NECESSARY EXISTENCE CRITERIA FOR MANDATORY REPRESENTATION DESIGNS MRD (4,k;v)

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Abstract

A mandatory representation design MRD $(K;v)$ is a pairwise balanced design on v points with block sizes from the set K in which for each $k \in K$ there is at least one block in the design of size k. A survey of known results on MRDs has been given in [R2] by Rees. MRDs with $K = \{3,k\}$ have been extensively studied by Mendelsohn and Rees [MR], Rees [R], Rees and Grüttmüller [GR2] and Grüttmüller [G, G2]. In this paper, we investigate necessary conditions for the existence of MRDs with $K = \{4,k\}$.

1 Introduction

A pairwise balanced design PBD is a pair $(X,\mathcal{B})$, where X is a set of *points* and $\mathcal{B}$ is a collection of subsets of X called *blocks*, such that each pair of distinct points from X occurs in a unique block. A PBD $(K;v)$ is a pairwise balanced design on v points in which each block has size an integer in the set K. A *mandatory representation design* MRD $(K;v)$ is a PBD $(K;v)$ in which for each $k \in K$ there is at least one block in the design of size k. These designs occur when considering the problem of constructing certain classes of pairwise balanced designs with pre-specified subdesigns. For example, a $(v,4,1)$-BIBD (balanced incomplete block design) is a PBD $(4;v)$; it is well known that a $(v,4,1)$-BIBD exists if and only if $v \equiv 1$ or $4 \bmod 12$. If $(X,\mathcal{B})$ is a $(v,4,1)$-BIBD, then $(X',\mathcal{B}')$ is a subdesign of $(X,\mathcal{B})$ if and only if $X' \subseteq X$, $\mathcal{B}' \subseteq \mathcal{B}$ and $(X',\mathcal{B}')$ is a $(w,4,1)$-BIBD. The following result is the culmination of the contributions of several authors (Brouwer and Lenz [BL1, BL2], Wei and Zhu [WZ], Rees and Stinson [RS]).

Theorem 1.1 *There exists a $(v,4,1)$-BIBD containing a sub-$(w,4,1)$-BIBD if and only if $v,w \equiv 1$ or 4 mod 12 and $v \geq 3w+1$.*

Theorem 1.1 is clearly equivalent to the assertion that for each $w \equiv 1$ or $4 \bmod 12$ there exists an MRD $(4,w;v)$ if and only if $v \geq 3w+1$ and $v \equiv 1$ or $4 \bmod 12$.

In the present paper, we study the general problem of determining reasonable necessary conditions for the existence of MRD $(4,k;v)$s. Expanding on the techniques used by the authors in [MR] to the analogous problem for MRD $(3,k;v)$s, we will prove the following result (we use the notation $\lceil x \rceil_{a;b}$ to mean the smallest integer not less than x which is congruent to a modulo b):

Theorem 1.2 *Suppose that there exists a mandatory representation design MRD $(4,k;v)$. Then the following conditions hold, where $p(t) = \min\{n>0 :$ the complete graph K_n contains t edge-disjoint $K_4s\}$*

(I) If $k \equiv 0 \bmod 3$, then $v \equiv 0$ or $1 \bmod 3$.

(i) If $k \equiv 0 \bmod 3$ and $v \equiv 1 \bmod 3$, then either

(a) $k \equiv 0$ or $9 \bmod 12, v \equiv 1$ or $4 \bmod 12$ and $v \geq \frac{1}{3}k(2k+1)$, or

(b) $k \equiv 3$ or $6 \bmod 12, v \equiv 1$ or $4 \bmod 12$ and $v \geq \frac{1}{3}k(2k+2)$, or

(c) $k \equiv 3$ or $6 \bmod 12, v \equiv 7$ or $10 \bmod 12$ and $v \geq \frac{1}{3}k(2k+1)$.

(ii) If $k \equiv 0 \bmod 3$ and $v \equiv 0 \bmod 3$, then either

(a) $k \equiv 0$ or $9 \bmod 12, v \equiv 0$ or $9 \bmod 12$ and $v \geq kp(t)-3t$, where $t = \lfloor \frac{kq-v}{3} \rfloor$ and $q = \lceil \frac{v}{k} \rceil$, or

(b) $k \equiv 3$ or $6 \bmod 12$ and $v \geq kp(t)-3t$, where $t = \lfloor \frac{kq-v}{3} \rfloor$, and $q = \lceil \frac{v}{k} \rceil_{0;2}$ when $v \equiv 0$ or $9 \bmod 12$ while $q = \lceil \frac{v}{k} \rceil_{1;2}$ when $v \equiv 3$ or $6 \bmod 12$.

(II) If $k \equiv 1 \bmod 3$, then $v \equiv 1 \bmod 3$.

(i) If $k \equiv 1$ or $4 \bmod 12$, then $v \equiv 1$ or $4 \bmod 12$ and $v \geq 3k+1$.

(ii) If $k \equiv 7$ or $10 \bmod 12$, then either $v \equiv 7$ or $10 \bmod 12$ and $v \geq 3k+1$, or $v \equiv 1$ or $4 \bmod 12$ and $v \geq 4k-3$.

(III) If $k \equiv 2 \bmod 3$, then $v \equiv 0, 1$ or $2 \bmod 3$.

(i) If $k \equiv 2 \bmod 3$ and $v \equiv 1 \bmod 3$, then either

(a) $k \equiv 2 \bmod 3$, $v \equiv 1$ or $4 \bmod 12$ and $v \geq \frac{1}{3}k(2k+2)$, or

(b) $k \equiv 2$ *or* 11 *mod* 12, $v \equiv 7$ *or* 10 *mod* 12 *and* $v \geq \frac{1}{3}(k(2k+4)-1)$.

(ii) *If* $k \equiv 2$ *mod* 3 *and* $v \equiv 2$ *mod* 3, *then either*

(a) $k \equiv 5$ *or* 8 *mod* 12, $v \equiv 5$ *or* 8 *mod* 12 *and* $v \geq kp(t) - 3t$, *where* $t = \lfloor \frac{kq-v}{3} \rfloor$ *and* $q = \lceil \frac{v}{k} \rceil_{1;3}$, *or*

(b) $k \equiv 2$ *or* 11 *mod* 12 *and* $v \geq kp(t) - 3t$, *where* $t = \lfloor \frac{kq-v}{3} \rfloor$, *and* $q = \lceil \frac{v}{k} \rceil_{1;6}$ *when* $v \equiv 2$ *or* 11 *mod* 12 *while* $q = \lceil \frac{v}{k} \rceil_{4;6}$ *when* $v \equiv 5$ *or* 8 *mod* 12, *with the possible exceptions* $(k,v) = (11, 113)$ *and* $(14, 161)$.

(iii) *If* $k \equiv 2$ *mod* 3 *and* $v \equiv 0$ *mod* 3, *then either* $k \equiv 2$ *or* 11 *mod* 12, *or* $k \equiv 5$ *or* 8 *mod* 12 *and* $v \equiv 0$ *or* 9 *mod* 12; *furthermore,*

$$v \geq \begin{cases} \frac{1}{2}k(k+1) & \text{if } k \equiv 2, 8, 17, 23 \text{ mod } 24, \\ \frac{1}{2}k(k+4) - \frac{3}{2}\lfloor \frac{k+4}{5} \rfloor & \text{if } k \equiv 5, 11, 14, 20 \text{ mod } 24 \text{ and } 5 \mid (k+4), \\ \frac{1}{2}k(k+4) - \frac{3}{2}\lfloor \frac{k}{5} \rfloor & \text{if } k \equiv 5, 11, 14, 20 \text{ mod } 24 \text{ and } 5 \nmid (k+4). \end{cases}$$

(IV) *If* $k \equiv 0, 3$ *or* 6 *mod* 12 *and* $k > 4$, *then* $v \neq 5k-3$ *or* $6k-3$, *except possibly for* $(k,v) = (6, 33)$.

We close this section by noting that in [MLAB] the authors confirm that the necessary conditions in Theorem 1.2 are sufficient for $5 \leq k \leq 9$ and $v \not\equiv 1$ or 4 mod 12, with at most 25 exceptions. Furthermore, in [RS] and [GR] it has been shown that for MRD$(4, k; v)$s with $k \equiv 1$ mod 3 (i.e. Case II of Theorem 1.2) the necessary conditions for existence are sufficient, with at most 56 exceptions.

2 Preliminaries

In this section, we give some definition and notations as well as some preliminary results which will be used in the sequel. We refer the reader to [BJL] and [CD] for undefined terms as well as a general overview of design theory.

In 1972, Wilson [W] proved the following necessary conditions for the existence of a PBD $(K; v)$. Let $\alpha(K) = \gcd\{k-1 \mid k \in K\}$ and $\beta(K) = \gcd\{k(k-1) \mid k \in K\}$. Then

$$v - 1 \equiv 0 \bmod \alpha(K) \quad \text{and} \quad v(v-1) \equiv 0 \bmod \beta(K).$$

The problem of determining the existence of an MRD $(4, k; v)$ can be separated via Wilson's conditions into the following sub-problems.

	Cases	Wilson's Condition on v
(I)	$k \equiv 0$ or 9 mod 12	$v \equiv 0, 1, 4$ or 9 mod 12
(II)	$k \equiv 1$ or 4 mod 12	$v \equiv 1$ or 4 mod 12
(III)	$k \equiv 2$ or 11 mod 12	none
(IV)	$k \equiv 3$ or 6 mod 12	$v \equiv 0$ or 1 mod 3
(V)	$k \equiv 5$ or 8 mod 12	$v \equiv 0$ or 1 mod 4
(VI)	$k \equiv 7$ or 10 mod 12	$v \equiv 1$ mod 3

Suppose that an MRD $(4, k; v)$ exists. Let β_4 and β_k denote the number of blocks of size 4 and k, respectively. Then by a simple counting argument we have

$$12\beta_4 + k(k-1)\beta_k = v(v-1). \tag{1}$$

Consider now an arbitrary point $x \in X$ and let $\gamma_4(x)$ and $\gamma_k(x)$ count the number of blocks through x with length 4 and k, respectively. Then

$$3\gamma_4(x) + (k-1)\gamma_k(x) = v - 1. \tag{2}$$

These two equations are useful to prove additional congruential conditions on the number of blocks of size k.

