta), \tag{18}$$

or equivalently, by using $(17)_3$,

$$\boldsymbol{T} = 2\rho(\partial\psi_1^*/\partial\boldsymbol{B})\boldsymbol{B} + \rho\frac{\psi_{20}\vartheta J}{\alpha^*(\vartheta)^2}(0.5|\boldsymbol{q}|^2\boldsymbol{I} - \boldsymbol{q}\otimes\boldsymbol{q}), \tag{19}$$

with $\alpha^*(\vartheta) = \rho_0\psi_{20}^*\vartheta^2 F_1'(\vartheta)$. Let us notice that even in this simplified case the stress–strain relation has an extra term due to the thermomechanical coupling; this term can be called an *extra thermal stress*. This contribution can have a substantial meaning in describing damage phenomena in deformable materials due to thermomechanical coupling.

4 Particular constitutive equations

4.1 Non–deformable case

Let us first consider a non–deformable case, in which mechanical fields $\boldsymbol{v}, \boldsymbol{F}$ and $\boldsymbol{S}$ are neglected. In [45] 3D thermal weak discontinuity waves were investigated. It was shown there, for example, that in the rigid case with (6) and

$$\psi = \psi_1^*(\vartheta) + 0.5\psi_2^*(\vartheta)|\nabla\beta|^2, \tag{20}$$

with the specific heat $c_v(\vartheta) = d\epsilon^*(\vartheta)/d\vartheta$, the governing system of equations for determining β and ϑ is composed of (5) with F given by (16), and

$$\begin{aligned} &\rho c_v(\vartheta)\ddot{\beta} - \alpha^{*\prime}(\vartheta)\nabla\dot{\beta}\cdot\nabla\beta - \alpha^*(\vartheta)F_1'(\vartheta)\triangle\beta \\ &-\rho_0 c_v(\vartheta)F_2'(\beta)\dot{\beta} + \alpha^{*\prime}(\vartheta)F_2'(\beta)\nabla\beta\cdot\nabla\beta = \rho_0 r F_1'(\vartheta) \end{aligned} \tag{21}$$

Here $\triangle\beta$ denotes the Laplacian of β and $\rho_0 r$ represents a body heat supply density. Under the monotonicity of F_1 the variable ϑ can be expressed in terms of β and $\dot{\beta}$ from (5), and consequently the equation (21) can be solved independently for β. The speed λ of a pure thermal wave is given by

$$\rho_0 c_v(\vartheta)\lambda^2 - \alpha^{*\prime}(\vartheta)\nabla\beta\cdot\boldsymbol{n}\lambda - \alpha^*(\vartheta)F_1'(\vartheta) = 0, \tag{22}$$

with $\boldsymbol{n}$ as a unit normal to the wave front. We can see that the wave propagation is not symmetric if $\nabla\beta$ differs from zero, and the hyperbolicity condition is

$$c_v(\vartheta)\alpha^*(\vartheta)F_1'(\vartheta) \geq 0 \quad \text{or} \quad c_v(\vartheta)\vartheta(F_1'(\vartheta))^2\psi_2^*(\vartheta) \geq 0 \;\;, \tag{23}$$

where the second equivalent form is due to the representation

$$\alpha^*(\vartheta) = \rho_0\vartheta F_1'(\vartheta)\psi_2^*(\vartheta). \tag{24}$$

In [44] it was shown that the hyperbolicity is a consequence of the second law of thermodynamics provided the specific heat $c_v(\vartheta)$ is positive, and solutions for β of (5) are asymptotically stable.

One can get the model of extended thermodynamics developed by Morro and Ruggeri [26] with the following evolution equation for the heat flux:

$$(\alpha(\vartheta)\boldsymbol{q})^{\cdot} + \nabla\ \nu(\vartheta) = -\frac{\nu'(\vartheta)}{k(\vartheta)}\boldsymbol{q}, \tag{25}$$

under identifications

$$\alpha(\vartheta) \equiv 1/\alpha^*(\vartheta)\ ,\ F_2'/\alpha^*(\vartheta) \equiv -\nu'(\vartheta)/k(\vartheta) \ \text{ and } \ \nu(\vartheta) \equiv F_1(\vartheta),$$

and $F_2(\beta)$ being linear in β. Similarly one can get a particular model with internal state variable developed by Kosiński and Perzyna [28].

