# PAIRWISE BALANCED DESIGNS PBD(v,K) WITH 3 $\in$ K

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**Abstract**

In this paper, we consider sufficient conditions for the existence of pairwise balanced designs PBD$(v, K)$ with $3 \in K$. Such conditions have been extensively studied by Wilson [W], Gronau, Mullin and Pietsch [GMP], and Grüttmüller [G1,G2]. Here, we show that the necessary conditions for the existence of a PBD$(v, K)$ with $3 \in K$ are sufficient when $v \geq v_0(K)$ where $v_0(K)$ is reasonable small.

# 1 Introduction

Let $K$ be a set of positive integers. Then a *pairwise balanced design* PBD$(v, K)$ of order $v$ with block sizes from $K$ is a pair $(V, \mathcal{B})$, where $V$ is a finite set (the *point set*) of cardinality $v$ and $\mathcal{B}$ is a family of subsets (called *blocks*) of $V$ which satisfy the following properties:

**(i)** if $B \in \mathcal{B}$, then $|B| \in K$;

**(ii)** every pair of distinct elements of $V$ occurs in exactly one block of $\mathcal{B}$.

The basic necessary condition for the existence of a PBD$(v, K)$ are due to Wilson [W]. Let $\alpha(K) = \gcd\{k - 1 \,|\, k \in K\}$ and $\beta(K) = \gcd\{k(k-1) \,|\, k \in K\}$. If there is a PBD$(v, K)$, then

$$v - 1 \equiv 0 \bmod \alpha(K) \ \text{ and } \ v(v-1) \equiv 0 \bmod \beta(K)\,. \tag{1}$$

Wilson also proved that these conditions are asymptotically sufficient, that is, there exists a constant $v_0 = v_0(K)$ such that if $v \geq v_0$ and (1) is satisfied, then there exists a PBD$(v, K)$. Unfortunately, the constant $v_0$ is not known in general. Although Wilson's proof is somehow constructive, the estimate of the constant is very large. Therefore, one attempts to determine $v_0$ and $B(K) = \{v : \exists \text{ a PBD}(v, K)\}$ for given $K$ as accurately as possible. For example, Kirkman [K] has shown that there exists a PBD$(v, \{3\})$ for all $v \equiv 1$ or 3 mod 6. Hence, $v_0(\{3\}) = 1$. In [GMP] the exact closure $B(K)$ for all subsets $K$ of $\{3, 4, \dots, 10\}$ which contain 3 are determined. This was extended by Grüttmüller [G1], who has determined the precise closure for all subsets of $\{3, 4, \dots, 22\}$ containing the element 3. An investigation on index-$\lambda$-closures of the subsets of $\{3, 4, \dots, 10\}$ which contain 3 has been done by Leck [L]. In [MLAB] the authors have considered the closure of all subsets of $\{4, 5, \dots, 9\}$ which contain 4.

It has been observed that in general the exact value $v_0(K)$ depends on relatively few elements of $K$. That means that the closure of a set $K$ is essentially determined by the closure of its subsets $K'$ of cardinality two and three. All other block sizes are only used to eliminate numbers that are in the list of exceptions for $B(K')$.

It is the purpose of the paper presented to determine reasonable small upper bounds for $v_0(K)$ for an arbitrary set $K$, where 3 in $K$.

# 2 Preliminaries

In this section, we give some definition and notations as well as some preliminary results which will be used in the sequel. We refer the reader to [BJL] and [CD] for undefined terms as well as a general overview of design theory.

First, the notion of a group divisible design is required. Let $K$ be a set of positive integers. A *group divisible design* $K$-GDD of order $v$ is a triple $(V, \mathcal{G}, \mathcal{B})$, where $V$ is a finite set of cardinality $v$, $\mathcal{G}$ is a partition of $V$ into parts (*groups*) and $\mathcal{B}$ is a family of subsets (*blocks*) of $V$ which satisfy the properties:

**(i)** if $B \in \mathcal{B}$, then $|B| \in K$;

**(ii)** every pair of distinct elements of $V$ occurs in exactly one block or one group, but not both.

We say that a $K$-GDD is of *type* $g_1^{a_1} g_2^{a_2} \dots g_s^{a_s}$ if $v = a_1 g_1 + a_2 g_2 + \dots + a_s g_s$ and there are $a_i$ groups of size $g_i$, $i = 1, 2, \dots, s$. We will denote a $\{k\}$-GDD as a $k$-GDD. Note that GDDs may be viewed as PBDs by considering groups to be blocks.