Theorem 2.1 *Suppose that there exists an MRD $(4, k; v)$. Then the following conditions for β_k and $\gamma_k(x)$ $(\forall x \in X)$ are satisfied:*

	Cases		β_k	$\gamma_k(x)$
(I)	$k \equiv 0$ *or* 9 *mod* 12			
	(a)	$v \equiv 0$ *or* 9 *mod* 12	*none*	$\gamma_k \equiv 1$ *mod* 3
	(b)	$v \equiv 1$ *or* 4 *mod* 12	*none*	$\gamma_k \equiv 0$ *mod* 3
(II)	$k \equiv 1$ *or* 4 *mod* 12			
	(a)	$v \equiv 1$ *or* 4 *mod* 12	*none*	*none*
(III)	$k \equiv 2$ *or* 11 *mod* 12			
	(a)	$v \equiv 0$ *or* 9 *mod* 12	$\beta_k \equiv 0$ *mod* 6	$\gamma_k \equiv 2$ *mod* 3
	(b)	$v \equiv 1$ *or* 4 *mod* 12	$\beta_k \equiv 0$ *mod* 6	$\gamma_k \equiv 0$ *mod* 3
	(c)	$v \equiv 2$ *or* 11 *mod* 12	$\beta_k \equiv 1$ *mod* 6	$\gamma_k \equiv 1$ *mod* 3
	(d)	$v \equiv 3$ *or* 6 *mod* 12	$\beta_k \equiv 3$ *mod* 6	$\gamma_k \equiv 2$ *mod* 3
	(e)	$v \equiv 5$ *or* 8 *mod* 12	$\beta_k \equiv 4$ *mod* 6	$\gamma_k \equiv 1$ *mod* 3
	(f)	$v \equiv 7$ *or* 10 *mod* 12	$\beta_k \equiv 3$ *mod* 6	$\gamma_k \equiv 0$ *mod* 3

	Cases (continued)		β_k	$\gamma_k(x)$
(IV)	$k \equiv 3$ *or* 6 *mod* 12			
	(a)	$v \equiv 0$ *or* 9 *mod* 12	$\beta_k \equiv 0$ *mod* 2	$\gamma_k \equiv 1$ *mod* 3
	(b)	$v \equiv 1$ *or* 4 *mod* 12	$\beta_k \equiv 0$ *mod* 2	$\gamma_k \equiv 0$ *mod* 3
	(c)	$v \equiv 3$ *or* 6 *mod* 12	$\beta_k \equiv 1$ *mod* 2	$\gamma_k \equiv 1$ *mod* 3
	(d)	$v \equiv 7$ *or* 10 *mod* 12	$\beta_k \equiv 1$ *mod* 2	$\gamma_k \equiv 0$ *mod* 3
(V)	$k \equiv 5$ *or* 8 *mod* 12			
	(a)	$v \equiv 0$ *or* 9 *mod* 12	$\beta_k \equiv 0$ *mod* 3	$\gamma_k \equiv 2$ *mod* 3
	(b)	$v \equiv 1$ *or* 4 *mod* 12	$\beta_k \equiv 0$ *mod* 3	$\gamma_k \equiv 0$ *mod* 3
	(c)	$v \equiv 5$ *or* 8 *mod* 12	$\beta_k \equiv 1$ *mod* 3	$\gamma_k \equiv 1$ *mod* 3
(VI)	$k \equiv 7$ *or* 10 *mod* 12			
	(a)	$v \equiv 1$ *or* 4 *mod* 12	$\beta_k \equiv 0$ *mod* 2	*none*
	(b)	$v \equiv 7$ *or* 10 *mod* 12	$\beta_k \equiv 1$ *mod* 2	*none.*

Proof.
We will handle just one case, namely $k \equiv 5$ or $8 \bmod 12$. In all other cases the same argument works.

First, observe that $k(k-1) \equiv 2 \bmod 3$ and $k-1 \equiv 1 \bmod 3$. If $v \equiv 0$ or $9 \bmod 12$, then $v(v-1) \equiv 0 \bmod 3$ and $v-1 \equiv 2 \bmod 3$. Hence, Equation (1) and Equation (2) imply that $\beta_k \equiv 0 \bmod 3$ and $\gamma_k(x) \equiv 2 \bmod 3$ for all $x \in X$. Similarly, it is easily seen that $v(v-1) \equiv 0 \bmod 3$ and $v-1 \equiv 0 \bmod 3$ if $v \equiv 1$ or $4 \bmod 12$. Therefore, $\beta_k \equiv 0 \bmod 3$ and $\gamma_k(x) \equiv 0 \bmod 3$. If $v \equiv 5$ or $8 \bmod 12$, then $v(v-1) \equiv 2 \bmod 3$ and $v-1 \equiv 1 \bmod 3$. Again Equation (1) and Equation (2) imply $\beta_k \equiv 1 \bmod 3$ and $\gamma_k(x) \equiv 1 \bmod 3$. $\square$

Another existence criteria is given in the following theorem, proved by Drake and Larson [DL]:

Theorem 2.2 *Let K be a set of positive integers whose smallest element is m. Suppose that there exists a PBD $(K; v)$ which contains blocks B_k and B_h of sizes k and h respectively, where $k \geq h$. Then $v \geq (m-1)k + h - m + 1$.*

As an immediate consequence we have the following

Corollary 2.3 *Suppose that there exists an MRD $(4, k; v)$. Then $v \geq 3k+1$. Moreover, if there are at least two blocks of size k in the MRD, then $v \geq 4k-3$.*

Note that all MRDs $(4, k; v)$ with $\beta_k \equiv 0 \bmod 2$, $\beta_k \equiv 0 \bmod 3$ or $\beta_k \equiv 0, 3$ or $4 \bmod 6$ contain at least two blocks of size k.

As noted in Section 1, the necessary conditions for the existence of MRDs $(4, k; v)$ have been shown to be sufficient when $k \equiv 1 \bmod 3$ (with at most

t	$p(t)$	t	$p(t)$	t	$p(t)$	t	$p(t)$
0	1	8	12	16	16	24	19
1	4	9	12	17	16	25	19
2	7	10	13	18	16	26	20
3	9	11	13	19	16	27	20
4	10	12	13	20	16	28	20
5	10	13	13	21	18	29	20
6	11	14	14	22	18	30	20
7	12	15	15	23	19	31	21

Table 1: $p(t)$, $0 \le t \le 31$

56 exceptions); thus in what follows we will assume that $k \equiv 0$ or $2 \bmod 3$, i.e. we are in Case I, III, IV or V of Theorem 2.1.

A pair $(X, \mathcal{B})$ consisting of a point set X ($|X| = v$) and a block set $\mathcal{B}$ with block sizes from K, so that any pair of distinct points occurs in at most one block, is called a *partial design* of order v. When $K = \{k\}$, the notation (v,k)-*packing* is in common use. The *packing number* $D(v,k)$ is the maximum number of blocks in any (v,k)-packing. For $k = 4$, the packing number is well known (see Hanani [H], Stinson [S]).

$$D(v,4) = \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \right\rfloor \right\rfloor - \epsilon \text{ , where}$$

$$\epsilon = \begin{cases} 1, & \text{if } v \equiv 7 \text{ or } 10 \bmod 12, \quad v \neq 10, 19; \\ 1, & \text{if } v = 9 \text{ or } 17; \\ 2, & \text{if } v = 8, 10 \text{ or } 11; \\ 3, & \text{if } v = 19; \\ 0, & \text{otherwise.} \end{cases}$$

The packing number can be used to compute the function $p(t)$ defined in Theorem 1.2 since $p(t) = \min\{v > 0 \mid D(v,4) \geq t\}$. We give Table 1 for the first few values of $p(t)$ and the following lemmata which develop lower bounds for the functions $D(n,4)$ and $p(t)$.

Lemma 2.4 *For each $v \geq 0$, except for $v = 8, 9, 10, 11$*

$$D(v,4) \geq \frac{1}{12}(v(v-3)-6). \tag{3}$$

Proof.
Clearly, the claim is true for all $v \neq 8, 9, 10, 11, 17, 19$ and $v \not\equiv 7, 10$

(mod 12) from the definition of $D(v,4)$. Notice that for $v \neq 8,9,10,11,17,19$ $D(v,4)$ can also be written as

$$D(v,4) = \begin{cases} \frac{1}{12}v(v-3), & \text{if } v \equiv 0 \text{ or } 3 \text{ mod } 12; \\ \frac{1}{12}v(v-1), & \text{if } v \equiv 1 \text{ or } 4 \text{ mod } 12; \\ \frac{1}{12}v(v-2), & \text{if } v \equiv 2 \text{ or } 8 \text{ mod } 12; \\ \frac{1}{12}(v(v-2)-3), & \text{if } v \equiv 5 \text{ or } 11 \text{ mod } 12; \\ \frac{1}{12}(v(v-3)-6), & \text{if } v \equiv 6 \text{ or } 9 \text{ mod } 12 \text{ and} \\ \frac{1}{12}(v(v-1)-18), & \text{if } v \equiv 7 \text{ or } 10 \text{ mod } 12. \end{cases}$$

Let $v \equiv 7$ or 10 mod 12. Then $12 \cdot D(v,4) = v(v-1) - 18 \geq v(v-3) - 6$ for all $v \geq 6$. Hence, it is sufficient to show the assumption for $v = 17$ and 19. Here,

$$D(17,4) = 20 > \frac{232}{12} \quad \text{and} \quad D(19,4) = 25 > \frac{298}{12}.$$

□

Lemma 2.5 *For each $t \geq 0$,*

$$\frac{(p(t)-1)(p(t)-3)}{2} - \frac{p(t)+5}{2} \leq 6t \tag{4}$$

with equality if and only if $t = 4$.