4.2 1D thermomechanical case

Now we consider a 3rd order elastic material with Debey law, with the internal energy ϵ of 3rd order in strain and the specific heat of 3rd order in temperature, all for the 1D case of a purely longitudinal motion. Then we will introduce:
displacement vector:

$$\boldsymbol{u} = (u, 0, 0)\,,$$

particle velocity:

$$\boldsymbol{v} = (v, 0, 0)\,,$$

strain component :

$$m = \partial u/\partial X\,,$$

component of spatial gradient of β :

$$p = \partial\beta/\partial x\,,$$

heat flux component in the reference configuration: q_κ,
S_1^1 – component of the Kirchhoff stress tensor: S ,
internal energy:

$$\epsilon^*(m,\vartheta) = 0.5\ c_1^2\ m^2 + 1/6\ A\ m^3 + 1/4\ c_{v3}\vartheta^4/\ \vartheta_0^3 + c_{v0}\theta\,,$$

specific heat:

$$\partial\epsilon^*/\partial\vartheta = c_v(\vartheta) = c_{v3}(\vartheta/\vartheta_0)^3 + c_{v0}\,,$$

free energy:

$$\psi(m,\vartheta,p) = \psi_1(m,\vartheta) + 0.5\vartheta\psi_{20}\ p^2\ (1+m)\,,$$

where:

$$\begin{aligned}\psi_1(m,\vartheta) = {} & 0.5\ c_1^2\ m^2 + 1/6\ A\ m^3 - 1/\rho_0\ \gamma\ (\vartheta-\vartheta_0)(m+0.5m^2)\\ & -c_{v0}\vartheta\ \ln(\vartheta/\vartheta_0) - 0.5\ c_{v3}\vartheta^4/\vartheta_0^3 - \eta_0\ (\vartheta-\vartheta_0),\end{aligned}$$

$c_1 = (\lambda + 2\mu)/\rho_0,\ A = 3(\lambda + 2\mu + 2l)/\rho_0,\ \lambda$ and μ – Lamé constants,
l – 3rd order elastic constant, γ – thermal expansion coefficient,
ϑ_0 – reference temperature,
$\psi_{20} = \tau_0 k_0/\vartheta_0^2, c_{v3}, c_{v0}, \gamma_0, \tau_0, k_0$ – material constants.

Then we get the following constitutive equations compatible with the thermodynamic consequences of the previous section:

stress:

$$\begin{aligned} S &= \rho_0 \partial\psi_1/\partial m - 0.5\rho_0\psi_{20}\vartheta\ p^2 \\ &= \rho_0\ c_1^2\ m + 0.5\rho_0\ A\ m^2\ - \gamma\ (\vartheta - \vartheta_0)(1+m) - 0.5\rho_0\ \psi_{20}\vartheta\ p^2\ , \end{aligned}$$

entropy:

$$\begin{aligned} \eta &= -\partial\psi/\partial\vartheta \\ &= \eta_0 + \gamma/\rho_0\ (m + 0.5\ m^2) + 1/3 c_{v3}\ (\vartheta/\vartheta_0)^3 + \\ &\quad c_{v0}\ \ln(\vartheta/\vartheta_0) - 0.5\psi_{20}\ p^2\ (1+m)\ , \end{aligned}$$

evolution equation for β (internal state variable): $\dot{\beta} = F_1(\vartheta) + F_2(\beta)$
with $F_1(\vartheta) = a((\vartheta_\lambda - \vartheta)/\vartheta_0)^\nu$, while $F_2(\beta)$ is to be derived from the quasi-stationary relation (7) and steady-state heat conduction experimental measurements implemented in the quasi-stationary heat conduction law (8),
modified Fourier law for heat flux component: $q_\kappa = -\rho_0\vartheta^2\ F_1'(\vartheta)\psi_{20}\ p$.

