Orthogonal Double Covers of $K_{n,n}$ by Small Graphs

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Abstract

An orthogonal double cover (ODC) of K_n is a collection of graphs such that each edge of K_n occurs in exactly two of the graphs and two graphs have precisely one edge in common. ODCs of K_n and their generalizations have been extensively studied by several authors (e.g. [1, 7, 16]). In this paper, we investigate ODCs where the graph to be covered twice is $K_{n,n}$ and all graphs in the collection are isomorphic to a given small graph G. We prove that there exists an ODC of $K_{n,n}$ by all proper subgraphs G of $K_{n,n}$ for $1 \leq n \leq 9$, with two genuine exceptions.

1 Introduction

Let $\mathcal{G} = \{G_1, \ldots, G_n\}$ be a collection of n simple spanning graphs on an n-element vertex set and let K_n denote the complete graph on this vertex set. We call $\mathcal{G}$ an *Orthogonal Double Cover (ODC)* of K_n, if the following properties hold.

(i) Every edge of K_n is covered by exactly two of the graphs $G_1, \ldots, G_n$.

(ii) For every choice of integers i, j with $1 \leq i < j \leq n$, the graphs G_i and G_j have exactly one common edge.

The graphs $G_1, \ldots, G_n$ are called *pages*. If all pages are isomorphic to a graph G, we speak about an *ODC of K_n by G.*

The original motivation for investigating ODCs comes from a problem of Demetrovics, Füredi and Katona [4] on minimal databases, and a problem of Hering and Rosenfeld [10] on the organization of statistical test plannings.

In our notation they were interested in ODCs of K_n. Many researchers have been involved in investigating ODCs of K_n by G for several classes of graphs G for example in [2, 5, 6, 8, 9, 11, 12, 13, 14, 15].

A generalization of the notion of an ODC to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_1, \ldots, G_n\}$ be a collection of n spanning subgraphs of H. $\mathcal{G}$ is called an ODC of H if there exists a bijective mapping $\varphi : V \to \mathcal{G}$ such that:

(i) Every edge of H is contained in exactly two of the graphs $G_1, \ldots, G_n$.

(ii) For every choice of distinct vertices i, j,

$$|E(\varphi(i)) \cap E(\varphi(j))| = \begin{cases} 1 & \text{if } \{i,j\} \in E(H), \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

If all of the G_i are isomorphic to a graph G, we call $\mathcal{G}$ an ODC of H by G.

In this paper, we consider the case that H is $K_{n,n}$ the complete bipartite graph on $2n$ vertices. It seems natural to extend the concept of ODC of $H = K_n$ to others graphs. According to the obvious properties of ODCs by a graph G, the underlying graph H has to be $|E(G)|$-regular. Therefore, $H = K_{n,n}$ is of special interest. On the other hand, the idea of constructing an ODC of K_{2n} by two ODCs of K_n requires an ODC of $K_{n,n}$.

We will label the vertices of $K_{n,n}$ by the elements of $\Gamma \times \mathbb{Z}_2$, where $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ is an additive group of order n, such that $\{a_i, b_i\} \notin E(K_{n,n})$ for $a, b \in \Gamma$ and fixed $i \in \mathbb{Z}_2$. Here, we first use the groups Γ and $\mathbb{Z}_2$ just for labeling the vertices. It will turn out later that these groups can be used for the construction of ODCs. If there is no chance of confusion we will write (a, b) instead of $\{a_0, b_1\}$ for the edge between the vertices a_0 and b_1. We denote $\varphi(a_i)$ by $G_{a,i}$ for all $a_i \in \Gamma \times \mathbb{Z}_2$. In Figure 1 an ODC of $K_{3,3}$ by $G = S_2 \cup K_2$ (the vertex disjoint union of a star with two edges and a single edge) is exhibited.

As an immediate consequence of the double cover property (i) follows that G contains exactly n edges. Moreover, the orthogonality property (ii) forces the graphs $G_{a,0}$ for all $a \in \Gamma$ and the graphs $G_{a,1}$ for all $a \in \Gamma$ to form two orthogonal edge-decompositions of $K_{n,n}$.

To every of these two edge decomposition we may associate an $n \times n$-square with entries belonging to Γ by $L_i = L_i(k, l) \quad i = 0, 1; k, l \in \Gamma$ with $L_i(k, l) = m$ if and only if $\{k_0, l_1\} \in E(G_{m,i})$. For example, the ODC given in Figure 1 is associated with the squares:

$$L_0 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

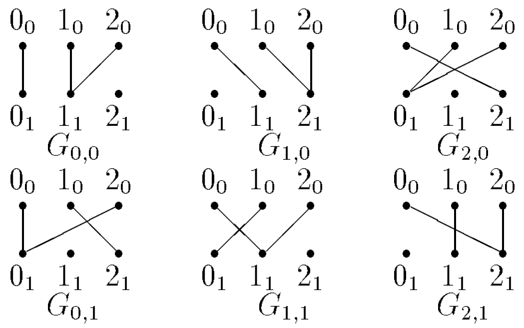

Figure 1: ODC of $K_{3,3}$ by $G = S_2 \cup K_2$ with $\Gamma = \mathbb{Z}_3$

n	1	2	3	4	5	6	7	8	9
number of graphs	1	2	4	9	18	43	96	239	595
number of symmetric starters	1	1	2	5	14	40	92	235	589
number of direct constructions	0	0	2	4	4	2	4	4	6
number of graphs with no solution	0	1	0	0	0	1	0	0	0

Table 1: Survey of results and construction methods used

The orthogonality condition (ii) reads for the squares as usually as $|\{(L_0(k,l), L_1(k,l)) : k,l \in \Gamma\}| = n^2$. Obviously, $G = nK_2$ corresponds bijectively to Latin squares. According to the existence of mutually orthogonal Latin squares, see [3], we have the following.

Theorem 1.1 *There exists an ODC of $K_{n,n}$ by nK_2 if and only if $n \geq 1$, except for $n = 2$ or $n = 6$.*

In this paper, we will prove the following result.

Theorem 1.2 *There exists an ODC of $K_{n,n}$ by all spanning subgraphs G of $K_{n,n}$ with n edges for $1 \leq n \leq 9$, with two genuine exceptions, namely $G = 2K_2$ or $G = 6K_2$.*

Our main construction tool for that result (called symmetric starter) will be investigated in Section 2. In some cases (see Table 1) we need to find direct constructions which are considered in Section 3.