Proof.
If $0 \leq t \leq 9$ the validity of this inequality can easily be checked by means of Table 1. Note that only for $t = 4$ equality holds in Equation (4). Let $t \geq 10$ and let $v = p(t) - 1$. Then $v \geq 12$ and $D(v,4) \geq \frac{1}{12}(v(v-3)-6)$. By the definition of $p(t)$ we have $D(v,4) < t$. Thus,

$$\begin{aligned} \frac{(p(t)-1)(p(t)-3)}{2} - \frac{p(t)+5}{2} &= \frac{v(v-2)}{2} - \frac{v+6}{2} \\ &= \frac{v(v-3)-6}{2} \leq 6D(v,4) < 6t \quad . \end{aligned}$$

□

Finally, we quote a result of Mendelsohn and Rees [MR] which gives a lower bound for the number of points v in a partial design.

Proposition 2.6 *Let $(X, \mathcal{B})$ be a partial design of order v, with block size k, in which there are b blocks. Then letting $r = 1 + \lfloor (b-1)/k \rfloor$*

$$v \geq b \cdot \frac{2rk - b + 1}{r^2 + r}.$$

3 MRDs (4,k;v) with $\gamma_k \equiv 0$ mod 3

In this section, we will consider mandatory representation designs where each point is contained in $3m$ blocks of size k (k-blocks for short) for some nonnegative integer m.

Theorem 3.1 *Suppose that there exists an MRD* $(4, k; v)$ *with* $\gamma_k(x) \equiv 0$ *mod* 3 *for all points* x *in* X. *Then the following conditions hold*

(i) If $k \equiv 0$ *or* 9 *mod* 12 *and* $v \equiv 1$ *or* 4 *mod* 12*, then* $v \geq \frac{1}{3}k(2k+1)$.

(ii) If $k \equiv 3$ *or* 6 *mod* 12 *and* $v \equiv 1$ *or* 4 *mod* 12*, then* $v \geq \frac{1}{3}k(2k+2)$.

(iii) If $k \equiv 3$ *or* 6 *mod* 12 *and* $v \equiv 7$ *or* 10 *mod* 12*, then* $v \geq \frac{1}{3}k(2k+1)$.

(iv) If $k \equiv 2$ *mod* 3 *and* $v \equiv 1$ *mod* 3*, then either*

(a) $k \equiv 2$ *mod* 3, $v \equiv 1$ *or* 4 *mod* 12 *and* $v \geq \frac{1}{3}k(2k+2)$*, or*

(b) $k \equiv 2$ *or* 11 *mod* 12, $v \equiv 7$ *or* 10 *mod* 12 *and* $v \geq \frac{1}{3}(k(2k+4)-1)$.

Proof.
Every point lies on no k-block or at least three k-blocks. Hence, $\beta_k \geq 2k+1$. Then Proposition 2.6 immediately implies that $v \geq \frac{1}{3}k(2k+1)$.

If $k \equiv 2, 3, 5, 6, 8$ or 11 mod 12 and $v \equiv 1$ or 4 mod 12, then the congruential conditions in Theorem 2.1 imply that $\beta_k \geq 2k+2$. If the block set contains exactly $\beta_k = 2k+2$ k-blocks, then every point is contained in either zero or three k-blocks. Let $U \subseteq V$ denote the set of points which lie on precisely three k-blocks. Counting points on k-blocks in two ways yields $3|U| = k(2k+2)$. Thus, $v \geq |U| = \frac{1}{3}k(2k+2)$. Otherwise, $\beta_k \geq 2k+4$ if $k \equiv 3$ or 6 mod 12 or $\beta_k \geq 2k+5$. Then again Proposition 2.6 shows that $v \geq \frac{1}{3}k(2k+2)$.

In the remaining case $k \equiv 2$ or 11 mod 12 and $v \equiv 7$ or 10 mod 12 we observe $\beta_k \geq 2k+5$, since $\beta_k \equiv 3$ mod 6. Let $\beta_k = 2k+5$. Here, counting points on k-blocks in two ways yields $3|U| + 6|W| = k(2k+5)$, where W contains all points which lie on precisely six k-blocks. Clearly, every k-block contains at most one point from W. Hence, $|W| \leq \lfloor \frac{2k+5}{6} \rfloor = \frac{2k+2}{6}$ and, therefore, $v \geq |U| + |W| = \frac{1}{3}(k(2k+5) - 3|W|) \geq \frac{1}{3}(k(2k+5) - (k+1))$. If $\beta_k \geq 2k+11$ and $k \geq 23$, then Proposition 2.6 confirms the claim. Finally, if $k = 11$ or 14, then a counting argument involving points which lie on $3, 6, 9$ or 12 k-blocks proves the result for $\beta_k = 2k+11$. $\square$

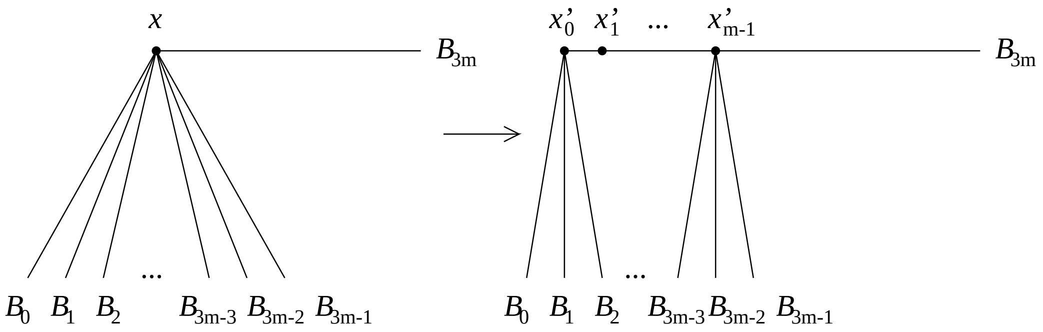

Figure 1: Shift-Construction

4 MRDs (4,k;v) with $\gamma_k \equiv 1$ mod 3

In this section, we will investigate the most difficult case of all, namely mandatory representation designs where each point is contained in $3m+1$ blocks of size k for some nonnegative integer m. First, we need a construction introduced by Mendelsohn and Rees [MR] for $K = \{3, k\}$ which can easily be adapted for $K = \{4, k\}$.

Lemma 4.1 (shift-construction) *Suppose that there exists a partial design $(X, \mathcal{B})$ with blocks of size k, where $\gamma_k(x) \equiv 1$ mod 3 for all points x in X. Furthermore, suppose that $|\mathcal{B}| \leq 3k+1$. Then there exists a partial design on the same point set with blocks of size k, where every point is contained in either one or four k-blocks.*

Proof.
Let x be a point in X. Then x is contained in $3m+1$ blocks for some $m \geq 0$. Let these blocks be $B_0, B_1, \ldots, B_{3m}$. If $m \geq 1$, then replace x by m new points $x'_0, x'_1, \ldots, x'_{m-1}$ in the following way: replace B_{3m} by $(B_{3m} \setminus \{x\}) \cup \{x'_0, \ldots, x'_{m-1}\}$ and for each $i = 0, 1, \ldots, m-1$ and $j = 0, 1, 2$ replace B_{3i+j} by $(B_{3i+j} \setminus \{x\}) \cup \{x'_i\}$. See Figure 1 for a graphical representation of the point exchange. If $m = 0$, remove x from the point set and from the (unique) block containing x. Perform this operation for all points in X to construct a new partial design $(X', \mathcal{B}')$. (Note that multiple blocks may occur in $\mathcal{B}'$. Moreover, we allow blocks with length 0.) Each point in X' is contained in exactly four blocks and no block has more than k elements, since $|\mathcal{B}'| = |\mathcal{B}| \leq 3k+1$. Finally, construct a partial design $(X'', \mathcal{B}'')$ by adding $k - |B|$ new points to each block $B \in \mathcal{B}'$. Obviously, each point lies on either one or four k-blocks.

To complete the proof, it is sufficient to show that $|X| = |X''|$. By counting the points on k-blocks in $\mathcal{B}$ and $\mathcal{B}''$ in two ways it follows that

$$\sum_{x \in X} \gamma_k(x) = k|\mathcal{B}| \quad \text{and} \quad \sum_{x \in X''} \gamma_k(x) = k|\mathcal{B}''|$$

and, therefore,

$$|X| = k|\mathcal{B}| - \sum_{x \in X} (\gamma_k(x) - 1) \quad \text{and} \quad |X''| = k|\mathcal{B}''| - \sum_{x \in X''} (\gamma_k(x) - 1).$$

Observing that $|\mathcal{B}| = |\mathcal{B}''|$ and

$$\sum_{x \in X} (\gamma_k(x) - 1) = 3 \sum_{x \in X} \frac{\gamma_k(x) - 1}{3} = 3 \sum_{x \in X'} 1 = \sum_{x \in X''} (\gamma_k(x) - 1)$$

gives the desired conclusion. $\square$

Note that the partial design obtained by applying the shift-construction is not uniquely determined.

Lemma 4.2 *Suppose that there exists an MRD* $(4, k; v)$ *with* $\beta_k \leq 3k + 1$ *where* $\gamma_k(x) \equiv 1$ *mod* 3 *for all points* x *in* X. *Let* $t = \frac{k \cdot \beta_k - v}{3}$. *Then*

$$D(\beta_k, 4) \geq t$$

or, equivalently,

$$v \geq k \cdot p(t) - 3t \ .$$

Proof.

Apply Lemma 4.1 on the partial design formed by all of the k-blocks to obtain a partial design on the same point set, where all points lie on either one or four k-blocks. Observe that there are precisely t points with $\gamma_k = 4$ (count the points in k-blocks in two ways). Take the dual of the new partial design and remove all blocks which contain only one point. The remaining blocks form a $(\beta_k, 4)$-packing. This packing contains exactly t blocks. Hence, $D(\beta_k, 4) \geq t$. The equivalence of the two statements follows from the definition of $p(t)$: $D(\beta_k, 4) \geq t$ if and only if $\beta_k \geq p(t)$, if and only if $0 \geq k \cdot (p(t) - \beta_k)$, if and only if $v \geq k \cdot p(t) - (k \cdot \beta_k - v)$. $\square$

Lemma 4.3 *There do not exist mandatory representation designs MRD* $(4, 11; 95)$, *MRD* $(4, 11; 98)$ *and MRD* $(4, 14; 152)$.