The first calculations of thermomechanical coupled waves have been made in the linearized case [40], when all quadratic and higher order terms have been dropped out, and the specific heat c_v with the non-equilibrium heat conductivity coefficient α^* were assumed constant. Then we have arrived at the coupled system of four equations

$$\begin{aligned} m,_t &= v,_X & \qquad \rho_0 v,_t &= \rho_0 c_1^2 m,_X - \gamma\vartheta,_X \\ \rho_0 c_v \vartheta,_t &= \alpha^* p,_X - \gamma\vartheta_0 v,_X & \qquad \tau p,_t &= \vartheta,_X - p. \end{aligned} \tag{26}$$

The following numerical values of material constants of NaF at $\vartheta_0 = 15K$ have been identified:

ρ_0	2.866	$[g/cm^3]$	γ	1.1464E-5	$[g/\mu s^2/cm/K]$
c_1	0.5477	$[cm/\mu s]$	$\rho_0 c_v$	7950	$[Ws/m^3/K]$
α^*	205	$[W/cm/K]$	c_T	0.19531	$[cm/\mu s]$
τ	0.675986	$[\mu s]$	c_{v0}	0	$[W/\mu s/g/K]$

where c_T is the second sound speed in a non-deformable case. The temperature constant ϑ_λ, called in [42,46] the *critical temperature*, corresponds to the temperature level at which a peak in the quasi–static heat conductivity coefficient has been experimentally observed for NaF and for Bi. In [42] for NaF the constant ϑ_λ was assumed equal to 17.5 K and $F_2(\beta) = -a(\beta/\vartheta_0)^\nu$.

A boundary value for the temperature at one end is assumed: the temperature is continuously but quickly increased, then it is kept constant for a while, and then decreased to the reference temperature ϑ_0. The second end is assumed to be in adiabatic condition, i. e., the heat flux is kept zero.

5 Numerical solution

While in the linear 1D case the model is determined by four quantities and only a 4×4–system has to be solved (cf. [40]), here we are confronted with a 5×5–system. It is no longer possible to calculate β afterwards.

We collect the equations and obtain for the unknown vector $\boldsymbol{w}(X,t) = (e, i, n, \pi, \beta)^T$ the system

$$\boldsymbol{w}_{,t} = \boldsymbol{f}(\boldsymbol{w})_{,X} + \boldsymbol{b}(\boldsymbol{w}) \ . \tag{27}$$

Here we denoted the flux by $\boldsymbol{f}$, the source (inhomogeneities) by $\boldsymbol{b}$ and the new dependent variables

$$\begin{aligned} e &= 1 + m \,, & i &= \rho_0 v \,, \\ n &= \rho_0 \eta \,, & \pi &= (1+m)p \,. \end{aligned} \tag{28}$$

The flux $\boldsymbol{f}$ and the inhomogeneities $\boldsymbol{b}$ are

$$\begin{aligned} f_1(\boldsymbol{w}) &= v & b_1(\boldsymbol{w}) &= 0 \\ f_2(\boldsymbol{w}) &= S & b_2(\boldsymbol{w}) &= 0 \\ f_3(\boldsymbol{w}) &= -q_\kappa/\vartheta & b_3(\boldsymbol{w}) &= -\rho_0 F_2'(\beta)\psi_{20}\pi \\ f_4(\boldsymbol{w}) &= F_1(\vartheta) & b_4(\boldsymbol{w}) &= F_2'(\beta)\pi \\ f_5(\boldsymbol{w}) &= 0 & b_5(\boldsymbol{w}) &= F_1(\vartheta) + F_2(\beta) \end{aligned} \tag{29}$$

Obviously, all fluxes and sources are functions of the basic unknowns, however, in order to calculate them we need auxiliary quantities such as the classical temperature ϑ, $\kappa(\vartheta)$ and $c_v(\vartheta)$. Further, we need the constants ρ_0, the elastic constants, but also a, ν and ϑ_λ.
Most information is available for the case of NaF, for which the authors of [38,39] obtained useful approximations to the thermal material functions, while the authors of [42,46] obtained formulae for a and ν.