2 Symmetric Starters

Let G be a spanning subgraph of $K_{n,n}$ and $x \in \Gamma$. Then the graph $G + x$ with $E(G + x) = \{(a + x, b + x) : (a, b) \in E(G)\}$ is called the *x-translate* of

G. The *length* of an edge $e = (a, b) \in E(G)$ is defined by $d(e) = b - a$. G is called *half starter* with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are mutually different, i.e. $\{d(e) : e \in E(G)\} = \Gamma$. It is easy to see that the following lemma is true.

Lemma 2.1 *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e. $\bigcup_{x\in\Gamma} E(G + x) = E(K_{n,n})$.*

In what follows, we will represent a half starter G by the vector $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_j} \in \Gamma$ and $(v_{\gamma_j})_0$ is the unique vertex of the edge of length γ_j. In general, we need to define an order of the elements of Γ. In this paper, we consider Abelian groups only represented by Cartesian products of cyclic groups. Therefore, we take the lexicographic order, i.e. in case of $\Gamma = \mathbb{Z}_n$ it is just the natural order $0, 1, \ldots, n-1$. (For example, the vector $(0, 2, 1)$ represents a star S_3 with the root at the vertex 0_1.) Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be *orthogonal* if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2.2 *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $\mathcal{G} = \{G_{a,i} : a_i \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = (G_i + a)$ is an ODC of $K_{n,n}$.*

Proof: Firstly, it is clear from Lemma 2.1 that if $(a, b) \in E(K_{n,n})$, then there are exactly two graphs $G_{x,0}$ and $G_{y,1}$ with $(a, b) \in E(G_{x,0})$ and $(a, b) \in E(G_{y,1})$. Thus, the double cover property (i) is satisfied.

Secondly, let $a, b \in \Gamma$ and $k, l \in \mathbb{Z}_2$. We want to prove that $|E(G_{a,k}) \cap E(G_{b,l})| = 0$ if $k = l$ and $a \neq b$, and $|E(G_{a,k}) \cap E(G_{b,l})| = 1$ if $k \neq l$. The first case follows immediately from construction. If otherwise $k \neq l$ (w.l.o.g. $k = 0, l = 1$), then there is exactly one γ_i with $v_{\gamma_i}(G_0) - v_{\gamma_i}(G_1) = b - a$. Thus, $v_{\gamma_i}(G_0) + a = v_{\gamma_i}(G_1) + b$, but this means that there is exactly one edge e with $e = (v_{\gamma_i}(G_0) + a, v_{\gamma_i}(G_0) + a + \gamma_i) = (v_{\gamma_i}(G_1) + b, v_{\gamma_i}(G_1) + b + \gamma_i)$. Therefore, $e \in E(G_0 + a) = E(G_{a,0})$ and $e \in E(G_1 + b) = E(G_{b,1})$. We conclude $|E(G_{a,0}) \cap E(G_{b,1})| = 1$. $\square$

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{b_0, a_1\} : \{a_0, b_1\} \in E(G)\}$ is called the *symmetric graph* of G. Note that if G is a half starter, then G_s is also a half starter. A half starter G is called a *symmetric starter* with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 2.3 *Let n be a positive integer and let G be a half starter represented by $v(G) = (v_{\gamma_0}, v_{\gamma_1}, \ldots, v_{\gamma_{n-1}})$. Then G is a symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

Proof: We know that G_s is a half starter and representable by $v(G_s)$. Since $(v_{\gamma_i}(G_s), v_{\gamma_i}(G_s)+\gamma_i)$ is an edge in $E(G_s)$ we have $(v_{\gamma_i}(G_s)+\gamma_i, v_{\gamma_i}(G_s))$ is an edge in $E(G)$ of length $-\gamma_i$. Therefore, $v_{-\gamma_i}(G) = v_{\gamma_i}(G_s)+\gamma_i$ and thus $v_{\gamma_i}(G_s) = v_{-\gamma_i}(G) - \gamma_i$. Consequently, $v(G)$ and $v(G_s)$ are orthogonal if and only if $\{v_\gamma(G) - v_\gamma(G_s) = v_\gamma - (v_{-\gamma} - \gamma) = v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$. $\square$

Example 2.4 *(i) Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then there is an ODC of $K_{4,4}$ by $4K_2$, since G with $v(G) = (00, 10, 11, 01)$ is a symmetric starter.*

(ii) Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then there is an ODC of $K_{8,8}$ by C_8 (the circle with 8 vertices), since G with $v(G) = (000, 000, 001, 011, 111, 011, 111, 001)$ is a symmetric starter.

For our investigations, we need to have a list with all non-isomorphic spanning subgraphs of $K_{n,n}$ which have exactly n edges. This list $L(n)$ (with a canonical order of the graphs) was created using the program MAKEBG of Brendan McKay [17]. All graphs in the list are non-isomorphic in the following sense: For two given graphs G_0 and G_1 there are no bijections $\psi_0 : \Gamma \times \{0\} \mapsto \Gamma \times \{0\}$ and $\psi_1 : \Gamma \times \{1\} \mapsto \Gamma \times \{1\}$ such that $E(G_0) = \{\{\psi_0(a_0), \psi_1(b_1)\} : (a, b) \in E(G_1)\}$. For our purpose, we need to define that graphs G_0 and G_1 are isomorphic if there is a bijection $\psi : \Gamma \times \mathbb{Z}_2 \mapsto \Gamma \times \mathbb{Z}_2$ such that $E(G_0) = \{\{\psi(a_0), \psi(b_1)\} : (a, b) \in E(G_1)\}$. Therefore, we performed an isomorphism check on the graphs from $L(n)$ using the canonical output of MAKEBG to compute a partition of L into classes $L_1, L_2, \ldots, L_m$ of isomorphic graphs. We apply Theorem 2.3 as described in Algorithm 1 to find a list of graphs which are symmetric starters. Please note that $G_{\pi_1}^{\pi_2}$ is defined by $E(G_{\pi_1}^{\pi_2}) := \{(\pi_1(a), \pi_2(b)) : (a, b) \in E(G)\}$ where π_1 and π_2 are permutations of the elements of Γ.