$k = 11, v = 95$			$k = 11, v = 98$			$k = 14, v = 152$		
β_{11}	t	$k \cdot p(t) - 3t$	β_{11}	t	$k \cdot p(t) - 3t$	β_{14}	t	$k \cdot p(t) - 3t$
13	16	128	13	15	120	16	24	194
19	38	139	19	37	131	22	52	208
25	60	128	25	59	142	28	80	208
31	82	117	31	81	131	34	108	194
						40	136	180

Table 2: $v < k \cdot p(t) - 3t$ where $t = \frac{k \cdot \beta_k - v}{3}$ for all $\lceil \frac{v}{k} \rceil \leq \beta_k \leq 3k + 1$

Proof.
There do not exist mandatory representation designs with the above parameters and $\beta_k \leq 3k+1$, see Lemma 4.2 and Table 2. Note that $\beta_{11} \equiv 1 \bmod 6$ and $\beta_{14} \equiv 4 \bmod 6$.

Now, suppose to the contrary that a mandatory representation design with $\beta_k > 3k+1$ blocks exists. We investigate the partial design built by all of the k-blocks. Let W denote the set of those points which are contained in seven blocks of size k, and let $w = |W|$. Without loss of generality, we may assume that every block D of size k that contains a point x from W contains no point y that lies on only one block of size k. Otherwise, apply the shift-construction (Lemma 4.1) with $B_6 = D$ to x and remove y. Repeat this operation for all blocks which contain both points from W and points with $\gamma_k = 1$ to obtain a partial design with the desired property.

Since $\beta_k > 3k+1$, one knows $\beta_{11} = 37 + 6n$ (for $n = 0, 1, \ldots$) and $\beta_{14} = 46 + 6n$ (for $n = 0, 1, \ldots$), respectively. Let us assume that no point lies on ten or more k-blocks. Then $w \geq \frac{11(37+6n)-4v}{3}$ or $w \geq \frac{14(46+6n)-4v}{3}$, respectively, since $4(v-w) + 7w \geq k \cdot \beta_k$. Moreover, every k-block contains at most $2n+1$ points from W since only $\beta_k - 1$ other blocks can intersect this block. Consider now the dual of the partial design built by all of the k-blocks. This partial design has exactly $37 + 6n$ (resp. $46 + 6n$) points and at least $\frac{11(37+6n)-4v}{3}$ (resp. $\frac{14(46+6n)-4v}{3}$) blocks of size 7 with the additional property that every point lies on at most $2n+1$ 7-blocks. Counting blocks spanned by the points gives

$$\frac{(37+6n)(2n+1)}{7} \quad \text{or} \quad \frac{(46+6n)(2n+1)}{7}$$

as an upper bound for the number of 7-blocks.

Let $v = 95$ and $k = 11$. Then the above upper bound is smaller than $9 + 22n$ (the minimum number of 7-blocks in the dual partial design) for all

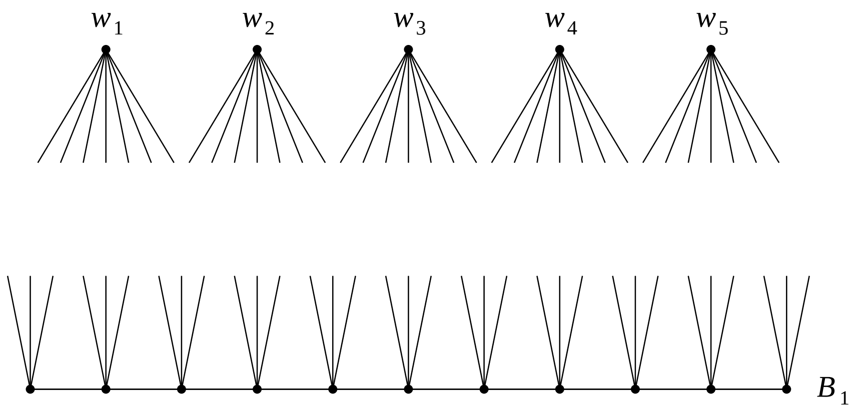

Figure 2: Intersection of blocks of size 11

$0 \leq n \leq 4$. If $n \geq 5$, then there are points which lie on at least ten 11-blocks, since $v \geq w$. Hence, there does not exist a required partial design with blocks of size 11 where no point has degree (number of blocks through a point) ten or higher.

Nearly the same argument works when $v = 98$ and $k = 11$. Then $\frac{(37+6n)(2n+1)}{7} < 5 + 22n$ for all $1 \leq n \leq 4$. If the partial design exists, then $n = 0$. This means $\beta_{11} = 37$ and no two points of W lie on a common 11-block. But then $w = 5$ since $37 \geq 7w$. It is easily checked that precisely 35 11-blocks contain a point from W. Moreover, each of these 35 blocks intersects every other 11-block, since all points not in W but in one of these 35 blocks lie on exactly four 11-blocks (see the argument above which applies the shift-construction). It follows that both of the 11-blocks, say B_1,B_2, which do not contain a point from W, are intersected by at least 35 11-blocks, but this is not possible (see Figure 2).

If $v = 152$ and $k = 14$, then the above upper bound $\frac{(46+6n)(2n+1)}{7}$ for the number of blocks in the dual partial design is smaller than $12 + 28n$ for all $0 \leq n \leq 5$. Hence, there does not exist a required partial design with blocks of size 14 where no point has degree ten or higher.

It remains to show that no point lies on ten or more k-blocks. If $k = 11$, then 11-blocks through such a point cover at least 101 points, a contradiction. Let $k = 14$. The possibility that there exists a point x of degree ten (or higher) is immediately ruled out. Otherwise, we find by a simple counting argument that there must be another point y of degree at least seven, but 14-blocks through x and y cover at least 155 (if x and y are contained in a common 14-block) or 153 points (see Figure 3), a contradiction. This completes the proof. $\square$

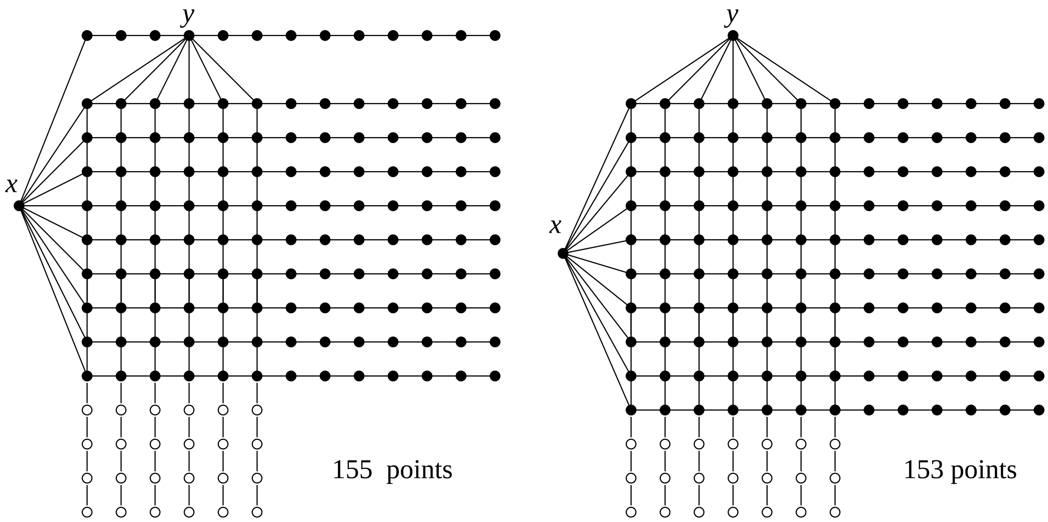

Figure 3: Points covered by blocks of size 14

Lemma 4.4 *Suppose that there exists an MRD* $(4, k; v)$ *with* β_k *blocks of size* k *where* $\gamma_k(x) \equiv 1 \bmod 3$ *for all points* x *in* X. *Then* $q \leq \beta_k$ *where* $q = q(v, k)$ *is defined as follows:*

(i) $q = \lceil \frac{v}{k} \rceil$ *if* $k \equiv 0$ *or* 9 *mod* 12 *and* $v \equiv 0$ *or* 9 *mod* 12 *, or*

(ii) $q = \lceil \frac{v}{k} \rceil_{0;2}$ *if* $k \equiv 3$ *or* 6 *mod* 12 *and* $v \equiv 0$ *or* 9 *mod* 12*, or*

(iii) $q = \lceil \frac{v}{k} \rceil_{1;2}$ *if* $k \equiv 3$ *or* 6 *mod* 12 *and* $v \equiv 3$ *or* 6 *mod* 12*, or*

(iv) $q = \lceil \frac{v}{k} \rceil_{1;3}$ *if* $k \equiv 5$ *or* 8 *mod* 12 *and* $v \equiv 5$ *or* 8 *mod* 12*, or*

(v) $q = \lceil \frac{v}{k} \rceil_{1;6}$ *if* $k \equiv 2$ *or* 11 *mod* 12 *and* $v \equiv 2$ *or* 11 *mod* 12*, or*

(vi) $q = \lceil \frac{v}{k} \rceil_{4;6}$ *if* $k \equiv 2$ *or* 11 *mod* 12 *and* $v \equiv 5$ *or* 8 *mod* 12*.*

Proof.
Every point must be contained in at least one k-block. Thus $\frac{v}{k} \leq \beta_k$. It is easily checked that the congruential conditions in Theorem 2.1 yield the result. □

We will use the definition of $q = q(v, k)$ in what follows.

Lemma 4.5 *Let* $(X, \mathcal{B})$ *be a partial design of order* v *with* β_k *blocks* ($\beta_k \leq 3k+1$) *of size* k*, where* $\gamma_k(x) \equiv 1 \bmod 3$ *for all points* x *in* X*. Furthermore, let* $d = q - \lceil \frac{v}{k} \rceil$ *and let* $c = \beta_k - \lceil \frac{v}{k} \rceil$*. Then*

$$d \leq c \leq \frac{3}{4}(3k+1).$$

Proof.
Since $q \leq \beta_k$, $d \leq c$ follows.