The following family of GDDs will be extremely useful.

**Theorem 2.1** (Colbourn, Hoffman, Rees [CHR]) *Let $g > 0$, $t \geq 3$ and $u \geq 0$. There exists a 3-GDD of type $g^t u^1$ if and only if the following conditions are all satisfied:*

**(i)** $u \leq g(t-1)$;

**(ii)** $g(t-1) + u \equiv 0 \ mod\ 2$;

**(iii)** $gt \equiv 0 \ mod\ 2$ *or* $u = 0$;

**(iv)** $\frac{1}{2}g^2 t(t-1) + gtu \equiv 0 \ mod\ 3$.

We use a particular type of design called a modified group divisible design as an essential tool in our constructions. A *modified group divisible design* MGD$(k, m, v)$ is a quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where $V$ is a finite set of cardinality $v$, $\mathcal{G}$ and $\mathcal{H}$ are two partitions of $V$ into parts (*groups* and *holes*) and $\mathcal{B}$ is a family of subsets (*blocks*) of $V$ which satisfy the properties:

**(i)** if $G \in \mathcal{G}$, then $|G| = m$;

**(ii)** if $B \in \mathcal{B}$, then $|B| = k$;

**(iii)** if $G \in \mathcal{G}$ and $H \in \mathcal{H}$, then $|G \cap H| = 1$;

**(iv)** every pair of distinct elements of $V$ occurs either in exactly one block, or exactly one group or one hole, but not both.

MGD$(3, m, v)$s were studied by Assaf in [A], in which their existence was completely determined as recorded in Theorem 2.2.

**Theorem 2.2** *Let $m$ and $v$ be positive integers. The necessary and sufficient conditions for the existence of a modified group divisible design $MGD(3, m, v)$ are that $v \equiv 0 \ mod\ m$, $v \geq 3m$, $m \geq 3$, $v + 1 - m - n \equiv 0 \ mod\ 2$ and $v(v + 1 - m - n) \equiv 0 \ mod\ 6$, where $n = v/m$.*

A *mandatory representation design* MRD$(v, K)$ is a pairwise balanced design on $v$ points with block sizes from $K$ in which for each $k \in K$ there is a block in the design of size $k$.

MRDs with $K = \{3, k\}$ have been studied by Mendelsohn and Rees [MR], Rees [R], and Grüttmüller [G2, G3]. We cite the following existence results.

**Theorem 2.3** [MR] *There exists an $MRD(v, \{3, k\})$ with $k$ odd if and only if either*

**(i)** $k \equiv 1$ *or* $3 \ mod\ 6$, $v \equiv 1$ *or* $3 \ mod\ 6$ *and* $v \geq 2k + 1$; *or*

**(ii)** $k \equiv 5$ *mod* $6$, $v \equiv 5$ *mod* $6$ *and* $v \geq 2k+1$*; or*

**(iii)** $k \equiv 5$ *mod* $6$, $v \equiv 1$ *or* $3$ *mod* $6$ *and* $v \geq 3k-2$.

**Theorem 2.4** [R] *There exists an MRD*$(v, \{3, k\})$ *with* $k$ *even and* $v$ *odd if*

**(i)** $k \equiv 0$ *or* $4$ *mod* $6$, $v \equiv 1$ *or* $3$ *mod* $6$ *and* $v \geq \frac{1}{2}k^2 + 6k$*; or*

**(ii)** $k \equiv 2$ *mod* $6$ *and* $v \geq \frac{1}{2}k^2 + 6k$.

We close this section by giving the following notation. Two integer sets $N_1$ and $N_2$ are called *coterminal* when there is an integer $s$ such that $\{t \in N_1 \mid t \geq s\} = \{t \in N_2 \mid t \geq s\}$. We use $\mathbb{N}$ to denote the set of positive integers, and $\mathbb{Z}_n$ to denote the cyclic group of order $n$. Let $\mathbb{N}_{b(m)}$ denote the set of positive integers $a$ which are congruent to $b$ modulo $m$, $\mathbb{N}_{b(m)} = \{a \in \mathbb{N} \,|\, a \equiv b \bmod m\}$, and let $\mathbb{N}_{b_1,\ldots,b_s(m)} = \bigcup_{i=1}^{s} \mathbb{N}_{b_i(m)}$.