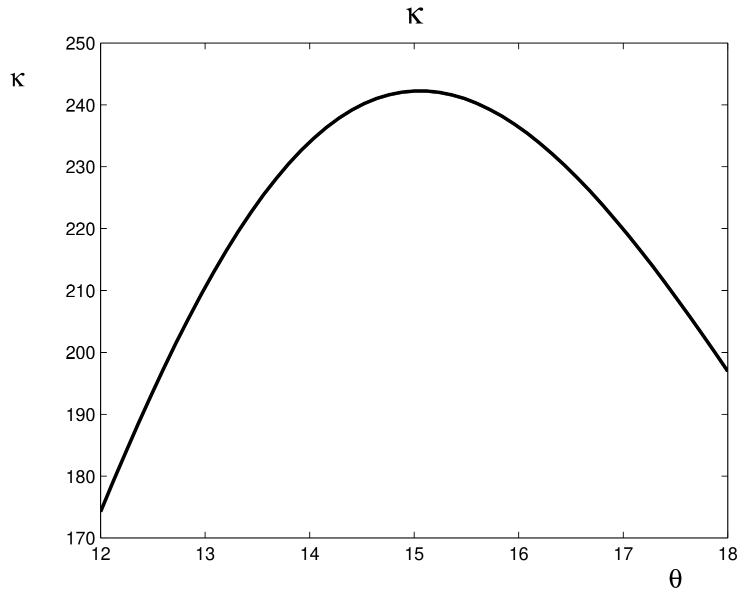

Fig. 1: Steady-state heat conductivity versus temperature.

Observe that for NaF (as well as for Bi) the thermal properties are extremely variable in the interesting temperature range (around ϑ_λ), while the elastic properties (e.g. first sound speed) are practically independent of ϑ.
Thus we conclude that we can use the procedure developed for the rigid heat conductor case in order to identify the above constants from available data on second sound speed (instead of using the exact sound speeds of the coupled system).

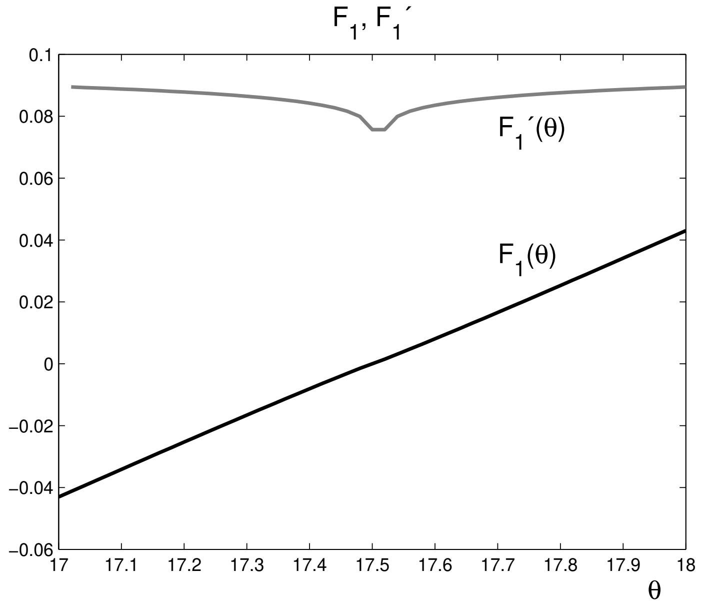

Fig. 2: F_1 and F_1' versus temperature.

In order to demonstrate the behaviour of the right-hand side of the system (29) we present here the temperature dependence of κ (Fig. 1) and F_1 (Fig. 2) as obtained by Frischmuth and Cimmelli [38,39], and by Frischmuth basing on results due to Kosiński and Saxton [42,46], respectively. Numerical solution of the thus specified model equations yields the temperature field of the type presented in Fig. 3. The two peaks, travelling at first and second sound speeds, are clearly visible. In the case we studied no mechanical disturbance of equilibrium was applied. Nevertheless, also the mechanical components of the solution exhibit waves corresponding to the ones we depicted.