In view of Lemma 4.1, we may assume that every point of the partial design is contained in either one or four blocks. Let $U \subseteq X$ consist of all of the points which lie on four k-blocks. Then

$$v = k\beta_k - \sum_{x \in X} (\gamma_k(x) - 1) = k\beta_k - 3|U|.$$

Replace β_k by $c + \lceil \frac{v}{k} \rceil$ to obtain $3|U| = s + ck$, where $s = k \cdot \lceil \frac{v}{k} \rceil - v$ or, equivalently, $v \equiv -s \bmod k$ and $0 \leq s < k$. By counting the blocks spanned by U we have

$$\frac{4}{3}ck \leq \frac{4}{3}(s + ck) = 4|U| \leq k\beta_k \leq k(3k+1).$$

This implies $c \leq \frac{3}{4}(3k+1)$. □

Corollary 4.6 *Let $(X, \mathcal{B})$ be a partial design of order v with at most $3k+1$ blocks of size k, where $\gamma_k(x) \equiv 1$ mod 3 for all points x in X. Let $v \equiv -s$ mod k, where $0 \leq s < k$ and let $d = q - \lceil \frac{v}{k} \rceil$. Define the integer set $C = \{c \mid d \leq c \leq \min\{\frac{3}{4}(3k+1), \frac{3v-s}{k}\},\ \frac{1}{3}(s+ck) \in \mathbb{N}\}$. Then*

$$v \geq \min \left\{ k \cdot p\left(\frac{s+ck}{3}\right) - (s+ck) \mid c \in C \right\}.$$

Proof.
In view of Lemma 4.1, we may assume that every point of the partial design lies on either one or four k-blocks. The proof of Lemma 4.5 implies that there are exactly $\frac{1}{3}(s+ck)$ points which are contained in four k-blocks for some c such that $d \leq c \leq \frac{3}{4}(3k+1)$. Clearly, $\frac{1}{3}(s+ck) \in \mathbb{N}$ and, moreover, $v \geq \frac{1}{3}(s+ck)$. Thus, $c \leq \frac{3v-s}{k}$.

The smallest number of blocks required to construct such a partial design is $p(\frac{1}{3}(s+ck))$. These blocks cover $k \cdot p(\frac{1}{3}(s+ck)) - 3 \cdot \frac{1}{3}(s+ck)$ distinct points. □

We can now prove the following result.

Theorem 4.7 *Suppose that there exists a mandatory representation design MRD $(4,k;v)$ where $\gamma_k(x) \equiv 1$ mod 3 for all points x in X. Then either $k \equiv 0$ or 9 mod 12 and $v \equiv 0$ or 9 mod 12 , or $k \equiv 3$ or 6 mod 12 and $v \equiv 0$ or 9 mod 12, or $k \equiv 3$ or 6 mod 12 and $v \equiv 3$ or 6 mod 12, or $k \equiv 5$ or 8 mod 12 and $v \equiv 5$ or 8 mod 12, or $k \equiv 2$ or 11 mod 12 and $v \equiv 2$ or 11 mod 12, or $k \equiv 2$ or 11 mod 12 and $v \equiv 5$ or 8 mod 12; in all cases*

$$v \geq kp(t) - 3t,$$

k	$\lfloor\frac{1}{4}k(3k+1)\rfloor$	$\max\{kp(t')-3t'\}$	k	$\lfloor\frac{1}{4}k(3k+1)\rfloor$	$\max\{kp(t')-3t'\}$
9	63	57	33	825	399
12	111	99	35	927	881
15	172	159	36	981	438
18	247	204	38	1092	974
20	305	272	39	1150	672
21	336	213	41	1271	829
24	438	267	42	1333	762
27	553	384	44	1463	971
29	638	502	45	1530	588
30	682	432	47	1668	1366
32	776	579			

Table 3: $\frac{1}{4}k(3k+1) \geq k \cdot p(t') - 3t'$ for $1 \leq t' \leq t_{\max}(k)$

where $t = \lfloor\frac{kq-v}{3}\rfloor$ *and* $q = q(v,k)$ *is as defined in Lemma 4.4, except possibly for* $(v,k) = (113, 11)$ *or* $(161, 14)$.

Proof.
We begin by determining lower and upper bounds for t. By Lemma 4.4 one knows

$$\left\lceil\frac{v}{k}\right\rceil \leq q \leq \left\lceil\frac{v}{k}\right\rceil + 5.$$

Thus,

$$0 \leq t \leq \frac{k \cdot (\lceil\frac{v}{k}\rceil + 5) - v}{3} < 2k.$$

If $t = 0$, then clearly the claim is true. Hence, we may assume that $1 \leq t \leq 2k-1$. In fact, it is possible to determine a more precise upper bound for t. Recall that q, and therefore t, depends on k. Let $t_{\max}(k)$ denote the maximum possible t. From Theorem 2.1, it is easy to check that $t_{\max} = \frac{1}{3}k-1$ if $k \equiv 0$ or $9 \bmod 12$, $t_{\max} = 2k-1$ if $k \equiv 2$ or $11 \bmod 12$, $t_{\max} = \frac{2}{3}k-1$ if $k \equiv 3$ or $6 \bmod 12$, and $t_{\max} = k-1$ if $k \equiv 5$ or $8 \bmod 12$.

Now, there are two cases to be considered: (1) $\beta_k \geq 3k+1$ and (2) $\beta_k < 3k+1$. We will handle each in turn.

(1) If $\beta_k \geq 3k+1$, then Proposition 2.6 implies that $v \geq \frac{1}{4}k(3k+1)$. Hence, it is sufficient to show that $\frac{1}{4}k(3k+1) \geq k \cdot p(t) - 3t$ for all possible $t > 0$.

For $k = 9, 12, 15, 18, 20, 21, 24, 27, 29, 30, 32, 33, 35, 36, 38, 39, 41, 42, 44, 45$ and 47, Table 3 is used to prove the claim.

$k=11, 93 \le v < 135$			$k=14, 150 \le v < 202$			$k=23, 402 \le v < 446$		
v	t	$k \cdot p(t) - 3t$	v	t	$k \cdot p(t) - 3t$	v	t	$k \cdot p(t) - 3t$
95	16	**128**	152	24	**194**	404	34	404
98	15	**120**	155	9	141	407	10	269
101	3	90	158	8	144	410	9	249
104	2	71	161	21	**189**	413	31	390
107	12	107	164	20	164	416	30	370
110	11	110	167	5	125	419	6	235
113	21	**135**	170	4	128	422	5	215
116	20	116	173	17	173	425	27	379
119	8	108	176	16	176	428	26	382
122	7	111	179	1	53	431	2	155
125	17	125	185	13	143	434	1	89
128	16	128	188	12	146	437	23	368
131	4	98	191	25	191	440	22	348
134	3	90	194	24	194	443	44	443
			197	9	141			
			200	8	144			

Table 4: $k = 11, 14, 23$; $v \ge k \cdot p(t) - 3t$ for
$\frac{1}{4}k(3k+1) \le v < \max\{k \cdot p(t') - 3t' \mid 1 \le t' \le t_{\max}(k)\}$
except $k = 11$ and $v = 95, 98, 113$, and $k = 14$ and $v = 152, 161$

For $k = 5, 6, 8, 11, 14, 17, 23$ and 26, $\frac{1}{4}k(3k+1) < k \cdot p(t') - 3t'$ holds for some t' between 1 and $t_{\max}(k)$. Therefore, it is necessary to verify that $v \ge k \cdot p(t) - 3t$ for all v with $\frac{1}{4}k(3k+1) \le v < \max\{k \cdot p(t') - 3t' \mid 1 \le t' \le t_{\max}(k)\}$, where $t > 0$. This is done in Table 4 and Table 5 with five exceptions. (Note that when $k = 17$ we have $v = 221$ and so $t = 0$, in which case there is nothing to check.) These exceptions, namely $v = 95, 98, 113$ and $k = 11$, and $v = 152, 161$ and $k = 14$ are separately considered in Lemma 4.3 and Remark 4.8.

Next, it is proved that $v \ge k \cdot p(t) - 3t$ for all $k \ge 48$. It is sufficient to show that

$$\left\lfloor \frac{1}{4}(3k+1) \right\rfloor \ge p(2k-1)$$

or, equivalently, that

$$D\left(\left\lfloor \frac{1}{4}(3k+1) \right\rfloor, 4 \right) \ge 2k - 1.$$

$k=5, 20 \leq v < 38$			$k=6, 28 \leq v < 45$			$k=8, 50 \leq v < 75$		
v	t	$k \cdot p(t) - 3t$	v	t	$k \cdot p(t) - 3t$	v	t	$k \cdot p(t) - 3t$
29	2	29	33	1	21	53	1	29
32	1	17	39	1	21	65	5	65
						68	4	68
						$k=26, 513 \leq v < 523$		
						v	t	$k \cdot p(t) - 3t$
						515	45	515
						518	44	518
						521	17	365

Table 5: $k = 5, 6, 8, 26$; $v \geq k \cdot p(t) - 3t$ for
$\frac{1}{4}k(3k+1) \leq v < \max\{k \cdot p(t') - 3t' \mid 1 \leq t' \leq t_{\max}(k)\}$

By Lemma 2.4, it follows that we are done if we can show

$$\frac{1}{12}\left(\left\lfloor \frac{1}{4}(3k+1)\right\rfloor \left(\left\lfloor \frac{1}{4}(3k+1)\right\rfloor - 3\right) - 6\right) \geq 2k - 1.$$

But this is true for all $k \geq 48$.