# 3 Upper bounds for $v_0(K)$

If 3 is an element of $K$ and $K$ is a subset of $\mathbb{N}_{1,3(6)}$, then obviously $v_0(K) = 1$, since $B(\{3\}) \subseteq B(K)$. Now, by Wilson's theory there are three cases for the closure of a set $K$ when $K$ includes 3 and $K$ is not a subset of $\mathbb{N}_{1,3(6)}$.

**(i)** If $K$ contains only odd integers and is not a subset of $\mathbb{N}_{1,3(6)}$, then $B(K)$ is coterminal with $\mathbb{N}_{1(2)}$.

**(ii)** If $K$ contains a positive even integer and all members of $K$ lie in $\mathbb{N}_{0,1(3)}$, then $B(K)$ is coterminal with $\mathbb{N}_{0,1(3)}$.

**(iii)** If $K$ contains a positive even integer and is not a subset of $\mathbb{N}_{0,1(3)}$, then $B(K)$ is coterminal with $\mathbb{N}$.

We now record our main theorem as

**Theorem 3.1** *Let* $K$ *be a set of positive integers with* $3 \in K$. *Let* $g$, $k$ *and* $h$ *be the smallest elements in* $K$ *with* $g \equiv 2$ *mod* $6$, $k \equiv 0$ *or* $4$ *mod* $6$ *and* $h \equiv 5$ *mod* $6$ *(if they exist). There exists a PBD*$(v, K)$

**(i)** *if* $K \subseteq \mathbb{N}_{1(2)}$, $K \cap \mathbb{N}_{5(6)} \neq \emptyset$, $v \equiv 1$ *mod* $2$ *and* $v \geq v_0 = \max(\{1\} \cup \{i+2 : i \in \mathbb{N}_{5(6)}, i < 2h+1, i \notin K\})$*; or*

**(ii)** *if* $K \subseteq \mathbb{N}_{0,1(3)}$, $K \cap \mathbb{N}_{0(2)} \neq \emptyset$, $v \equiv 0$ *or* $1$ *mod* $3$ *and* $v \geq 2k^2 - 3k + 4$*; or*

**(iii)** *if* $K \cap \mathbb{N}_{0(2)} \neq \emptyset$, $K \cap \mathbb{N}_{2(3)} \neq \emptyset$ *and* $v \geq \min\{6g^2 - 11g + 8\,,\, 12hk + 9k^2 - 29k - 8h + 20\}$.

**Proof.**
Since there is a PBD$(v, \{3\})$ for all $v \equiv 1$ or $3 \bmod 6$ by Kirkmans result we only need to consider those parameters $v$ with $v \notin \mathbb{N}_{1,3(6)}$. Therefore, in case (i) we need PBD$(v, K)$s for $v \equiv 5 \bmod 6$ with at least one block of size congruent to 5 modulo 6. But such designs exist by Theorem 2.3 for all $v \geq 2h + 1$ or, trivially, when $v \in K \cap \mathbb{N}_{5(6)}$. In case (ii) we need PBD$(v, K)$s for $v \equiv 0$ or $4 \bmod 6$ with at least one block of size congruent 0 or 4 modulo 6. In Section 4 (Lemma 4.1) we will show the existence of MRD$(v, \{3, k\})$s for all $v \equiv 0$ or $4 \bmod 6$ with $v \geq 2k^2 - 3k + 4$. Finally, in case (iii) we consider $v \equiv 0, 2, 4$ or $5 \bmod 6$. The existence of PBD$(v, K)$s with $v \equiv 5 \bmod 6$, $v \geq \min\{2h + 1\,,\, \frac{1}{2}g^2 + 6g\}$ can be established by using MRDs either from Theorem 2.3 or Theorem 2.4. MRDs from Lemma 4.1 prove the existence of PBD$(v, K)$s with $v \equiv 0$ or $4 \bmod 6$ and $v \geq \min\{2k^2 - 3k + 4\,,\, 6g^2 - 11g + 8\}$. It remains to present suitable MRDs for PBD$(v, K)$s with $v \equiv 2 \bmod 6$. These designs are constructed in Lemma 4.1 or Lemma 4.2 for all parameters $v$ with $v \geq \min\{6g^2 - 11g + 8\,,\, 12hk + 9k^2 - 29k - 8h + 20\}$. $\square$

