On the orthogonal product of simplices and direct products of truncated Boolean lattices

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Abstract

The initial point of this paper are two Kruskal-Katona type theorems. The first is the Colored Kruskal-Katona Theorem which can be stated as follows: Direct products of the form $B^1_{k_1} \times B^1_{k_2} \times \cdots \times B^1_{k_n}$ belong to the class of Macaulay posets, where B^t_k denotes the poset consisting of the $t+1$ lowest levels of the Boolean lattice B_k. The second one is a recent result saying that also the products $B^{k_1-1}_{k_1} \times B^{k_2-1}_{k_2} \times \cdots \times B^{k_n-1}_{k_n}$ are Macaulay posets. The main result of this paper is that the natural common generalization to products of truncated Boolean lattices does not hold, i.e. that $(B^t_k)^n$ is a Macaulay poset only if $t \in \{0, 1, k-1, k\}$.

1   Introduction

In two recent papers [11, 12], we studied the poset $P(N; A_1, A_2, \ldots, A_n)$ of all subsets of a finite set N which do not contain any of the non-empty, pairwise disjoint subsets $A_1, A_2, \ldots, A_n \subset N$. The elements of P are ordered by inclusion. The main result presented there is that P belongs to the class of Macaulay posets, i.e. there an analogue of the Kruskal-Katona Theorem [9, 8] for P. This is closely related to the well-known Colored Kruskal-Katona Theorem (see below). Here, we investigate possible common generalizations.

In order to define what we mean by a Macaulay poset, we need a few notions. Let $(P, \leq)$, briefly P, be a ranked poset with rank function r such that $r(x) = 0$

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for some minimal element $x \in P$. The i-th *level* of P is the set $N_i(P) = P_i := \{x \in P \mid r(x) = i\}$. The *(lower) shadow* of an element $x \in P$ is the set $\Delta(x) := \{y \in P \mid y \le x \text{ and } r(y) = r(x) - 1\}$, its *upper shadow* $\nabla(x) := \{y \in P \mid x \le y \text{ and } r(y) = r(x) + 1\}$. The lower and upper shadows of a subset $X \subseteq P$ are $\Delta(X) := \bigcup_{x \in X} \Delta(x)$ and $\nabla(X) := \bigcup_{x \in X} \nabla(x)$, respectively.

Consider a linear ordering $\prec$ of the elements of P. For $X \subseteq P_i$ let $C(X)$ denote the set of the first $|X|$ elements of P_i w.r.t. $\prec$. The set $C(X)$ is called the *compression* of X, and if $X = C(X)$ holds, then X is called *compressed*. For $\emptyset \subset X \subseteq P$ and $1 \le m \le |X|$, the set of the first resp. last m elements of X w.r.t. $\prec$ is denoted by $C(m, X)$ resp. $L(m, X)$. A subset $X \subseteq P$ is called *left-compressed* (resp. *right-compressed*) if $C(X \cap P_i) = X \cap P_i$ (resp. $L(X \cap P_i) = X \cap P_i$) for all i. The notation $X \prec Y$ for $X, Y \subseteq P$ is used to indicate that the last element of X precedes the first element of Y in the linear order $\prec$.

The poset P is said to be a *Macaulay poset* if the ordering $\prec$ can chosen such that for all $i \in \{1, 2, \ldots, r(P)\}$ and all $X \subseteq P_i$ the following inclusion holds:

$$\Delta(C(X)) \subseteq C(\Delta(X)) \quad . \tag{1}$$

In this case, we also say that $(P, \le, \prec)$ is a *Macaulay structure*.