Remark 2.5 *We remark that we do not need to consider all graphs in an isomorphism class L_k since $G \in L_k$ has a symmetric starter if and only if $G_s \in L_k$ has a symmetric starter. But please note that there are graphs G, G' in some L_k such that G has a symmetric starter and G' does not. For example, let G and G' be isomorphic to the vertex disjoint union of three stars S_2, where in G one star has its root in $\mathbb{Z}_6 \times \{0\}$ and two stars have their roots in $\mathbb{Z}_6 \times \{1\}$ while in G' all three stars have their roots in $\mathbb{Z}_6 \times \{0\}$. Then $(0, 2, 2, 4, 3, 1)$ is a symmetric starter for G but there is no one for G'.*

The list of all symmetric starters derived that way with $\Gamma = \mathbb{Z}_n$ is shown in the Appendix. Graphs which do not admit a symmetric starter with $\Gamma = \mathbb{Z}_n$ are considered in the following section.

Algorithm 1 Algorithm for constructing symmetric starters

Require: Positive integer n, additive group Γ of order n

Ensure: list of symmetric starters, list of graphs which do not have a symmetric starter with respect to Γ

Compute a list $L(n)$ with all subgraphs of $K_{n,n}$ with n edges using MAKEBG
Compute a partition of the graphs of L into isomorphism classes L_1, L_2, ... , L_m
for all isomorphism classes L_k **do**
  for all graphs G in L_k **do**
    for all permutations $\pi_1, \pi_2 \in S_\Gamma$ **do**
      if $G_{\pi_1}^{\pi_2}$ is a half starter **then**
        Compute $v(G_{\pi_1}^{\pi_2})$
        if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$ **then**
          Output: $v(G_{\pi_1}^{\pi_2})$ {found a symmetric starter}
          Goto NEXT {next isomorphism class}
        end if
      end if
    end for $\{\text{all } \pi_1, \pi_2 \in S_\Gamma\}$
  end for $\{\text{all } G \in L_k\}$
  Output: No symmetric starter for graphs in L_k
  NEXT {label: NEXT}
end for $\{\text{all } L_k\}$

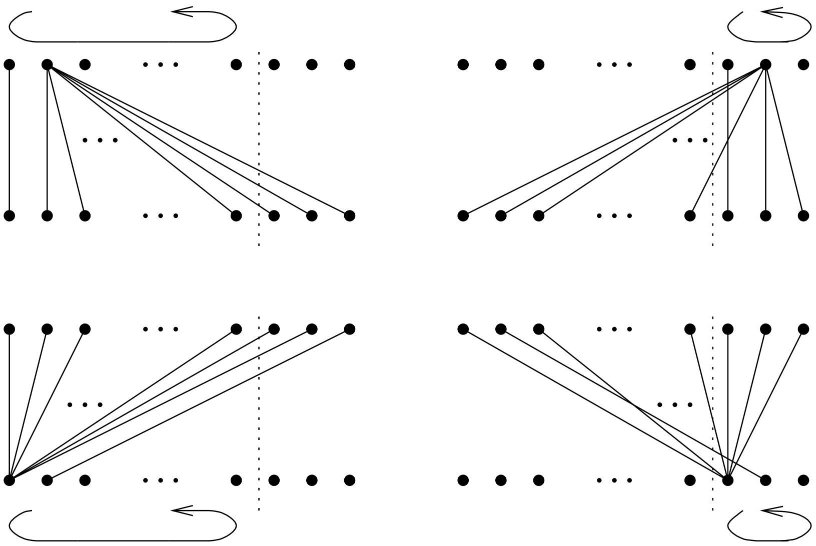

Figure 2: Graphs $G_{0,0}$, $G_{n-3,0}$, $G_{0,1}$ and $G_{n-3,1}$ (top left to bottom right)

3 Direct Constructions

In this section, we will investigate all remaining graphs which do not have a symmetric starter. In the sequel, we will use the notion $G^{n,m}$ to denote a subgraph of $K_{n,n}$ that is element of the class L_m. We first give three results that yield infinitely many values of n for which there is an ODC of $K_{n,n}$ by graphs of a special type which do not admit symmetric starters.

Lemma 3.1 *Let $n \geq 5$, $\Gamma = \mathbb{Z}_n$ and $G = K_2 \cup S_{n-1}$. Then there exists an ODC of $K_{n,n}$ by G.*

Proof: We define the pages as follows. (Here, $x \mod y$ always means the smallest nonnegative residue class modulo y.) For $j \in \{0, \ldots, n-4\}$ let $E(G_{j,0}) = \{(j,0)\} \cup \{((j+1) \mod (n-3), i) : i \in \{1, \ldots, n-1\}\}$. For $j \in \{n-3, n-2, n-1\}$ let $E(G_{j,0}) = \{(j, n-3)\} \cup \{(((j-n+4) \mod 3) + n-3, i) : i \in \{0, \ldots, n-4, n-2, n-1\}\}$. Moreover, for $j \in \{0, \ldots, n-4\}$ let $E(G_{j,1}) = \{(n-1, (j+1) \mod (n-3))\} \cup \{(i,j) : i \in \{0, \ldots, n-2\}\}$ and for $j \in \{n-3, n-2, n-1\}$ let $E(G_{j,1}) = \{(1, ((j-n+4) \mod 3)+n-3)\} \cup \{(i,j) : i \in \{0, 2, 3 \ldots, n-1\}\}$. A graphical representation of the pages is given in Figure 2.

For the sake of brevity we define sets $A = \{0, \ldots, n-4\}$ and $B = \{n-3, n-2, n-1\}$. To see that conditions (i) and (ii) of the ODC definition are satisfied, please note that an arbitrary edge (a, b) is contained exactly in the graphs $G_{j,0}$ and $G_{j',1}$, where

$$j = \begin{cases} (a-1) \mod (n-3) & \text{if } b \neq 0 \text{ and } a \in A; \\ ((a-n+2) \mod 3) + n - 3 & \text{if } b \neq n-3 \text{ and } a \in B; \text{ or} \\ a & \text{otherwise,} \end{cases}$$

and

$$j' = \begin{cases} (b-1) \mod (n-3) & \text{if } a = n-1 \text{ and } b \in A; \\ ((b-n+2) \mod 3) + n - 3 & \text{if } a = 1 \text{ and } b \in B; \text{ or} \\ b & \text{otherwise.} \end{cases}$$

Furthermore, arbitrary graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge. More precisely, $E(G_{j,0}) \cap E(G_{j',1}) =$

$$\begin{cases} \{(j,0)\} & \text{if } j \in A \text{ and } j' = 0; \\ \{((j+1) \mod (n-3), j')\} & \text{if } j \in A \text{ and } j' \in A \setminus \{0\}, \text{ or} \\ & j \in A \setminus \{0\} \text{ and } j' \in B; \\ \{(1, ((j'-n+4) \mod 3) + n - 3)\} & \text{if } j = 0 \text{ and } j' \in B; \\ \{(n-1, (j'+1) \mod (n-3)\} & \text{if } j = n-2 \text{ and } j' \in A; \\ \{(((j-n+4) \mod 3) + n - 3, j')\} & \text{if } j \in \{n-3, n-1\} \text{ and } j' \in A, \text{ or} \\ & j \in B \text{ and } j' \in \{n-2, n-1\}; \\ \{(j, n-3)\} & \text{if } j \in B \text{ and } j' = n-3. \end{cases}$$ □