Note that in case (i) Theorem 3.1 provides the exact value for $v_0(K)$. We remark that all designs involved in the above proof have at most three different block sizes. In case (iii) we need to have PBDs which contain blocks of three different sizes if there is no block size congruent to 2 modulo 6 contained in $K$.

# 4 Constructions

In this section, we develop the constructions for MRD$(v, \{3, k\})$s and MRD$(v, \{3, k, h\})$s for even $v$ required in the proof of Theorem 3.1. In order to facilitate this, we observe that there are additional necessary conditions to be satisfied.

Assume that there exists an MRD$(v, \{3, k\})$ where both $v$ and $k$ are even. Let $x$ be an arbitrary point and let $\gamma_k$ denote the number of blocks of size $k$ which contain $x$. Then we must have $v - 1 - (k - 1)\gamma_k \equiv 0 \bmod 2$ which reduces to $\gamma_k \equiv 1 \bmod 2$. Hence, we want to construct MRDs where the blocks of size $k$ cover all points.

**Lemma 4.1** *Let $k$ be an even integer. There exists an MRD$(v, \{3, k\})$*

**(i)** *if* $k \equiv 0 \bmod 6$, $v \equiv 0$ *or* $4 \bmod 6$ *and* $v \geq 2k^2 - 7k + 8$;

**(ii)** *if* $k \equiv 4 \bmod 6$, $v \equiv 0$ *or* $4 \bmod 6$ *and* $v \geq 2k^2 - 3k + 4$; *and*

**(iii)** *if* $k \equiv 2$ *mod* $6$, $v$ *is even and* $v \geq 6k^2 - 11k + 8$.

**Proof.**
All group divisible designs used in this proof are obtained from Theorem 2.1.

If $k \equiv 0 \bmod 6$, then there exists a 3-GDD of type $k^s w^1$ for all even $w$ with $0 \leq w \leq k(s-1)$. Hence, if there exists an MRD$(w, \{3, k\})$, then by breaking up the group of size $w$ there exists an MRD$(v, \{3, k\})$ for all $v \equiv w \bmod k$ with $v \geq 2w + k$. If $k \equiv 0 \bmod 6$, then $v \equiv 0$ or $4 \bmod 6$. Let $w_t \equiv -2t \bmod k$. One has to find a representative MRD$(w_t, \{3, k\})$ for each $0 \leq t < \frac{k}{2}$ with $t \equiv 0$ or $1 \bmod 3$ to establish the result. There is a 3-GDD of type $(k-1)^s$ for all $s \geq 1$, $s \equiv 1$ or $3 \bmod 6$. Adding a common point to each group yields an MRD$(s(k-1)+1, \{3, k\})$. Then $s = 1, 3, 7, \ldots, k-3$ gives the desired representatives modulo $k$ since $s(k-1)+1 \equiv -s+1 \bmod k$ and $-s+1 \equiv -2t \bmod k$ with $0 \leq t < \frac{k}{2}$, $t \equiv 0, 1 \bmod 3$. Clearly, all "ingredient" designs have order at most $(k-3)(k-1)+1$. Thus, a mandatory representation design can be constructed for all

$$v \geq 2(k^2 - 4k + 4) + k = 2k^2 - 7k + 8, \quad v \equiv 0 \text{ or } 4 \bmod 6.$$