It is well-known that (1) holds for all i and $X \subseteq P_i$ if and only if for $i \in \{1, 2, \ldots, r(P)\}$ and $X \subseteq P_i$ the two conditions

$$|\Delta(C(X))| \le |\Delta(X)| \tag{2}$$

and

$$C(\Delta(C(X))) = \Delta(C(X)) \tag{3}$$

are satisfied (cf. [3, 5]). By (2), compressed subsets have minimum-sized shadow among all subsets of the same level with fixed cardinality. That means, the solutions to the *Shadow Minimization Problem (SMP)* form a *nested structure* since $C(m, P_i) \subset C(m + 1, P_i)$ for $1 \le m < |P_i|$. By (3), shadows of compressed subsets are compressed as well. Therefore, we speak of the *continuity* of the solutions to the SMP.

Assume that the Hasse diagram of P is connected. The dual of P is the poset P^* on the same elements with $x \le^* y$ whenever $y \le x$ holds in P. Obviously, P^* is ranked with the rank-function $r^*(x) = r(P) - r(x)$ for $x \in P$. Let further be $\prec^*$ be the reverse of $\prec$, i.e. we have $x \prec^* y$ whenever $y \prec x$. The following is well-known (see [3] or [5] for proof):

Proposition 1 $(P^*, \leq^*, \prec^*)$ *is a Macaulay structure if and only if* $(P, \leq, \prec)$ *is a Macaulay structure.*

By the *poset induced by* $X \subseteq P$ we mean the set X together with the restriction of the order relation $\leq$ to X. In the sequel, such a poset will also be called a *subposet* of P. A subset $X \subseteq P$ is an *ideal* (resp. *filter*) if $y \in X$ is implied by $x \in X$ and $y \leq x$ (resp. $x \leq y$). The ideal generated by $x \in P$ and the filter generated by $x \in P$ are defined to be the sets $I(x) = \{y \in P \mid y \leq x\}$ and $F(x) = \{y \in P \mid x \leq y\}$, respectively. In the next section, we will make use of the following fact which follows from the above definition of Macaulay posets and Proposition 1.

Proposition 2 *Let* P *be a Macaulay poset, and let* I *and* F *be a left-compressed ideal and a right-compressed filter in* P, *respectively. Then* I *and* F *are Macaulay posets.*

Let B'_n denote the Boolean lattice of order n without its minimal element. The problem to decide whether the direct product $O = B'_{k_1} \times B'_{k_2} \times \cdots \times B'_{k_n}$ is a Macaulay poset for all $2 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ was raised by Harper. He introduced the name *orthogonal product of simplices* for O because it arises in geometry that way when trying to extend Lindsey's Theorem [14] to higher-dimensional faces. Some partial results toward a solution of the above problem were obtained by Moghadam [17], Vasta [20] gave a solution to the related easier *Maximum Rank Ideal Problem (MRI)*. In the special case $n = 2$ the poset O was studied by Sali [18, 19] following a proposal of Katona and Sós. In this case, O can be interpreted as the poset of submatrices of a matrix, ordered by containment. Sali proved several theorems corresponding to classical ones like Sperner's Theorem and the Erdős-Ko-Rado Theorem. Concerning the SMP, he determined subsets of a given level with minimum shadow into the *first* level of O.

The papers [11] and [12] contain a proof that O indeed is a Macaulay poset for all $k_1, \ldots, k_n$. In fact, a slightly more general class of posets is considered. Moreover, the dual O^* is considered rather than O. Clearly, O^* is the direct product of Boolean lattices without their maximal elements. We allowed the case that there is another factor which is a complete Boolean lattice. Let us choose the following representation of this poset: Let N be a finite set, and let $A_1, A_2, \ldots, A_n$ be mutually disjoint subsets of N such that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_n$, where $k_i := |A_i|$ for $i = 1, 2, \ldots, n$. Then our poset $P = P(N; A_1, A_2, \ldots, A_n)$ consists of all $F \subseteq N$ satisfying $A_i \not\subseteq F$ for $i = 1, 2, \ldots, n$, ordered by inclusion.