Corollary 3.2 *There exists an ODC of $K_{5,5}$ by $G^{5,3}$, an ODC of $K_{6,6}$ by $G^{6,3}$, an ODC of $K_{7,7}$ by $G^{7,3}$, an ODC of $K_{8,8}$ by $G^{8,3}$, and an ODC of $K_{9,9}$ by $G^{9,3}$.*

Remark 3.3 *Notice that for none of the graphs in Lemma 3.1 there exists a symmetric starter since for an arbitrary group Γ and any labeling of the vertices there is an edge in the star of the same length as the length of the single edge.*

Lemma 3.4 *Let $n \geq 5$ be an odd integer, $\Gamma = \mathbb{Z}_n$ and $G = S_2 \cup S_{n-2}$. Then there exists an ODC of $K_{n,n}$ by G.*

Proof: We define the pages as follows. For $j \in \mathbb{Z}_n$ let $E(G_{j,0}) = \{(i,j) : i \in \{0, \ldots, n-3\}\} \cup \{(n-2, j+1), (n-1, j+1)\}$. Moreover, let $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, n-3\}\} \cup \{((j+1) \mod (n-2), i) : i \in \{n-2, n-1\}\}$ for $j \in \{0, \ldots, n-3\}$ and $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, n-3\}\} \cup \{(((j-n+3) \mod 2) + n - 2, i) : i \in \{n-2, n-1\}\}$ for $j \in \{n-2, n-1\}$. See Figure 3.

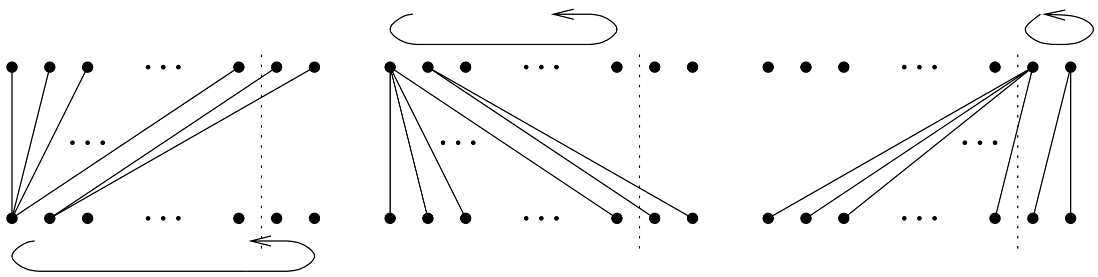

Figure 3: Graphs $G_{0,0}$, $G_{0,1}$ and $G_{n-2,1}$ (left to right)

Now, let $A = \{0, \dots, n-3\}$, $B = \{n-2, n-1\}$, $C = \{n-3, n-2\}$ and $D = \{0, \dots, n-4, n-1\}$. An arbitrary edge (a, b) is contained exactly in the graphs $G_{j,0}$ and $G_{j',1}$, where

$$j = \begin{cases} b & \text{if } a \in A, \text{ or} \\ b-1 & \text{otherwise,} \end{cases}$$

and

$$j' = \begin{cases} ((a-n+3) \mod 2) + n - 2 & \text{if } a, b \in B, \\ a & \text{if } b \in A, \\ (a-1) \mod (n-2) & \text{otherwise.} \end{cases}$$

Furthermore, arbitrary graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge: $E(G_{j,0}) \cap E(G_{j',1}) =$

$$\begin{cases} \{(j', j)\} & \text{if } j, j' \in A; \\ \{((j'+1) \mod (n-2), j)\} & \text{if } j' \in A \text{ and } j \in B; \\ \{(((j'-n+3) \mod 2) + n - 2, j+1)\} & \text{if } j' \in B \text{ and } j \in C; \text{or} \\ \{(j', j+1)\} & \text{if } j \in D \text{ and } j' \in B. \end{cases} \quad \square$$

Corollary 3.5 *There exists an ODC of $K_{5,5}$ by $G^{5,6}$, an ODC of $K_{7,7}$ by $G^{7,6}$, and an ODC of $K_{9,9}$ by $G^{9,6}$.*

Remark 3.6 *If $G = S_2 \cup S_{n-2}$, then there is for even n always an ODC of $K_{n,n}$ created by the symmetric starter $(0, n/2, \dots, n/2, v_{n/2} = 0, n/2, \dots, n/2,)$ with $\Gamma = \mathbb{Z}_n$. On the other hand, if n is odd, then for any group Γ and any labeling of the vertices the stars S_2 and S_{n-2} have edges of the same length. Thus, there is no symmetric starter for odd n.*

Lemma 3.7 *Let $n \geq 5$ be an odd integer, $\Gamma = \mathbb{Z}_n$ and $G = S_2 \cup (n-2)K_2$. Then there exists an ODC of $K_{n,n}$ by G.*

Proof: For $j \in \mathbb{Z}_n$ define $E(G_{j,0}) = \{(n-1,j)\} \cup \{(i,j+i) : i \in \{0,\dots,n-2\}\}$ and $E(G_{j,1}) = \{(n-1,j+n-1)\} \cup \{(i,j+2i) : i \in \{0,\dots,n-2\}\}$. Here, an arbitrary edge (a,b) is contained exactly in the graphs $G_{(b-a),0}$ and $G_{(b-2a),1}$ if $a \neq n-1$, or in $G_{(b-a+1),0}$ and $G_{(b-a),1}$ if $a = n-1$. Thus, condition (i) is satisfied. Furthermore, graphs $G_{j,0}$ and $G_{j',1}$ share a unique edge which is $(n-1,j)$ if $j = j'-1$, or $(j-j', 2j-j')$ otherwise. Therefore, condition (ii) is satisfied. $\square$

Corollary 3.8 *There exists an ODC of $K_{5,5}$ by $G^{5,17}$, an ODC of $K_{7,7}$ by $G^{7,95}$, and an ODC of $K_{9,9}$ by $G^{9,594}$.*

As a consequence of Theorem 1.1 we have the following.