If $k \equiv 4 \bmod 6$, then there is a 3-GDD of type $k^{3s} w^1$ for all even $w$ with $0 \leq w \leq k(3s-1)$, a 3-GDD of type $k^{3s+1} w^1$ for all $w \equiv 0 \bmod 6$ with $0 \leq w \leq 3ks$, and a 3-GDD of type $k^{3s+2} w^1$ for all $w \equiv 4 \bmod 6$ with $0 < w \leq k(3s+1)$. Since $v \equiv 0$ or $4 \bmod 6$, we have that if there exists an MRD$(w, \{3, k\})$, then there exists an MRD$(v, \{3, k\})$ for all $v \equiv w \bmod k$ with $v \geq 2w + k$. Therefore, one needs a representative MRD$(w_t, \{3, k\})$ with $w_t \equiv -2t \bmod k$ for each $0 \leq t < \frac{1}{2}k$ to establish the result. Again, we use a 3-GDD of type $(k-1)^s$ which exists for all $s \geq 1$, $s \equiv 1 \bmod 2$ to obtain an MRD$(s(k-1)+1, \{3, k\})$. Choose $s = 1, 3, 5, \ldots, k-1$ to get the required "ingredient" designs, since $s(k-1)+1 \equiv -s+1 \bmod k$. Therefore, there exists an MRD$(v, \{3, k\})$ for all

$$v \geq 2((k-1)(k-1)+1) + k = 2k^2 - 3k + 4, \quad v \equiv 0 \text{ or } 4 \bmod 6.$$

Similarly, if $k \equiv 2 \bmod 6$, then there exists a 3-GDD of type $k^{3s} w^1$ for all even $w$ with $0 \leq w \leq k(3s-1)$, a 3-GDD of type $k^{3s+1} w^1$ for all $w \equiv 0 \bmod 6$ with $0 \leq w \leq 3ks$, and a 3-GDD of type $k^{3s+2} w^1$ for all $w \equiv 2 \bmod 6$ with $0 < w \leq k(3s+1)$. Thus, if there exists an MRD$(w, \{3, k\})$ with $w \equiv 2 \bmod 6$, then there exists an MRD$(v, \{3, k\})$ for all $v \equiv w \bmod 3k$ with $v \geq 2w + k$ and for all $v \equiv w + 2k \bmod 3k$ with $v \geq 2w + k$. Moreover, if there is an MRD$(w, \{3, k\})$ with $w \equiv 4 \bmod 6$, then there is an MRD$(v, \{3, k\})$ for all $v \equiv w \bmod 3k$ with $v \geq 2w + k$. Therefore, we only need a representative MRD$(w_t, \{3, k\})$ with $w_t \equiv -2t \bmod 3k$ for each $0 \leq t < \frac{3}{2}k$ with $t \equiv$

1 or 2 mod 3. An MRD$(s(k-1)+1,\{3,k\})$ can be constructed from a 3-GDD of type $(k-1)^s$ for all $s \equiv 1$ or 3 mod 6. Take $s = 1, 3, 7, \ldots, 3k-3$ to obtain a representative design in each required fibre $-2t$ mod $3k$. Thus, we get an MRD$(v,\{3,k\})$ for all even $v$ with

$$v \geq 2((3k-3)(k-1)+1)+k = 6k^2-11k+8. \qquad \square$$

When constructing MRD$(v,\{3,k,h\})$s we have a further necessary condition in addition to $\gamma_k \equiv 1$ mod 2. If there are $\beta_k$ blocks of size $k$ and $\beta_h$ blocks of size $h$ in an MRD$(v,\{3,k,h\})$, then we must have $v(v-1)-\beta_k k(k-1)-\beta_h h(h-1) \equiv 0$ mod 6 which, when $v \equiv 2$ mod 6, $k \equiv 0$ or 4 mod 6 and $h \equiv 5$ mod 6, reduces to $\beta_h \equiv 1$ mod 3. Thus, we will construct MRDs having exactly one block of size $h$ in which all points are covered by blocks of size $k$.