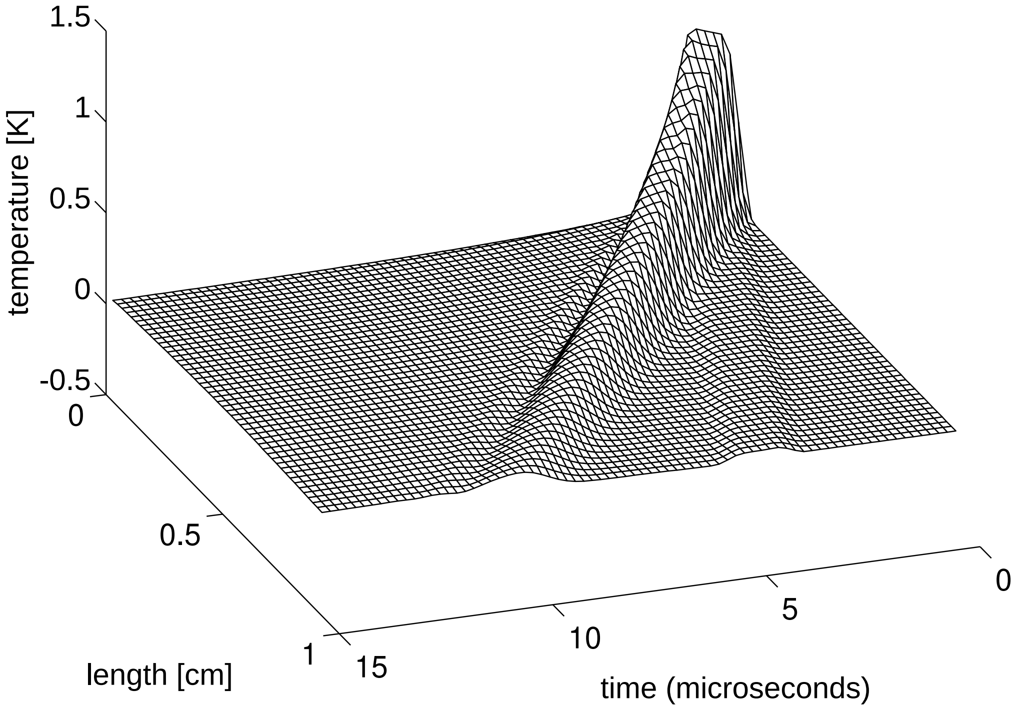

Fig. 3: Solution for temperature versus space and time.

The nonlinearity of the model does not lead to the formation of visible shocks, given the resolution of the applied numerical method, [38]. In fact, the characteristic speeds are almost constant for all states in the range of the solution, i.e, characteristics do not intersect (at reasonable angles). However, the nonlinearity shows in the varying wave speeds for different reference levels of ϑ and the quickly decaying amplitudes with increasing distance from the critical temperature. Thus we can observe *quantitatively* a transition into the classical diffusive regime in the sense that the applied boundary pulses fade away before reaching the end of the specimen (at room temperature).

6 Conclusions

Remark: Note that numerical experiments cannot be carried out at room temperature because some of the material functions are not reasonably defined outside the neighborhood of the critical temperature where measurements have been performed. For NaF, our approximations yield values that are in accordance with the second law (residual inequality) for up to 25 K (which does not mean that they are accurate, since they are based on data up to 20 K, only). However, already at this temperature the model solutions do not differ numerically from solutions obtained on the basis of Fourier's law. This is not true around the critical temperature, where the error reaches about 1000% of the Fourier result.

We should mention that all particular initial-boundary value problems solved for NaF or Bi crystals were very close to linear problems – although the model in general is clearly nonlinear.

This is due to the fact that realistic initial conditions at very low temperatures are spatially homogeneous for each case, and the heat pulses applied as boundary conditions have just a small amplitude. Otherwise we would leave the temperature range where second sound is observable.

The amplitudes of the mechanical variables are extremely small since – according to the description of experimental setups – no external forcing was applied. As a consequence the distinction between quantities for material and reference configurations has no quantitative relevance, neither the introduction of third order terms in the stress-strain relation.

Note, however, that a global linearization of the model would not be acceptable, instead, for each level of the reference temperature, a separate linearization would be required. In that case, a nonlinear model is preferable. The thermal submodel was shown to contain cubic terms, hence we chose to keep the same order for the mechanical part as well.

References

[1] C. C. Ackerman, B. Bertman, H. A. Fairbank and R. A. Guyer (1966), Second sound in solid helium, *Phys. Rev. Letters*, 16, 789-791 (1966).