Corollary 3.9 *There exists an ODC of $K_{3,3}$ by $G^{3,4}$, an ODC of $K_{8,8}$ by $G^{8,239}$, and an ODC of $K_{9,9}$ by $G^{9,595}$.*

Lemma 3.10 *There exists an ODC of $K_{3,3}$ by $G^{3,3}$.*

Proof: See Figure 1. $\square$

Lemma 3.11 *There exists an ODC of $K_{4,4}$ by $G^{4,3} = S_3 \cup K_2$.*

Proof: Let $E(G_{0,0}) = \{(0,0)\} \cup \{(1,i) : i \in \{1,2,3\}\}$, $E(G_{1,0}) = \{(1,0)\} \cup \{(0,i) : i \in \{1,2,3\}\}$, $E(G_{2,0}) = \{(2,2)\} \cup \{(3,i) : i \in \{0,1,3\}\}$, $E(G_{3,0}) = \{(3,2)\} \cup \{(2,i) : i \in \{0,1,3\}\}$, $E(G_{0,1}) = \{(3,1)\} \cup \{(i,0) : i \in \{0,1,2\}\}$, $E(G_{1,1}) = \{(3,0)\} \cup \{(i,1) : i \in \{0,1,2\}\}$, $E(G_{2,1}) = \{(1,3)\} \cup \{(i,2) : i \in \{0,2,3\}\}$, and $E(G_{3,1}) = \{(1,2)\} \cup \{(i,3) : i \in \{0,2,3\}\}$. $\square$

Lemma 3.12 *There exists an ODC of $K_{4,4}$ by $G^{4,5} = P_5$ (the path with 5 vertices).*

Proof: The ODC required can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,0,3,3)$ and $v(G_1) = (0,3,1,0)$. $\square$

Lemma 3.13 *There exists an ODC of $K_{4,4}$ by $G^{4,8} = S_2 \cup 2K_2$.*

Proof: An ODC of $K_{4,4}$ by $S_2 \cup 2K_2$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,3,1,2)$ and $v(G_1) = (0,2,3,3)$. $\square$

Lemma 3.14 *There exists an ODC of $K_{6,6}$ by $G^{6,26} = C_6$.*

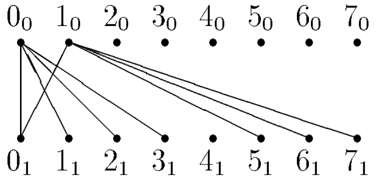

Figure 4: Graph $G^{8,14}$

Proof: The pages of the first edge decomposition can be defined as follows. Let $E(G_{0,0}) = \{(0,0),\ (0,1),\ (3,0),\ (3,5),\ (4,1),\ (4,5)\}$, $E(G_{1,0}) = \{(1,1), (1,2), (3,2), (3,4), (5,1), (5,4)\}$, $E(G_{2,0}) = \{(2,0), (2,2), (4,0), (4,3), (5,2), (5,3)\}$, $E(G_{3,0}) = \{(0,3), (0,4), (2,1), (2,4), (3,1), (3,3)\}$, $E(G_{4,0}) = \{(0,2), (0,5), (1,4), (1,5), (4,2), (4,4)\}$, $E(G_{5,0}) = \{(1,0), (1,3), (2,3), (2,5), (5,0), (5,5)\}$. While the pages of the second edge decomposition are obtained by taking the symmetric graphs $G_{j,1} = (G_{j,0})_s$ for $j \in \{0, \ldots, 5\}$. $\square$

Lemma 3.15 *There exists an ODC of $K_{7,7}$ by $G^{7,10} = S_3 \cup S_4$.*

Proof: For $j \in \mathbb{Z}_7$ let $E(G_{j,0}) = \{(i,j) : i \in \{0,1,2,3\}\} \cup \{(i,j+1) : i \in \{4,5,6\}\}$. Furthermore, for $j \in \{0,1,2,3\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0,1,2,3\}\} \cup \{((j+1) \mod 4, i) : i \in \{4,5,6\}\}$ and for $j \in \{4,5,6\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0,1,2,3\}\} \cup \{(((j-2) \mod 3)+4, i) : i \in \{4,5,6\}\}$. $\square$

Lemma 3.16 *There exists an ODC of $K_{8,8}$ by $G^{8,10} = S_3 \cup S_5$.*

Proof: The pages of an ODC of $K_{8,8}$ by $S_3 \cup S_5$ can be defined as follows. For $j \in \mathbb{Z}_8$ let $E(G_{j,0}) = \{(i,j) : i \in \{0, \ldots, 4\}\} \cup \{(i,j+1) : i \in \{5,6,7\}\}$. Furthermore, for $j \in \{0, \ldots, 4\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, 4\}\} \cup \{((j+1) \mod 5, i) : i \in \{5,6,7\}\}$ and for $j \in \{5,6,7\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, 4\}\} \cup \{(((j-3) \mod 3)+5, i) : i \in \{5,6,7\}\}$. $\square$

Lemma 3.17 *There exists an ODC of $K_{8,8}$ by $G^{8,14}$ (see Figure 4).*

Proof: An ODC of $K_{8,8}$ by $G^{8,14}$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0,0,0,0,1,1,1,1)$ and $v(G_1) = (0,7,6,5,5,4,3,2)$. $\square$

Lemma 3.18 *There exists an ODC of $K_{9,9}$ by $G^{9,15} = S_4 \cup S_5$.*

Proof: Define the pages as follows. For $j \in \mathbb{Z}_9$ let $E(G_{j,0}) = \{(i,j) : i \in \{0, \ldots, 4\}\} \cup \{(i,j+1) : i \in \{5, \ldots, 8\}\}$. For $j \in \{0, \ldots, 4\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, 4\}\} \cup \{((j+1) \mod 5, i) : i \in \{5, \ldots, 8\}\}$, and for $j \in \{5, \ldots, 8\}$ let $E(G_{j,1}) = \{(j,i) : i \in \{0, \ldots, 4\}\} \cup \{(((4-j) \mod 4)+5, i) : i \in \{5, \ldots, 8\}\}$. $\square$