**Lemma 4.2** *Let $k \equiv 0$ or 4 mod 6 and $h \equiv 5$ mod 6 be positive integers. There exists an MRD$(v,\{3,k,h\})$*

**(i)** *if $k \equiv 0$ mod 6, $v \equiv 2$ mod 6 and $v \geq 12hk+3k^2-25k-8h+20$;*

**(ii)** *if $k \equiv 4$ mod 6, $v \equiv 2$ mod 6 and $v \geq 12hk+9k^2-29k-8h+20$.*

**Proof.**
We start with two classes of GDDs from Theorem 2.1. If $k \equiv 0$ mod 6 and $w \equiv 2$ mod 6, then there exists a 3-GDD of type $k^s w^1$ for all integers $s$ with $s \geq \max\{3, w/k+1\}$. Furthermore, if $k \equiv 4$ mod 6, $w \equiv 2$ mod 6 and $s \equiv 0$ mod 3, then there exists a 3-GDD of type $k^s w^1$ for all integers $s$ with $s \geq \max\{3, w/k+1\}$. Hence, if there exists an MRD$(w,\{3,k,h\})$ with $w \equiv 2$ mod 6 we obtain, by breaking up the group of order $w$, an MRD$(v,\{3,k,h\})$ for all $v \geq 2w+k$ with $v \equiv w$ mod $k$ if $k \equiv 0$ mod 6 or $v \equiv w$ mod $3k$ if $k \equiv 4$ mod 6. Thus, we need a representative MRD$(w,\{3,k,h\})$ in each residue class modulo $k$ or $3k$, respectively. Since we want $w \equiv 2$ mod 6, we need an MRD$(w_t,\{3,k,h\})$ with $w_t \equiv -2t$ mod $k$ ($w_t \equiv -2t$ mod $3k$) for all $0 \leq t < k/2$ ($0 \leq t < 3k/2$) with $t \equiv 2$ mod 3.

We construct the required MRDs in the following way. Let $2h-2 \leq t$ with $t \equiv 2$ mod 3. Put $p=t$, $q=t$ if $t \equiv 5$ mod 6 or $p=t+1$, $q=t+3$ if $t \equiv 2$ mod 6. We take an MGD$(3,k-1,(k-1)3p)$ and add $q$ new points from $H_0=\{x_j : j \in \mathbb{Z}_q\}$ to the groups such that there are exactly $t$ new points each of which is contained in precisely 3 groups (see Figure 1 for the case $t \equiv 2$ mod 6; for $t \equiv 5$ mod 6 just remove $G_{0,t}$, $G_{1,t}$ and $G_{2,t}$ from the figure). We denote the groups by $G_{i,j}$, $(i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_p$ and assume w.l.o.g. that $G_{i,j} \cap G_{i+1,j} = x_j$ for $j = 0, \ldots, t-1$ and that, when $t \equiv 2$ mod 6, $x_{t+i} \in G_{i,t}$