[2] H. E. Jackson, C. T. Walker and T. F. McNelly, Second sound in NaF, *Phys. Rev. Letters*, 25 (1), 26-28 (1970).

[3] V. Narayanamurti, R. C. Dynes, Observation of second sound in Bismuth, *Phys. Rev. Letters*, 28 (22), 1461-1465 (1972).

[4] B. D. Coleman, D. C. Newman, Implication of a nonlinearity in the theory of second sound in solids, *Phys. Rev. B.* 32, 1492-1498 (1988).

[5] W. Dreyer, H. Struchtrup, Heat pulse experiments revisited, *Continuum Mech. Thermodyn.* 5, 3-50 (1993).

[6] D.Y. Tzou, Shock wave formation around a moving heat source in a solid with finite speed of heat propagation, *Int. J. Heat Mass Transfer* 32 (10), 1979 (1989).

[7] L. G. Hector Jr. , W.-S. Kim, M. N. Özisik, Hyperbolic heat conduction due to a mode locked laser pulse train, *Int.J.Engng.Sci.* 30 (12), 1731-1744 (1992).

[8] H. E. Jackson and C. T. Walker, Thermal conductivity, second sound, and phonon-phonon interactions in NaF, *Physical Review B*, 3 (4), 1428-1439 (1971).

[9] T. F. McNelly, S. J. Rogers, D. J. Chamin, R. J. Rollefson, W. M. Goubau, G. E. Schmidt, J. A. Krumhansi and R. O. Pohl, Heat pulses in NaF: onset of second sound, *Phys. Rev. Letters*, 24 (3), 100-102 (1970).

[10] Y. M. Pao, D. K. Banerjee, Thermal pulses in dielectric crystals, *Lett. Appl. Engng. Sci.* 1, 33-41 (1973).

[11] D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: A review of recent literature, *Applied Mechanics Review* 51, 12, 1, 705-729, (1998).

[12] D. D. Joseph, L. Preziosi, Heat waves, *Rev. Mod. Phys.* 61 (41), 1-62, (1989), Addendum, *Rev. Mod. Phys.* 62 (2), 375-392 (1990).

[13] D. Jou, J. Casas-Vázquez, G. Lebon, Extended irreversible thermodynamics revisited (1988–98), *Rep. Prog. Phys.* 62, 1035-1142 (1999).

[14] J. C. Maxwell, On the dynamical theory of gases, *Phil. Trans. Royal. Soc. London* 157, 49-101 (1867).

[15] C. Cattaneo, Sulla conduzione del calore, *Atti Sem. Mat. Fis. Univ. Modena* 3, 83-101 (1948).

[16] C. Cattaneo, Sur une forme de l'equation de la chaleur, éliminant le paradoxe d'une propagation instantanée, *Comp. Rend. Sci.* 247, 431-433 (1958).

[17] M. P. Vernotte, Les paradoxes de la theorie continue de l'equation de la chaleur, *Comp. Rend. Sci.* 246, 3154-3155 (1958).

[18] S. Kaliski, Wave equation of heat conduction, *Bull. Acad. Polon. Sci., Série Sci. Tech.* 13 (4), 211-219 (1965).

[19] M. E. Gurtin, A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Arch. Rat. Mech. Anal.* 31, 113-126 (1968).

[20] K. J. Baumeister, T. D. Hamil, Hyperbolic heat conduction equation – a solution for the semi-infinite body problem, *J.Heat Transfer* 91, 543 (1969).

[21] N. Fox, Generalized thermoelasticity, *Int. J. Eng. Sci.* 57 (1), 437-445 (1969).

[22] H. W. Lord, Y. Shulman, A generalized dynamical theory of thermoelasticity, *J. Mech. Phys. Solids* 15, 299-309 (1967).

[23] M. F. McCarthy, Thermodynamics of materials of differential type, *Proc. Vibr. Probl* now *J. Techn. Phys.* 13, 137-143, 1972.

[24] J. Ignaczak, Generalized thermoelasticity and its applications, in *"Thermal Stresses III"*, R.Hetnarski (ed.), Elsevier, Amsterdam, pp. 279-354 (1989).

[25] P. Mazilu, The equation of heat conduction, *Rev. Roum. Math. Pures et Appl.* 23 (3), 419-435 (1978).