(2) If $\beta_k < 3k+1$, then following Corollary 4.6 we will be done if we can show that for each $d \leq c \leq \frac{3}{4}(3k+1)$

$$k \cdot p(\frac{s+ck}{3}) - (s+ck) \geq k \cdot p(t) - 3t,$$

where $d = q - \lceil \frac{v}{k} \rceil$ and $s = k \cdot \lceil \frac{v}{k} \rceil - v$ or, equivalently,

$$p(\frac{s+ck}{3}) \geq p(\frac{s+dk}{3}) + c - d \tag{5}$$

since

$$t = \frac{k \cdot q - v}{3} = \frac{k \cdot (d + \lceil \frac{v}{k} \rceil) - v}{3} = \frac{s+dk}{3}.$$

This is true for $d = c$, so we may assume $d < c$. It is obvious from the definition that $p(m) > n$ whenever $6m > \binom{n}{2}$. Thus, to prove inequality (5) it is sufficient to show that

$$\begin{aligned} 6\left(\frac{s+ck}{3}\right) &> \frac{(p(\frac{s+dk}{3}) + c - d - 1)(p(\frac{s+dk}{3}) + c - d - 2)}{2} \\ &= \frac{(p(\frac{s+dk}{3}) - 1)(p(\frac{s+dk}{3}) - 3)}{2} - \frac{p(\frac{s+dk}{3}) + 5}{2} \\ &\quad + (c-d+1)p(\frac{s+dk}{3}) - \frac{3}{2}(c-d) + \frac{c^2 - 2cd + d^2 + 4}{2}. \end{aligned}$$

$k=5, 16 \le v < 20$			$k=6, 19 \le v \le 28$			$k=8, 25 \le v < 50$		
v	$q(v,k)$	$\lceil \frac{v}{k} \rceil$	v	$q(v,k)$	$\lceil \frac{v}{k} \rceil$	v	$q(v,k)$	$\lceil \frac{v}{k} \rceil$
17	4	4	21	4	4	29	4	4
			24	4	4	32	4	4
			27	5	5	41	7	6
						44	7	6

Table 6: $k = 5, 6, 8; \quad d = q(v,k) - \lceil \frac{v}{k} \rceil$ for $3k+1 \le v < \frac{1}{4}k(3k+1)$

Lemma 2.5 gives

$$\frac{(p(\frac{s+dk}{3})-1)(p(\frac{s+dk}{3})-3)}{2} - \frac{p(\frac{s+dk}{3})+5}{2} \le 6\left(\frac{s+dk}{3}\right), \qquad (6)$$

so that we need only show

$$2(s+ck) > 2(s+dk) + (c-d+1)p(\frac{s+dk}{3}) - \frac{3(c-d)}{2} + \frac{c^2+d^2}{2} - cd + 2$$

where $d+1 \le c \le \frac{2}{3}(2k+1)$. This is equivalent to

$$0 > c^2 - (4k - 2p(\frac{s+dk}{3}) + 2d + 3)c + 2(1-d)p(\frac{s+dk}{3}) + 4dk + 3d + d^2 + 4.$$

Observe that

$$\frac{1}{2}k + 2d + a(k,d) \ge p\left(\left\lfloor \frac{(d+1)k-1}{3} \right\rfloor\right) \ge p\left(\frac{s+dk}{3}\right)$$

for all $k \ge 5$ and suitable d, where

$$a(k,d) = \begin{cases} 3, & \text{if } 3 \le d \le 5, \text{ except } d=4 \text{ and } k = 14, 23, 26, \\ & \text{or } d = 5 \text{ and } k = 23, 35; \\ 4, & \text{if } d = 4 \text{ and } k = 14, 23, 26, \text{ or } d = 5 \text{ and } k = 23, 35; \\ 4, & \text{if } d \le 2, \text{ except } d = 1 \text{ and } k = 11; \\ 5, & \text{if } d = 1 \text{ and } k = 11. \end{cases}$$

Suitable d (for given k) means that $d \le q(v,k) - \lceil \frac{v}{k} \rceil$ for all $3k+1 \le v < \frac{1}{4}k(3k+1)$. For example, Lemma 4.4 implies that $d = 0$ whenever $k \equiv 0$ or 9 mod 12. Furthermore, Table 6 shows that if $k = 5$ or $k = 6$, then $d = 0$, or if $k = 8$, then $d = 0$ or $d = 1$.

For the sake of brevity, we write a instead of $a(k,d)$. The roots of the quadratic

$$c^2 - (3k - 2d - 2a + 3)c + (3d+1)k - 3d^2 + 7d - 2ad + 2a + 4$$

are

$$\frac{1}{2}(3k - 2d - 2a + 3) \pm$$

$$\frac{1}{2}\sqrt{9k^2 - (24d + 12a - 14)k + 16d^2 + (16a - 40)d + 4a^2 - 20a - 7}.$$

Since

$$9k^2 - (24d + 12a - 14)k + 16d^2 + (16a - 40)d + 4a^2 - 20a - 7$$

$$\geq \ (3k - 4d - 2a + 1)^2$$

for all $k \geq 4d + 2a + 1$, inequality (5) is satisfied for all

$$d + 1 \leq c \leq \frac{3}{4}(3k + 1) \leq 3k - 3d - 2a + 2$$

whenever $k > 4d + 2a + 1$ or whenever $k = 4d + 2a + 1$ and either $d = 0$ and $k \leq 12$ or $dk > 12$. The latter conditions ensure that $\frac{1}{3}(s + dk) \neq 4$ and, therefore, in (6) strict inequality holds.

Finally, we show that inequality (5) is satisfied whenever $k < 4d + 2a + 1$. Possible values for the parameter k are in detail:

$$k = \begin{cases} 5, 6, 8, & \text{if } d = 0; \\ 8, 11, & \text{if } d = 1; \\ 11, 14, & \text{if } d = 2; \\ 11, 14, & \text{if } d = 3; \\ 11, 14, 23, & \text{if } d = 4; \text{ and} \\ 11, 14, 23, 26, & \text{if } d = 5. \end{cases}$$

If $k \geq 2d + a + 1$, then

$$9k^2 - (24d + 12a - 14)k + 16d^2 + (16a - 40)d + 4a^2 - 20a - 7$$

$$\geq \ (3k - 4d - 2a - 3)^2.$$

Hence, (5) is true for $d + 3 < c < 3k - 3d - 2a$. For $k = 5, 6, 8, 23$ and 26, we prove in Table 7 that (5) is also satisfied for $c = d + 1, d + 2, d + 3$ and $3k - 3d - 2a \leq c \leq \frac{3}{4}(3k + 1)$, by showing that

$$\min\left\{p(\frac{s + ck}{3}) - p(\frac{s + dk}{3}) \ \mid \ s = 0, 1, \ldots, k - 1\right\} \geq c - d.$$

Note that both $\frac{s+ck}{3}$ and $\frac{s+dk}{3}$ must be integers.

$k=5, d=0$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
3	3	7	3
9	3	12	9
12	3	14	12

$k=6, d=0$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
1	3	5	1
2	3	6	2
3	3	8	3
10	3	14	10
11	3	15	11
12	3	15	12
13	3	16	13
14	3	16	14

$k=8, d=0$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
3	6	6	3
18	6	18	18

$k=8, d=1$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
4	4	3	3
13	4	12	12
16	4	15	15

$k=23, d=4$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
7	22	5	3
49	4	47	45
52	4	48	48

$k=23, d=5$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
8	2	5	3
47	14	42	42
50	14	45	45

$k=26, d=5$			
c	s^*	$p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3})$	c-d
8	2	6	3
59	2	54	54

Table 7: $p(\frac{s^*+ck}{3})-p(\frac{s^*+dk}{3}) \geq c-d$,
where $s^* \in \{0,1,\ldots,k-1\}$ minimizes $p(\frac{s+ck}{3})-p(\frac{s+dk}{3})$

v	s	d	c^*	$k \cdot p(\frac{1}{3}(s+c^*k)) - (s+c^*k)$	t	$k \cdot p(t) - 3t$
35	9	3	6	134	14	112
38	6	3	6	137	13	104
47	8	2	5	135	10	113
50	5	2	5	116	9	105
53	2	5	8	130	19	119
56	10	4	7	133	18	122
59	7	1	4	125	6	103
62	4	1	4	128	5	95
65	1	4	10	131	15	120
68	9	3	6	134	14	112
77	0	3	21	121	11	110
80	8	2	20	113	10	113
83	5	5	20	116	20	116
86	2	5	17	119	19	119
89	10	1	4	122	7	111
92	7	1	4	125	6	103

Table 8: $k = 11; k \cdot p(\frac{1}{3}(s + c^*k)) - (s + c^*k) \geq k \cdot p(t) - 3t$
for all $3k + 1 \leq v < \frac{1}{4}k(3k + 1)$ with $1 \leq d \leq 5$
where c^* minimizes $k \cdot p(\frac{1}{3}(s + ck)) - (s + ck)$

We complete the proof by verifying that the claim is also true if $k = 11$ or $k = 14$. Since $p(\frac{s+ck}{3}) - p(\frac{s+dk}{3}) < c - d$ for some s (e.g. $k = 11$, $d = 1$, $c = 25$, $s = 10$), we need to apply the complete formula of Corollary 4.6

$$v \geq \min\left\{ k \cdot p\left(\frac{s+ck}{3}\right) - (s+ck) \mid c \in C \right\},$$

where $C = \{c \mid d \leq c \leq \min\{\frac{3}{4}(3k+1), \frac{3v-s}{k}\},\ \frac{1}{3}(s+ck) \in \mathbb{N}\}$. Until now, we assumed only that $c \leq \frac{3}{4}(3k+1)$ and that $\frac{1}{3}(s+ck) \in \mathbb{N}$. The proof will be completed if we can show that for $k = 11$ and 14

$$\min\left\{ k \cdot p\left(\frac{s+ck}{3}\right) - (s+ck) \mid c \in C \right\} \geq k \cdot p(t) - 3t$$

for all $3k + 1 \leq v \leq \frac{1}{4}k(3k+1)$. This is done in Table 8 and Table 9. □

Remark 4.8 *Although the condition $v \geq k \cdot p(t) - 3t$ is violated, it was not possible to show the non-existence of an MRD $(4, 11; 113)$ or of an MRD $(4, 14; 161)$. In both cases there exists a partial design built by blocks of size 11 or 14, respectively.*

v	s	d	c^*	$k \cdot p(\frac{1}{3}(s + c^*k)) - (s + c^*k)$	t	$k \cdot p(t) - 3t$
47	9	3	6	201	17	173
50	6	3	6	190	16	176
59	11	2	5	199	13	143
62	8	2	5	202	12	146
65	5	5	8	205	25	191
68	2	5	8	208	24	194
77	7	4	7	203	21	189
80	4	4	7	206	20	164
89	9	3	6	201	17	173
92	6	3	6	190	16	176
101	11	2	5	199	13	143
104	8	2	5	202	12	146
107	5	5	8	205	25	191
110	2	5	23	194	24	194
119	7	4	7	203	21	189
122	4	4	7	206	20	164
131	9	3	27	173	17	173
134	6	3	27	176	16	176
143	11	2	23	185	13	143
146	8	2	23	188	12	146
149	5	5	23	191	25	191

Table 9: $k = 14; k \cdot p(\frac{1}{3}(s + c^*k)) - (s + c^*k) \geq k \cdot p(t) - 3t$
for all $3k + 1 \leq v < \frac{1}{4}k(3k + 1)$ with $2 \leq d \leq 5$
where c^* minimizes $k \cdot p(\frac{1}{3}(s + ck)) - (s + ck)$

If $v = 113$ and $k = 11$, then take a resolvable $(40, 4, 1)$ BIBD [HRW] which has exactly 130 blocks of size 4, and remove two parallel classes of blocks and one further block. Take the dual of the partial design built in this way. The dual design contains 109 points, 36 blocks of size 11 and four blocks of size 10. Add to each 10-block an additional point to obtain a partial design with 113 points and $40 \equiv 4$ mod 6 blocks of size 11, where every point in contained in either one or four blocks of size 11.