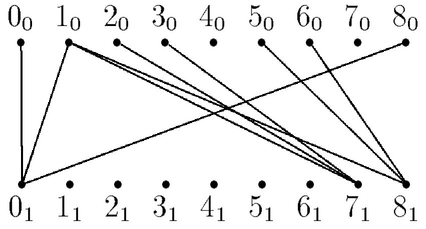

Figure 5: Graph $G^{9,113}$

Lemma 3.19 *There exists an ODC of $K_{9,9}$ by $G^{9,113}$ (see Figure 5).*

Proof: An ODC of $K_{9,9}$ by $G^{9,113}$ can be obtained using the following two orthogonal (but not symmetric) half starters $v(G_0) = (0, 8, 6, 5, 3, 2, 1, 1, 1)$ and $v(G_1) = (0, 0, 3, 3, 6, 6, 6, 3, 0)$. $\square$

Appendix

We list all symmetric starter vectors for $1 \leq n \leq 9$. The order of the vectors corresponds to the order of the isomorphism classes L_m. For brevity, we omit commas inside of a vector.

n=1: 1:(0)
n=2: 1:(00) 2: no solution
n=3: 1:(021) 2:(011) 3:direct construction 4:direct construction
n=4: 1:(0321) 2:(0311) 3:direct construction 4:(0022) 5:direct construction 6:(0123) 7:(0232) 8:direct construction 9:(00,10,11,01)
n=5: 1:(04321) 2:(04221) 3:direct construction 4:(03311) 5:(04211) 6:direct construction 7:(04111) 8:(04224) 9:(03121) 10:(03421) 11:(04114) 12:(04232) 13:(03232) 14:(03422) 15:(01314) 16:(01034) 17:direct construction 18:(03142)
n=6: 1:(054321) 2:(054221) 3:direct construction 4:(053311) 5:(014543) 6:(051324) 7:(004422) 8:(044211) 9:(053211) 10:(034125) 11:(052221) 12:(005232) 13:(041321) 14:(045321) 15:(044111) 16:(053111) 17:(052201) 18:(053431) 19:(010441) 20:(042320) 21:(044120) 22:(034225) 23:(055231) 24:(022231) 25:(043101) 26:direct construction 27:(015333) 28:(053115) 29:(052132) 30:(022431) 31:(013533) 32:(003532) 33:(034312) 34:(024315) 35:(051215) 36:(001443) 37:(024215) 38:(031220) 39:(034412) 40:(012354) 41:(042353) 42:(034512) 43:no solution
n=7: 1:(0654321) 2:(0653321) 3:direct construction 4:(0644311) 5:(0653221) 6:direct construction 7:(0553311) 8:(0643311) 9:(0665431) 10:direct construction 11:(0652221) 12:(0105543) 13:(0642321) 14:(0654231) 15:(0063332) 16:(0006542) 17:(0633121) 18:(0653431) 19:(0644226) 20:(0544121) 21:(0045332) 22:(0643121) 23:(0634211) 24:(0465321) 25:(0552220) 26:(0005542) 27:(0552211) 28:(0452226) 29:(0642211) 30:(0565421) 31:(0116644) 32:(0534220) 33:(0655412) 34:(6141150) 35:(0633111) 36:(0643431) 37:(0044222) 38:(0444201) 39:(0044232) 40:(0642111) 41:(0633201) 42:(0464132) 43:(0532320) 44:(0606532) 45:(0166531) 46:(0645232) 47:(4513160) 48:(0622221) 49:(0641111) 50:(0651115) 51:(0010544) 52:(0010643) 53:(0106530)

54:(0623326) 55:(0531321) 56:(0562432) 57:(0652314) 58:(0531311) 59:(0042322)
60:(0045542) 61:(0631211) 62:(0624216) 63:(0146433) 64:(0463421) 65:(0353110)
66:(0042332) 67:(0642432) 68:(0163443) 69:(0055236) 70:(0353216) 71:(0455312)
72:(0465231) 73:(0453412) 74:(0441201) 75:(0533125) 76:(0343101) 77:(0055243)
78:(0424110) 79:(0631216) 80:(0145333) 81:(0623116) 82:(0663542) 83:(0532614)
84:(0514161) 85:(0240534) 86:(0634615) 87:(0316542) 88:(0510542) 89:(0234260)
90:(0635012) 91:(0423350) 92:(0536442) 93:(0145623) 94:(0313562) 95:direct construction
96:(0531642)

n=8: 1:(07654321) 2:(07653321) 3:direct construction 4:(07554311) 5:(07634121)
6:(07614325) 7:(07553311) 8:(07765422) 9:(07643221) 10:direct construction 11:(00664422)
12:(07764422) 13:(07544211) 14:direct construction 15:(05634127) 16:(07633321)
17:(00763432) 18:(07524321) 19:(07654231) 20:(07552221) 21:(07642221) 22:(07633121)
23:(07645421) 24:(06652320) 25:(07544121) 26:(07552311) 27:(06514321) 28:(07655231)
29:(05764321) 30:(00663332) 31:(00753332) 32:(01055343) 33:(05633327) 34:(07664232)
35:(07635431) 36:(00662332) 37:(00655232) 38:(00663422) 39:(06553121) 40:(07763422)
41:(07543121) 42:(07453211) 43:(34233076) 44:(07444111) 45:(07434431) 46:(07514315)
47:(07552227) 48:(06454111) 49:(05652211) 50:(07534111) 51:(00756532) 52:(07652132)
53:(06632320) 54:(06552121) 55:(07542121) 56:(01766343) 57:(05762321) 58:(01634527)
59:(00555222) 60:(05553111) 61:(06633111) 62:(07533111) 63:(07663232) 64:(06541021)
65:(01064441) 66:(05454110) 67:(07745222) 68:(06763322) 69:(00563322) 70:(06453111)
71:(07443201) 72:(07534117) 73:(01765343) 74:(76313176) 75:(05454217) 76:(07745231)
77:(07214563) 78:(07622221) 79:(00733332) 80:(02155554) 81:(07333021) 82:(07733322)
83:(06551121) 84:(07541121) 85:(04622226) 86:(06622211) 87:(05622227) 88:(07522211)
89:(06766421) 90:(01046443) 91:(07522321) 92:(00742432) 93:(07524327) 94:(01746543)
95:(07624315) 96:(00542212) 97:(06631311) 98:(07531311) 99:(01773533) 100:(05651310)
101:(07451211) 102:(07343121) 103:(01575443) 104:(07565341) 105:(02270755)
106:(01063443) 107:(01773443) 108:(00562412) 109:(01745543) 110:(07423311)
111:(05765231) 112:(06735422) 113:(01576423) 114:(00046442) 115:(04644110)
116:(06762322) 117:(05522301) 118:(06076432) 119:(01166644) 120:(04554110)
121:(00562322) 122:(00454232) 123:(06056531) 124:(04644217) 125:(06762421)
126:(05624210) 127:(05644137) 128:(00534422) 129:(06734422) 130:(05424317)
131:(05764231) 132:(07733131) 133:(07531111) 134:(00744542) 135:(07441117)
136:(07073433) 137:(06634430) 138:(06531101) 139:(05622210) 140:(01773531)
141:(06631120) 142:(01763343) 143:(07541127) 144:(06733422) 145:(01575423)
146:(06066332) 147:(06735330) 148:(05551317) 149:(00575343) 150:(07642412)
151:(06635430) 152:(07773441) 153:(06775521) 154:(05551201) 155:(06464330)
156:(06662330) 157:(00562320) 158:(07441201) 159:(00053542) 160:(05523301)
161:(06742421) 162:(06451300) 163:(00646532) 164:(06635412) 165:(06531301)
166:(01745541) 167:(01063441) 168:(05524217) 169:(07463232) 170:(07423317)
171:(07155643) 172:(07465212) 173:(06234715) 174:(02064642) 175:(02166441)
176:(06724236) 177:(00556242) 178:(03156555) 179:(06752202) 180:(04622217)
181:(06232360) 182:(02074452) 183:(06422316) 184:(01164453) 185:(02756534)
186:(01234567) 187:(01575553) 188:(03634316) 189:(02146454) 190:(03700543)
191:(07363437) 192:(06613171) 193:(04612117) 194:(06312321) 195:(03652316)
196:(05614271) 197:(07536421) 198:(01463443) 199:(02507542) 200:(01746644)
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206:(06312301) 207:(00524427) 208:(07425412) 209:(00425312) 210:(00345122)
211:(05613271) 212:(04513327) 213:(06174352) 214:(07363202) 215:(06725201)
216:(05675413) 217:(03443171) 218:(03453116) 219:(02747542) 220:(03641177)