[26] A. Morro, T. Ruggeri, Second sound and internal energy in solids, *Int. J. Non–Linear Mech.* 22, 27-36 (1987).

[27] I. Müller, Zum Paradoxen der Wärmeleitungstheorie, *Z. Phys.* 198, 329 (1967).

[28] W. Kosiński, P. Perzyna, Analysis of acceleration waves in materials with internal parameters, *Arch. Mech.* 24 (4), 1972, 629-643 (1972).

[29] W. Kosiński, Thermal waves in inelastic bodies, *Arch. Mech.* 23, 733-748 (1975).

[30] B. D. Coleman, M. Fabrizio, R. Owen, On the thermodynamics of second sound in dielectric crystals, *Arch. Rat. Mech. Anal.* 80 (2), 135-158 (1982).

[31] W. Kosiński, Elastic waves in the presence of a new temperature scale, in *"Elastic Wave Propagation"*, M. F. McCarthy and M. Hayes (eds.), Elsevier Science (North Holland), New York, pp. 629-634 (1989).

[32] M. Arcisz, W. Kosiński, Hugoniot relations and heat conduction laws, *J. Thermal Stresses* 19 (1), 17-37 (1996).

[33] V. A. Cimmelli, W. Kosiński, Nonequilibrium semi–empirical temperature in materials with thermal relaxation, *Arch. Mech.* 43 (6), 753-767 (1991).

[34] V. A. Cimmelli, W. Kosiński, K. Saxton, Modified Fourier law – comparison of two approaches, *Arch. Mech.* 44 (4), 409-415 (1992).

[35] W. Kosiński, K. Saxton, The effect on finite time breakdown due to modified Fourier laws, *Quart. Appl. Math.* 51 (1), 55-68 (1993).

[36] W. Kosiński, K. Saxton, Weak discontinuity waves in materials with semi-empirical temperature scale, *Arch. Mech.* 43 (4), 547-559 (1991).

[37] Cz. Bajer, W. Kosiński, Numerical modelling of thermal waves via internal state variable approach, *Comp. Assisted Mech. and Engng. Sciences* 2 (4), 307-319 (1995).

[38] K. Frischmuth, V. A. Cimmelli, Numerical reconstruction of heat pulse experiments, *Int. J. Engng. Sci.* 33 (2), 209-215 (1995).

[39] K. Frischmuth, V. A. Cimmelli, Hyperbolic heat conduction with variable relaxation time, *J. Theor. Appl. Mechanics* 34 (1), 67-76 (1996).

[40] K. Frischmuth, V. A. Cimmelli, Coupling in thermo-mechanical wave propagation in NaF at low temperature, *Arch. Mech.* 50 (4), 703-713 (1998).

[41] V. A. Cimmelli, W. Kosiński, Well-posedness results for a nonlinear hyperbolic heat equation, *Ricerche di Matematica* XLII, 1-21 (1993).

[42] W. Kosiński, K. Saxton, R. Saxton, Second sound speed in a crystal of NaF at low temperature, *Arch. Mech.* 49 (1), 188-196 (1997).

[43] W. Kosiński, Thermodynamics of continua with heat waves, in *"Thermoelastic Problems and Thermodynamics of Continua", Proc. of the 1995 Joint ASME Applied Mechanics and Materials Summer Meeting, Los Angeles, California, June 28-30, 1995, UCLA*, L. Brock (ed.), AMD Vol. 198, American Society of Mechanical Engineers, New York, N.Y., pp. 19-25 (1995).

[44] W. Kosiński, Hyperbolic framework for thermoelastic materials, *Arch. Mech.* 50 (3), 423-450 (1998).

[45] W. Kosiński, Thermal waves in the kinematic theory, in *Proc. IUTAM Symposium on Nonlinear Waves in Solids*, August 15-20, 1993, Victoria B.C., Canada J. Wagner and F. Norwood, (eds.), *ASME Book, Appl. Mech. Rev.* 137, Fairfield, NJ, pp. 247-252 (1994).

[46] K. Saxton, R. Saxton, W. Kosiński, On second sound at the critical temperature, *Quart. Appl. Math.*, 57 (4) December, 723-740 (1999).