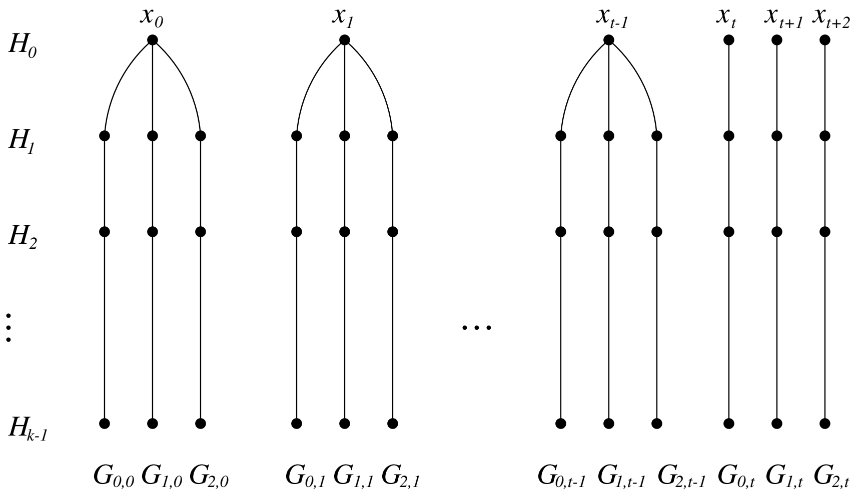


Figure 1: MGD$(3, k-1, (k-1)(3t+3))$ with $t+3$ new points, $t \equiv 2 \bmod 6$

for $i = 0, 1, 2$. Let $H_1, \ldots, H_{k-1}$ denote the holes in the MGD and let a point $x$ ($x \notin H_0$) be labeled by $(a, b, c)$ if $x \in G_{a,b} \cap H_c$. The design obtained has $w_t = (k-1)3t+t = 3tk-2t$ or $w_t = (k-1)(3t+3)+t+3 = 3tk+3k-2t$ points. Thus in each case $w_t \equiv -2t \bmod k$ and also $w_t \equiv -2t \bmod 3k$. Moreover, since $k$ is even, we have $w_t \equiv -2t \bmod 6$ and, therefore, $w_t \equiv 2 \bmod 6$ as required.

It remains to fill the holes with blocks of size 3 and $h$, that is, to cover pairs of points from the same hole and pairs of points with at least one point from $H_0$ which are not a subset of one of the groups.

First, we cover those pairs containing a point from $H_c$ ($1 \leq c \leq k-1$). Let $c$ be fixed, and define $G'_a = \{(a, b, c) : b \in \mathbb{Z}_p\}$ for $a \in \mathbb{Z}_3$. If $t \equiv 5 \bmod 6$ use the blocks of a 3-GDD of type $p^3$ where the groups are formed by $G'_0, G'_1$, and $G'_2$. Furthermore, construct on each of the groups $G'_a$ a factorization into $p$ near-1-factors. Add to all pairs of the near-1-factor with the isolated point $(a, b, c)$ the point $x_b$ to obtain the required triples. If $t \equiv 2 \bmod 6$ and, therefore, $p = t+1$ we use the blocks of an MGD$(3, p, 3p)$ where the groups are formed by $G'_0, G'_1$, and $G'_2$. Moreover, construct on each of the groups $G'_a$ a factorization into $p$ near-1-factors. Add to all pairs of the near-1-factor with the isolated point $(a, b, c)$ ($b < t$) the point $x_b$ and adjoin the point $x_{t+a}$ to all pairs of the near-1-factor with the isolated point $(a, t, c)$. Now form triples $\{(a, b, c), (a+1, b, c), x_{t+a+2}\}$ with $(a, b) \in \mathbb{Z}_3 \times \mathbb{Z}_p$ to cover the remaining pairs with a point from $H_c$.

Finally, we cover all pairs having both points from $H_0$ by the blocks of an MRD$(q, \{3, h\})$ which always exists by Theorem 2.3 since $q \equiv 5 \bmod 6$ and

$q \geq 2h+1$.

We have thus constructed an MRD of order $w_t$ for all $2h-2 \leq t \leq t_{\max} = 2h-5+\frac{a}{2}k$ where

$$w_t = \begin{cases} 3(t+1)k-2t & \text{if } t \equiv 2 \bmod 6, \\ 3tk-2t & \text{if } t \equiv 5 \bmod 6, \end{cases} \quad \text{and} \quad a = \begin{cases} 1 & \text{if } k \equiv 0 \bmod 6, \\ 3 & \text{if } k \equiv 4 \bmod 6. \end{cases}$$

It is easily checked that all ingredient designs have order at most $w_{t_{\max}}$ where

$$w_{t_{\max}} = \begin{cases} 3(2h-5+\frac{1}{2}k+1)k-2(2h-5+\frac{1}{2}k) & \text{if } \ k \equiv 0 \bmod 6, \\ 3(2h-5+\frac{3}{2}k+1)k-2(2h-5+\frac{3}{2}k) & \text{if } \ k \equiv 4 \bmod 6. \end{cases}$$

Using the class of 3-GDDs previously mentioned, we will obtain an MRD$(v,\{3,k,h\})$ for all $v \geq 2w_{t_{\max}}+k$. $\square$

# 5 Conclusions

We conclude this study by noting that the bounds obtained in Theorem 3.1, cases (ii) and (iii), can be improved by obtaining smaller bounds on the MRDs of the types constructed in Lemmas 4.1 and 4.2. In Table 1 we exhibit the gap between the upper bounds described in this paper (known bounds) and the bounds obtained from the necessary conditions for MRDs (see [MR,R,G3]) (expected bounds). We remark that computer constructions in [G2] have shown that the expected bound from the necessary conditions was never exceeded.