If $v = 161$ and $k = 14$, then a regular $(46, 4)$-packing exists [BBS, Thm. 3.4 (x)] where any point is contained in exactly 14 blocks. The dual of this packing gives the desired partial design with blocks of size 14.

Note that we can construct a partial design with blocks of size k where each point is contained in either one or four blocks if the conditions in Theorem 4.7 are satisfied. Here, a similar method as described in [MR,p.357] can be applied.

5 Non-existence of Pairwise Balanced Designs of Order 5k-3 and 6k-3

It is a simple observation that for a particular class of parameter sets the necessary conditions in Theorem 4.7 are not sufficient. To show this, we will first need the following lemma.

Lemma 5.1 *There does not exist an MRD $(4, 12; 69)$ with more than six blocks of size 12.*

Proof.
Possible point types $(\gamma_4(x), \gamma_{12}(x))$ are $(8, 4)$ and $(19, 1)$. This gives $\beta_{12} \leq 23$. Apply Lemma 4.2 for $7 \leq \beta_{12} \leq 23$ to establish the lemma. □

Theorem 5.2 *Suppose that there exists an MRD $(4, k; v)$ with $k \equiv 0, 3$ or 6 mod 12 and $k > 4$. Then $v \neq 5k - 3$ or $6k - 3$, except possibly for $(k, v) = (6, 33)$.*

Proof.
Suppose that there exists an MRD $(4, k; 5k - 3)$ or an MRD $(4, k; 6k - 3)$, respectively. It is easily verified from Theorem 2.1 that $\gamma_k \equiv 1$ mod 3 for all points in V. There is no point that lies on more than four k-blocks since otherwise the blocks through this point would cover at least $1 + 7(k - 1)$ points. Since k divides neither $5k - 3$ nor $6k - 3$, there is at least one point x which is contained in four k-blocks, say $B_1, \ldots, B_4$. Furthermore, x is

contained in a block of size 4 and at least one other point contained in this block lies on a k-block that intersects at least one of the blocks $B_1, \ldots, B_4$. Let y be such an intersection point. Then the blocks spanned by x and y cover at least $1 + 4(k-1) + 3(k-4)$ points. But this is larger than $5k - 3$ if $k > 6$ and larger than $6k - 3$ if $k > 12$, a contradiction. If there exists an MRD $(4, 12; 69)$, then by Lemma 5.1 it contains at most six blocks of size 12, contradicting the fact that x and y are contained in seven 12-blocks. The non-existence of an MRD $(4, 6; 27)$ was proved in [MLAB]; however, it is not known whether or not an MRD $(4, 6; 33)$ exists. $\square$

We remark that a hypothetical MRD $(4, 6; 33)$ contains either 20 or 22 blocks of size 6. A partial design on 33 points with 22 blocks of size 6 can be obtained by taking the dual of a configuration $(22_6, 33_4)$. The following example was taken from Gropp [GRO,Example 3.8]: Develop the base blocks $\{0_1, 1_0, 2_0, 3_1\}$, $\{0_0, 0_1, 4_1, 5_1\}$ and $\{0_0, 2_0, 5_0, 8_1\}$ under $G = \mathbb{Z}_{11}$ where the indices are fixed. We would like to point out that the survey paper [GRO2] and related references are a valuable source of information when searching for regular partial designs.

6 MRDs (4,k;v) with $\gamma_k \equiv 2$ mod 3

Now, we turn our attention to mandatory representation designs where every point is contained in precisely $3m + 2$ blocks of size k for some nonnegative integer m. Recall (Theorem 1.2) that this means $k \equiv 2$ or 11 mod 12 and $v \equiv 0$ mod 3, or $k \equiv 5$ or 8 mod 12 and $v \equiv 0$ or 9 mod 12. First, a lower bound for the number of points is given.

Theorem 6.1 *Let v and k be positive integers where $v \equiv 0 \bmod 3$ and $k \equiv 2$ or $11 \bmod 12$, or $v \equiv 0$ or $9 \bmod 12$ and $k \equiv 5$ or $8 \bmod 12$. Suppose that there exists an MRD $(4, k; v)$. Then*

$$v \geq \begin{cases} \frac{1}{2}k(k+1), & \text{if } k \equiv 2, 8, 17 \text{ or } 23 \bmod 24; \\ \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k+4}{5} \right\rfloor, & \text{if } k \equiv 5, 11, 14 \text{ or } 20 \bmod 24 \text{ and } 5 \mid (k+4); \\ \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k}{5} \right\rfloor, & \text{if } k \equiv 5, 11, 14 \text{ or } 20 \bmod 24 \text{ and } 5 \not\mid (k+4). \end{cases}$$

Proof.
Evidently, $\beta_k \geq k+1$ because every point lies on at least two k-blocks. Hence, Proposition 2.6 gives

$$v \geq \frac{1}{2}k(k+1).$$

Furthermore, it is easily checked from the following argument that $k \equiv 5, 11, 14$ or $20 \bmod 24$ implies $\beta_k \geq k+4$. Suppose that $k \equiv 14 \bmod 24$ and $v = \frac{1}{2}k(k+1)$. Then Theorem 2.1 implies $\beta_k \equiv 0 \bmod 6$, but $k+1 \equiv 3 \bmod 6$, a contradiction. Similarly, if $k \equiv 11 \bmod 24$ and $v = \frac{1}{2}k(k+1)$, then Theorem 2.1 implies $\beta_k \equiv 3 \bmod 6$, but $k+1 \equiv 0 \bmod 6$, again a contradiction. Therefore, $v > \frac{1}{2}k(k+1)$. If $k \equiv 5$ or $20 \bmod 24$, then $v > \frac{1}{2}k(k+1)$ follows by applying Wilson's necessary conditions. Now, $\beta_k \geq \lceil \frac{2v}{k} \rceil \geq k+2$, since every point must be covered twice by k-blocks. Thus, $\beta_k \equiv 0 \bmod 3$ gives $\beta_k = k+4+3m$ for some $m = 0, 1, \ldots$.

Next, it is shown that $v \geq \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k+4}{5} \right\rfloor$ if $\beta_k = k+4$. Letting w denote the number of points which are contained in exactly five blocks of size k and by counting points on k-blocks, we obtain $2(v-w)+5w = k(k+4)$. Hence, $v = \frac{1}{2}k(k+4) - \frac{3}{2}w$. Moreover, observe that $w \leq \left\lfloor \frac{k+4}{5} \right\rfloor$ since every k-block contains at most one point of degree five. Notice that a k-block that intersects four other k-blocks in one point must intersect all of the k-blocks. If we assume that there exists a k-block that has no point of degree five (which is always the case when $5 \not| (k+4)$), then $w \leq \left\lfloor \frac{k}{5} \right\rfloor$; that gives the desired conclusion for $\beta_k = k+4$.

Finally, consider $\beta_k \geq k+7$. For $k \geq 11$, Proposition 2.6 gives

$$v \geq \frac{1}{2}(k+7)(k-2) = \frac{1}{2}k(k+4) + \frac{1}{2}(k-14) \geq \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k}{5} \right\rfloor .$$

However, if $k = 5$ this lemma gives only $v \geq 18$. But, $v \equiv 0$ or $9 \bmod 12$. Thus, $v \geq 21 = \frac{5 \cdot 9}{2} - \frac{3}{2}$. □

Corollary 6.2 *Let $k \equiv 2, 8, 17$ or $23 \bmod 24$ and suppose that there exists an MRD $(4, k; v)$. If $v \neq \frac{1}{2}k(k+1)$, then*

$$v \geq \begin{cases} \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k+4}{5} \right\rfloor, & \text{if } 5 \mid (k+4); \\ \frac{1}{2}k(k+4) - \frac{3}{2}\left\lfloor \frac{k}{5} \right\rfloor, & \text{if } 5 \not| (k+4). \end{cases}$$

Proof.
For $k \geq 17$, or $k = 8$ and $\beta_8 = k+4$, the inequalities are an immediate consequence of the proof of Theorem 6.1. If $k = 8$ and $\beta_8 \geq k+10$, then again Proposition 2.6 gives the desired conclusion. It remains to examine $\beta_8 = k+7$. Every 8-block contains at most two points of degree five, hence $w \leq \left\lfloor \frac{2(k+7)}{5} \right\rfloor$. Then, the equality $v = \frac{1}{2}k(k+7) - \frac{3}{2}w$ is again obtained by counting points and blocks. Observe that

$$v = \frac{1}{2}k(k+7) - \frac{3}{2}w \geq \frac{1}{2}k(k+7) - \frac{3}{2}\left\lfloor \frac{2(k+7)}{5} \right\rfloor \geq \frac{1}{2}k(k+4)$$

for $k = 8$. This completes the proof. □

7 Conclusion

Now, Theorem 1.2 is a consequence of Theorem 1.1, Corollary 2.3, Theorem 3.1, Theorem 4.7, Theorem 5.2 and Theorem 6.1.