221:(02577442) 222:(04635410) 223:(00425310) 224:(05623430) 225:(04253117) 226:(05746123) 227:(03515645) 228:(03465236) 229:(05167253) 230:(05216734) 231:(01720675) 232:(01366237) 233:(04265217) 234:(02344160) 235:(03525417) 236:(02534716) 237:(05713645) 238:(01342675) 239:direct construction

n=9: 1:(087654321) 2:(087644321) 3:direct construction 4:(086654311) 5:(087643321) 6:direct construction 7:(086644311) 8:(088765422) 9:(088765431) 10:(084651327) 11:(077553311) 12:(086553311) 13:(086554211) 14:(087543221) 15:direct construction 16:(087633321) 17:(010866543) 18:(087534321) 19:(087654231) 20:(008744432) 21:(000876542) 22:(087442321) 23:(087645421) 24:(010776443) 25:(086554121) 26:(008564432) 27:(087542321) 28:(087453221) 29:(087652431) 30:(086633311) 31:(000866542) 32:(086633221) 33:(067454428) 34:(087533221) 35:(086765321) 36:(086644228) 37:(076644121) 38:(077453311) 39:(086544121) 40:(008564332) 41:(078765421) 42:(087765412) 43:(081674523) 44:(007744422) 45:(088744422) 46:(007744332) 47:(088744431) 48:(008644332) 49:(017855523) 50:(008876532) 51:(086543121) 52:(076763232) 53:(077453220) 54:(076554120) 55:(086542311) 56:(051624387) 57:(085554111) 58:(087442221) 59:(087545421) 60:(077454111) 61:(086454111) 62:(008553432) 63:(087532221) 64:(087443121) 65:(017086543) 66:(077632320) 67:(078755232) 68:(075653121) 69:(018776343) 70:(085762432) 71:(078653421) 72:(077444111) 73:(086444111) 74:(085544111) 75:(077765420) 76:(076434310) 77:(087543287) 78:(006655222) 79:(088565222) 80:(007743332) 81:(086633228) 82:(086765222) 83:(075644111) 84:(076633121) 85:(018855443) 86:(077765231) 87:(086533121) 88:(085633211) 89:(018765343) 90:(088545422) 91:(007743422) 92:(088743431) 93:(088743422) 94:(086434318) 95:(086764232) 96:(018766343) 97:(078663422) 98:(078563422) 99:(067542310) 100:(856242017) 101:(000666333) 102:(066633111) 103:(085533111) 104:(074641317) 105:(005452422) 106:(018666333) 107:(086444118) 108:(085544118) 109:(086433111) 110:(085533201) 111:(086753232) 112:(078743322) 113:direct construction 114:(078565221) 115:(081857643) 116:(051627384) 117:(087333321) 118:(086622221) 119:(087522221) 120:(087333021) 121:(077622320) 122:(001087544) 123:(001087643) 124:(007733332) 125:(008633332) 126:(008876440) 127:(067333328) 128:(008876530) 129:(075432010) 130:(008866440) 131:(001077544) 132:(088733431) 133:(000785634) 134:(008565532) 135:(010876530) 136:(010566543) 137:(087531321) 138:(085641321) 139:(037684521) 140:(077633327) 141:(085524221) 142:(020087552) 143:(086424221) 144:(077453327) 145:(008464232) 146:(087423221) 147:(086765402) 148:(018077543) 149:(085745321) 150:(087744338) 151:(008642432) 152:(008534432) 153:(086762432) 154:(006752432) 155:(005763422) 156:(076451321) 157:(067524311) 158:(084653127) 159:(020066644) 160:(075551311) 161:(066551311) 162:(085541311) 163:(084642211) 164:(087535431) 165:(020077552) 166:(077531311) 167:(077423220) 168:(007552332) 169:(086531311) 170:(008534332) 171:(020867644) 172:(077353120) 173:(067541311) 174:(084643201) 175:(076554127) 176:(064534128) 177:(005753322) 178:(077423320) 179:(076424320) 180:(067344310) 181:(077423311) 182:(070876432) 183:(076451221) 184:(086423311) 185:(087562432) 186:(075641320) 187:(086752431) 188:(000087633) 189:(087885551) 190:(086622228) 191:(010744443) 192:(067622211) 193:(000087453) 194:(010755430) 195:(077332320) 196:(076622121) 197:(077332311) 198:(010575441) 199:(086522121) 200:(086332311) 201:(067622310) 202:(018844533) 203:(017845443) 204:(067353228) 205:(001876552) 206:(078445321) 207:(000077633) 208:(066622201) 209:(086333318) 210:(001066643) 211:(087733232) 212:(078775521) 213:(000077552) 214:(080844433) 215:(067333310) 216:(006733322) 217:(067522228) 218:(008776340) 219:(010764530) 220:(010757551) 221:(001066544) 222:(007733420) 223:(066424201) 224:(007876430)