In the last case of Table 1 there are no reasonable necessary conditions known; therefore, we provide the exact value of $v_0(\{3,k,h\})$ where $k,h \leq 22$ in Table 2.


# 6 Acknowledgements

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| Case | Necessary conditions | Bounds for $v_0(K)$ known | Bounds for $v_0(K)$ expected |
|---|---|---|---|
| $K \subseteq \mathbb{N}_{1,3(6)}$ | $v \equiv 1$ or $3 \bmod 6$ | 1 | 1 |
| $K \subseteq \mathbb{N}_{1(2)}$, $K \cap \mathbb{N}_{5(6)} \neq \emptyset$ | $v \equiv 1$ or $3 \bmod 6$ | 1 | 1 |
| $h = \min K \cap \mathbb{N}_{5(6)}$ | $v \equiv 5 \bmod 6$ | 2h+1 | 2h+1 |
| $K \subseteq \mathbb{N}_{0,1(3)}$, $K \cap \mathbb{N}_{0(2)} \neq \emptyset$ | $v \equiv 1$ or $3 \bmod 6$ | 1 | 1 |
| $k = \min K \cap \mathbb{N}_{0(2)}$ | $v \equiv 0$ or $4 \bmod 6$ | $2k^2 - 3k + 4$ | $\sim \sqrt{3}k^{\frac{3}{2}}$ |
| $K \cap \mathbb{N}_{2(6)} \neq \emptyset$ | $v \equiv 1$ or $3 \bmod 6$ | 1 | 1 |
| $g = \min K \cap \mathbb{N}_{2(6)}$ | $v \equiv 5 \bmod 6$ | $\frac{1}{2}g^2 + 6g$ | $\frac{1}{2}g^2 + g + 1$ |
| | $v \equiv 0 \bmod 2$ | $6g^2 - 11g + 8$ | $\sim 3g^{\frac{3}{2}}$ |
| $K \cap \mathbb{N}_{0,4(6)} \neq \emptyset$ | $v \equiv 1$ or $3 \bmod 6$ | 1 | 1 |
| $K \cap \mathbb{N}_{5(6)} \neq \emptyset$ | $v \equiv 5 \bmod 6$ | $2h + 1$ | $2h + 1$ |
| $h = \min K \cap \mathbb{N}_{5(6)}$ | $v \equiv 0$ or $4 \bmod 6$ | $2k^2 - 3k + 4$ | $\sim \sqrt{3}k^{\frac{3}{2}}$ |
| $k = \min K \cap \mathbb{N}_{0,4(6)}$ | $v \equiv 2 \bmod 6$ | $12kh + 9k^2$ | ? |

Table 1: Known and expected bounds for $v_0(K)$.

| $K$ | $v_0(K)$ | $K$ | $v_0(K)$ | $K$ | $v_0(K)$ |
|---|---|---|---|---|---|
| $\{3, 4, 5\}$ | 9 | $\{3, 4, 11\}$ | 21 | $\{3, 4, 17\}$ | 33 |
| $\{3, 6, 5\}$ | 23 | $\{3, 6, 11\}$ | 33 | $\{3, 6, 17\}$ | 45 |
| $\{3, 10, 5\}$ | 45 | $\{3, 10, 11\}$ | 69 | $\{3, 10, 17\}$ | 75 |
| $\{3, 12, 5\}$ | 63 | $\{3, 12, 11\}$ | 93 | $\{3, 12, 17\}$ | 105 |
| $\{3, 16, 5\}$ | 87 | $\{3, 16, 11\}$ | 141 | $\{3, 16, 17\}$ | 177 |
| $\{3, 18, 5\}$ | 111 | $\{3, 18, 11\}$ | 159 | $\{3, 18, 17\}$ | 213 |
| $\{3, 22, 5\}$ | 159 | $\{3, 22, 11\}$ | 207 | $\{3, 22, 17\}$ | 285 |

Table 2: Exact values for $v_0(\{3, k, h\})$ where $k, h \leq 22$.

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