The accompanying table in the appendix gives a summary of non-existence results obtained from Theorem 1.2 for $5 \leq k \leq 21$. We point out that for $5 \leq k \leq 9$ and $v \not\equiv 1$ or 4 mod 12 these parameters are the same as in [MLAB]. The uniquely known definite exception which is not covered by Theorem 1.2, namely MRD $(4, 6; 24)$, is also included. Note that this design (with exactly four blocks of size 6) is equivalent to a $TD[4, 6]$, or a pair of orthogonal Latin squares of order 6; a structure which is well-known not to exist. If $\beta_6 \geq 6$, then Lemma 4.2 can be applied to show the non-existence.

From the determination of the non-existence table the following question arises. For which values of v and k can one expect that there does not exist an MRD $(4, k; v)$? For manadatory representation designs with the property that $\gamma_k \equiv 0$ or 2 mod 3 the lower bounds in Theorem 3.1, Theorem 6.1 and Corollary 6.2 up to which one can expect exceptions are quite simple. Considerably more interesting is the case $\gamma_k \equiv 1$ mod 3. The least integer for which an MRD $(4, k; v)$ can exist is $v = 4k - 3$. Such a design can be constructed by adjoining a new point to each group in a TD $(4, k - 1)$. On the other hand, one can expect that MRDs $(4, k; v)$ do not exist for many values of v between $4k$ and, respectively,

$$\begin{aligned}
v &\leq 2k^{3/2} + 2k && \text{if } k \equiv 0 \text{ or } 9 \text{ mod } 12, \\
v &\leq \sqrt{8}k^{3/2} + 2k && \text{if } k \equiv 3 \text{ or } 6 \text{ mod } 12, \\
v &\leq \sqrt{12}k^{3/2} + 2k && \text{if } k \equiv 5 \text{ or } 8 \text{ mod } 12, \text{and} \\
v &\leq \sqrt{24}k^{3/2} + 2k && \text{if } k \equiv 2 \text{ or } 11 \text{ mod } 12
\end{aligned}$$

satisfying Wilson's necessary conditions. We get these bounds by observing that the expression $kp(t) - 3t$ does not exceed $k(\sqrt{12t_{\max}(k)} + 2)$ where $t_{\max}(k)$ is as defined in the proof of Theorem 4.7.

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References

[BBS] J.-C. Bermond, J. Bond, and D. Sotteau. On regular packings and coverings. *Annals of Discrete Math.*, 34 (1987), 81-99.

[BJL] T. Beth, D. Jungnickel, and H. Lenz. Design theory. Bibliographisches Institut, Zürich, 1985 [D. Jungnickel, Design Theory: An Update, *Ars. Comb.*, 28 (1989), 129-199].

[BL1] A. Brouwer and H. Lenz. Unterräume von Blockplänen, "Geometrie-Symposium Siegen 1977, Birkhäuser Verlag, Basel", 383-389.

[BL2] A. Brouwer and H. Lenz. Subspaces of linear spaces of line size four. *European J. Combin.*, 2 (4) (1981), 323-330.

[CD] C.J. Colbourn and J.H. Dinitz (Eds.). *CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, FL, 1996.

[DL] D.A. Drake and J.A. Larson. Pairwise balanced designs whose line sizes do not divide six. *J. of Combin. Theory Ser. A*, 34 (1983), 266-300.

[GRO] H. Gropp. Nonsymmetric configurations with natural index. *Discrete Math.*, 124 (1994), 87-98.

[GRO2] H. Gropp. Configurations. In C.J. Colbourn and J.H. Dinitz, editors, *The CRC Handbook of Combinatorial Designs*, 253-255. CRC Press, 1996.

[G] M. Grüttmüller. On the existence of mandatory representation designs with block sizes 3 and k. *J. Comb. Designs,* 8 (2000), 122-131.

[G2] M. Grüttmüller. On the existence of pairwise balanced designs. Ph.D. Thesis, University of Rostock, 1999.

[GR] M. Grüttmüller and R.S. Rees. Mandatory representation designs MRD $(4, k; v)$ with $k \equiv 1$ mod 3. *Utilitas Math.*, to appear.

[GR2] M. Grüttmüller and R.S. Rees. Pairwise balanced designs PBD(v,K) with $3 \in K$. *Congr. Numer.*, submitted.

[H] H. Hanani. Balanced incomplete block designs and related designs. *Discrete Math.*, 11 (1975), 255-369.

[HRW] H. Hanani, D.K. Ray-Chaudhuri, and R.M. Wilson. On resolvable designs. *Discrete Math.*, 3 (1972), 343-357.

[MR] E. Mendelsohn and R. Rees. Mandatory representation designs. *J. Comb. Theory,* Ser. A, 49 (1988), 349-362.

[MLAB] R.C. Mullin, A.C.H. Ling, R.J.R. Abel, and F.E. Bennett. On the closure of subsets of $\{4, 5, \ldots, 9\}$ which contain $\{4\}$. *Ars. Comb.*, 45 (1997), 33-76.

[R] R.S. Rees. Mandatory representation designs MR $(3, k; v)$ with k even and v odd. *J. Stat. Plan. and Infer.*, 86 (2000), 567-594.

[R2] R.S. Rees. A survey of results on mandatory representation designs. *J. Comb. Math. and Comb. Computing*, to appear.

[RS] R.S. Rees and D.R. Stinson. On the existence of incomplete designs of block size four having one hole. *Utilitas Math.*, 35 (1989), 119-152.

[S] D.R. Stinson. Determination of a packing number. *Ars Combinatoria*, 3 (1977), 89-114.

[W] R.M. Wilson. An existence theory for pairwise balanced designs II: The structure of PBD-closed sets and the existence conjectures. *J. of Combin. Theory Ser. A*, 13 (1972), 246-273.

[WZ] R. Wei and L. Zhu. Embeddings of Steiner systems $S(2, 4, v)$. *Annals of Discrete Math.*, 34 (1987), 465-470.

Appendix

We apply the necessary criteria obtained in Theorem 1.2 to list those values of v which satisfy the necessary conditions of Wilson but do not admit an MRD $(4, k; v)$ where $5 \leq k \leq 21$. This table contains *no* existence results.

Set	Closure	Genuine Exceptions
4 5	$\mathbb{N}_{0,1(4)}$	8 9 12 13 16
4 6	$\mathbb{N}_{0,1(3)}$	7 9 10 12 13 15 16 18 19 22 24 25 27
4 7	$\mathbb{N}_{1(3)}$	10 13 16 19
4 8	$\mathbb{N}_{0,1(4)}$	5 9 12 13 16 17 20 21 24 25 28 33 37 40 41 44 45
4 9	$\mathbb{N}_{0,1,4,9(12)}$	12 13 16 21 24 25 28 37 40 48 49 52
4 10	$\mathbb{N}_{1(3)}$	7 13 16 19 22 25 28
4 11	$\mathbb{N}$	5 6 7 8 9 10 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 42 43 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 67 68 69 70 72 73 75 76 77 79 80 81 82 83 85 86 89 91 92 94 95 98
4 12	$\mathbb{N}_{0,1,4,9(12)}$	9 13 16 21 24 25 28 33 36 37 40 49 52 57 61 64 73 76 85 88 97
4 13	$\mathbb{N}_{1,4(12)}$	13 16 25 28 37
4 14	$\mathbb{N}$	5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 54 55 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 96 97 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 124 127 130 131 133 134 136 139 142 149 152
4 15	$\mathbb{N}_{0,1(3)}$	6 7 9 10 12 13 15 16 18 19 21 22 24 25 27 28 30 31 33 34 36 37 39 40 42 43 45 46 48 49 51 52 54 55 58 61 63 64 66 67 69 70 72 73 76 78 79 81 82 84 85 87 88 90 91 93 94 96 97 100 103 105 106 108 109 111 112 114 115 118 121 123 124 127 129 130 132 133 136 138 139 142 145 148 151 154 157
4 16	$\mathbb{N}_{1,4(12)}$	13 16 25 28 37 40
4 17	$\mathbb{N}_{0,1(4)}$	5 8 9 12 13 16 17 20 21 24 25 28 29 32 33 36 37 40 41 44 45 48 49 52 53 56 57 60 61 64 69 72 73 76 77 80 81 84 85 88 89 92 93 96 97 100 101 104 105 108 109 112 117 120 121 124 125 128 129 132 133 136 137 140 141 144 145 148 149 152 156 157 160 165 168 169 172 173 176 181 184 193 196
4 18	$\mathbb{N}_{0,1(3)}$	6 7 9 10 12 13 15 16 18 19 21 22 24 25 27 28 30

31 33 34 36 37 39 40 42 43 45 46 48 49 51 52 54 55 57 58 60 61 63 64 66 67 70 73 75 76 78 79 81 82 84 85 87 88 91 93 94 96 97 99 100 102 103 105 106 109 111 112 114 115 117 118 120 121 124 127 129 130 132 133 135 136 138 139 142 145 147 148 150 151 153 154 156 157 160 163 166 169 171 172 174 175 178 181 184 187 190 193 196 199 202 205 208 211 214 217 220

$\boxed{4\ 19}$ $\boxed{\mathbb{N}_{1(3)}}$ 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52 55 61 64

$\boxed{4\ 20}$ $\boxed{\mathbb{N}_{0,1(4)}}$ 5 8 9 12 13 16 17 20 21 24 25 28 29 32 33 36 37 40 41 44 45 48 49 52 53 56 57 60 61 64 65 68 69 72 73 76 81 84 85 88 89 92 93 96 97 100 101 104 105 108 109 112 113 116 117 120 121 124 125 128 129 132 133 136 141 144 145 148 149 152 153 156 157 160 161 164 165 168 169 172 173 176 177 180 181 184 189 192 193 196 201 204 205 208 209 212 213 216 217 220 225 228 229 232 241 244 253 256 265 268 277

$\boxed{4\ 21}$ $\boxed{\mathbb{N}_{0,1,4,9(12)}}$ 9 12 13 16 21 24 25 28 33 36 37 40 45 48 49 52 57 60 61 64 69 72 73 76 85 88 93 96 97 100 108 109 112 117 120 121 124 129 132 133 136 145 148 153 156 157 160 169 172 177 181 184 192 193 196 205 208 217 220 229 232 241 244 253 256 265 268 277 280 289 292