Next, let us introduce the Macaulay order $\prec$ on P. The definition of $\prec$ involves the *reverse-lexicographic order* $\prec_{r\ell}$ on 2^N which is defined putting $F \prec_{r\ell} G$ whenever $\max(F \setminus G) < \max(G \setminus F)$. It is convenient to use the notations $A_0 := N \setminus (A_1 \cup A_2 \cup \cdots \cup A_n)$ and $k_0 := |A_0|$, and to assume $A_i = \{a_i^1, a_i^2, \ldots, a_i^{k_i}\}$ with $a_i^1 < a_i^2 < \cdots < a_i^{k_i}$ for $i = 0, 1, \ldots, n$ such that $a_{i-1}^{k_{i-1}} < a_i^1$ for $i = 1, 2, \ldots, n$. Note that $k_0 = 0$ is allowed and corresponds to Harper's original proposal. For $i = 1, 2, \ldots, n$ and $F \in P$ define $a_i(F) := \max(A_i \setminus F)$ and $A(F) := \{a_1(F), a_2(F), \ldots, a_n(F)\}$. The linear order $\prec$ is established on P putting $F \prec G$ whenever one of the following two conditions is satisfied:

(a) $A(F) \neq A(G)$ and $\min\big(A(F) \setminus A(G)\big) > \min\big(A(G) \setminus A(F)\big)$, or

(b) $A(F) = A(G)$ and $F \prec_{r\ell} G$.

Theorem 3 $\big(P(N; A_1, A_2, \ldots, A_n), \subseteq, \prec\big)$ *is a Macaulay structure.*

The proof was given in two parts: In [11] we settled the case $m \leq 2$ which is used as the basis for the inductive proof for $m \geq 3$ in [12].

In the special case $k_0 = 0$, $k_1 = k_2 = \cdots = k_n = 2$ the poset P^* is isomorphic to the poset formed by all subcubes of an n-cube ordered by inclusion. In this case, a linear ordering $\prec$ for which (1) holds has been introduced by Lindström [15]. His result has been generalized to powers of stars by Leeb [13] and, independently, by Bezrukov [1]. Essentially the same, but in the dual version, has been found in [6]. The *colored complexes* introduced there are direct products of stars of *almost* equal size. This case, however, is somehow covered by the result for powers of stars because colored complexes occur as left-compressed ideals there, as one can easily derive from the definition of the corresponding ordering $\prec$. The observation that colored complexes are the duals of the star powers in [13, 1] is due to Engel [5]. Finally, it has been shown in [10] that products of stars of arbitrary sizes are Macaulay posets.

To formulate this result in a more detailed way, let us introduce the notation $\mathrm{Col}(A_1, A_2, \ldots, A_n)$ for the poset in question, where $A_i = \{kn + i \mid k = 0, 1, \ldots, k_i - 1\}$, $k_1 \geq k_2 \geq \cdots \geq k_n \geq 1$, and $\mathrm{Col}(A_1, A_2, \ldots, A_n)$ consists of all subsets $F \subseteq \bigcup_{i=1}^n A_i$ satisfying $|F \cap A_i| \leq 1$ for $i = 1, 2, \ldots, n$, ordered by inclusion.

Theorem 4 (Colored Kruskal-Katona Theorem)
$\big(\mathrm{Col}(A_1, A_2, \ldots, A_n), \subseteq, \prec_{r\ell}\big)$ *is a Macaulay structure.*

2 Products of truncated Boolean lattices

The Colored Kruskal-Katona Theorem and Theorem 3 suggest that there might be a common generalization to products of truncated Boolean lattices. More precisely, for $1 \leq t < k$ let the *truncated Boolean lattice* B_k^t be the subposet of the Boolean lattice B_k formed by the levels $N_0(B_k)$, $N_1(B_k)$, ... , $N_t(B_k)$, and consider posets of the form $B_{k_1}^{t_1} \times B_{k_2}^{t_2} \times \cdots \times B_{k_n}^{t_n}$. Theorem 4 covers the case $t_1 = t_2 = \cdots = t_n = 1$, and Theorem 3 corresponds to $k_1 - t_1 = k_2 - t_2 = \cdots = k_n - t_n = 1$. It is natural to ask for generalizations to $t_1 = \cdots = t_n$ or to $k_1 - t_1 = \cdots = k_n - t_n$, or at least to powers of truncated Boolean lattices.