225:(076663402)	226:(085441318)	227:(078733421)	228:(085353208)	229:(001877552)
230:(087733412)	231:(001876354)	232:(020065554)	233:(085524228)	234:(084642228)
235:(085441311)	236:(008464230)	237:(076455410)	238:(087642412)	239:(066354300)
240:(028885554)	241:(010767643)	242:(021775554)	243:(006543122)	244:(070773443)
245:(007535430)	246:(008565228)	247:(084552228)	248:(084534111)	249:(085523121)
250:(008642430)	251:(085523301)	252:(018663543)	253:(006752430)	254:(086441328)
255:(086535421)	256:(076762332)	257:(011767644)	258:(018085653)	259:(005645232)
260:(007463232)	261:(075623121)	262:(076762431)	263:(078535422)	264:(010875634)
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270:(075554117)	271:(076531301)	272:(070774433)	273:(008534330)	274:(056552201)
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315:(080843433)	316:(077875440)	317:(077331120)	318:(085562232)	319:(055641120)
320:(066752231)	321:(057531111)	322:(005553402)	323:(078733302)	324:(010575330)
325:(076331201)	326:(070755440)	327:(056451111)	328:(084441201)	329:(086421111)
330:(000857652)	331:(000857634)	332:(000677633)	333:(078773343)	334:(075752232)
335:(020055642)	336:(088445202)	337:(084542388)	338:(078733401)	339:(086733402)
340:(076331101)	341:(008443428)	342:(057522110)	343:(007733238)	344:(077421120)
345:(018743343)	346:(086521128)	347:(005642232)	348:(067322310)	349:(007733418)
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355:(056631201)	356:(067463412)	357:(080763513)	358:(081785643)	359:(027186345)
360:(087222221)	361:(000087640)	362:(087585551)	363:(076611121)	364:(086511121)
365:(067222228)	366:(086222211)	367:(086411111)	368:(008444542)	369:(067222210)
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375:(076334430)	376:(020646454)	377:(007335230)	378:(085511201)	379:(021065540)
380:(011065640)	381:(018086450)	382:(002806754)	383:(084252221)	384:(087242021)
385:(020108735)	386:(000576502)	387:(005363332)	388:(007465030)	389:(084252211)
390:(010565503)	391:(086313121)	392:(084744338)	393:(020088455)	394:(056613121)
395:(076314121)	396:(004744302)	397:(074252220)	398:(005363322)	399:(020008635)
400:(067724232)	401:(066541326)	402:(007424232)	403:(067625232)	404:(064614221)
405:(087253028)	406:(075314321)	407:(047634281)	408:(078634521)	409:(046544018)
410:(064614211)	411:(077420320)	412:(064614310)	413:(085241221)	414:(047542021)
415:(016835533)	416:(083453011)	417:(086425322)	418:(087512317)	419:(067254280)
420:(083643127)	421:(075725332)	422:(046624120)	423:(074514301)	424:(004752412)
425:(078836443)	426:(086425432)	427:(078425332)	428:(016475533)	429:(047524318)
430:(067243381)	431:(046235871)	432:(075620220)	433:(021006762)	434:(077353170)
435:(006736443)	436:(085313211)	437:(020587542)	438:(015845433)	439:(010587534)
440:(077423370)	441:(077420211)	442:(011845644)	443:(008623428)	444:(057413310)
445:(075314211)	446:(015865413)	447:(057244218)	448:(012087465)	449:(022867755)
450:(028576644)	451:(018475233)	452:(017845623)	453:(005524232)	454:(056252201)
455:(006553038)	456:(086312111)	457:(084252218)	458:(076232327)	459:(030167663)
460:(030868655)	461:(018535343)	462:(086313128)	463:(046451318)	464:(075745337)
465:(005623232)	466:(080745548)	467:(011565544)	468:(070445533)	469:(077312120)

470:(075625232) 471:(085723232) 472:(026067542) 473:(078835343) 474:(004742312) 475:(057232310) 476:(078866501) 477:(075763427) 478:(084745318) 479:(056242301) 480:(078363122) 481:(067353412) 482:(070834433) 483:(083443028) 484:(016834433) 485:(087412217) 486:(075340111) 487:(084612218) 488:(015665243) 489:(011708744) 490:(005846522) 491:(045552281) 492:(066352271) 493:(004734310) 494:(075512021) 495:(031706663) 496:(056612201) 497:(004535222) 498:(006736523) 499:(000734513) 500:(074735420) 501:(000475313) 502:(075724430) 503:(075512381) 504:(007345120) 505:(067525430) 506:(083642117) 507:(068623420) 508:(046534307) 509:(080735513) 510:(084513308) 511:(075623126) 512:(018567653) 513:(075253181) 514:(056414228) 515:(067413281) 516:(086512307) 517:(081567342) 518:(055662230) 519:(075224117) 520:(007742243) 521:(001878752) 522:(066875312) 523:(004645118) 524:(075856430) 525:(032886632) 526:(077466530) 527:(018475220) 528:(066723302) 529:(074732231) 530:(075875420) 531:(055314218) 532:(088847534) 533:(005623400) 534:(046351118) 535:(074413307) 536:(078352202) 537:(066525131) 538:(001567343) 539:(074412380) 540:(001845623) 541:(031784562) 542:(058262232) 543:(037340328) 544:(014742543) 545:(040708856) 546:(068363543) 547:(068373443) 548:(030208456) 549:(068346543) 550:(047134281) 551:(064873542) 552:(036541278) 553:(074235422) 554:(087142817) 555:(022578455) 556:(045124310) 557:(031578455) 558:(083142317) 559:(005263427) 560:(037453286) 561:(008253182) 562:(033780566) 563:(068253482) 564:(012588635) 565:(062551370) 566:(056134271) 567:(011467253) 568:(042653187) 569:(072532071) 570:(048018564) 571:(015785613) 572:(058573142) 573:(082441278) 574:(058263120) 575:(045502281) 576:(042633117) 577:(078486633) 578:(026475534) 579:(074152800) 580:(025866414) 581:(053764208) 582:(053764218) 583:(053836413) 584:(046875147) 585:(063725415) 586:(074682531) 587:(013571438) 588:(025144286) 589:(024153278) 590:(072268853) 591:(051638270) 592:(061423875) 593:(034782506) 594:direct construction 595:direct construction

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