A closer inspection of the Macaulay orders on colored complexes and on the duals of orthogonal products of simplices, respectively, shows that it could be problematic to find a common generalization. For instance: In a colored complex the last element of the first level comes from the largest factor, whereas it is in the smallest factor of $P(N; A_1, \ldots, A_n)$. In fact, it turns out that this is the right intuition. In the sequel, we will prove that all powers of truncated Boolean lattices which are Macaulay posets are already given by Theorems 4 and 3.

First we study the product of two factors one of which is a star. This turns out to be an important special case with further consequences.

Lemma 5 *Let $1 \leq t < k$ and $2 \leq m$ be integers. $P = B_k^t \times B_m^1$ is a Macaulay poset if and only if $t \in \{1, k-1\}$.*

Proof. Throughout the proof, P will be looked at as the collection of all subsets $F \subset A \cup B$ with $|F \cap A| \leq t$ and $|F \cap B| \leq 1$, where $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ are disjoint. These subsets are ordered by inclusion.

1. Let $t \notin \{1, k-1\}$. We have to show that P is not a Macaulay poset. Assume on the contrary that there exists a Macaulay order $\prec$ on P.

Obviously, all elements of the level P_{t+1} are of the form $F \cup \{b\}$, where $F \subset A$ and $b \in B$. Without loss of generality, let $G := \{b_1, a_1, a_2, \ldots, a_t\}$ be the first element of P_{t+1} w.r.t. $\prec$. Now by the continuity condition (3), we obtain

$$C(t + 1, P_1) = \big\{\{b_1\}, \{a_1\}, \{a_2\}, \ldots, \{a_t\}\big\} \quad .$$

Consider an element $F \in P_1$. By Proposition 1 and

$$|\nabla(F)| = \left\{ \begin{array}{ll} k + m - 1 & \text{if } F \subset A, \\ k & \text{if } F \subset B, \end{array} \right.$$

the last element of P_1 w.r.t. $\prec$ is a subset of B, without loss of generality it is $\{b_m\}$.

Let $\mathcal{F}(\{b_m\})$ denote the filter generated by $\{b_m\}$, and consider $P' := P \setminus \mathcal{F}(\{b_m\})$. Clearly, P' is a left-compressed ideal in P consisting of all $F \subset A \cup (B \setminus \{b_m\})$ with $|F \cap A| \leq t$ and $|F \cap (B \setminus \{b_m\})| \leq 1$, ordered by inclusion. By Proposition 2, P' is a Macaulay poset with the Macaulay order given by the restriction of $\prec$ to P'. If $m \geq 3$, then we can argue exactly like in the previous paragraph to show that the last element of P_1' is in $B \setminus \{b_m\}$. Further iteration gives

$$L(m - 1, P_1) = B \setminus \{b_1\}   .$$

Without loss of generality we assume

$$\{a_{t+1}\} \prec \{a_{t+2}\} \prec \cdots \prec \{a_k\} \prec \{b_2\} \prec \{b_3\} \prec \cdots \prec \{b_m\}$$

for the last $k + m - t - 1$ elements of P_1 w.r.t. $\prec$.

Now by Proposition 1, for $F_1, F_2 \in P_{t+1}$ with $F_1 \subset G \cup \{a_{t+1}\}$ and $F_2 \not\subset G \cup \{a_{t+1}\}$ we have $F_1 \prec F_2$. Consequently,

$$C(t + 1, P_{t+1}) = \binom{G \cup \{a_{t+1}\}}{t + 1} \setminus \{(G \setminus \{b_1\}) \cup \{a_{t+1}\}\}$$

holds which yields $|\Delta(C(t + 1, P_{t+1}))| = \binom{t+2}{2}$.

Consider the segment $\mathcal{S} := C(2t + 2, P_{t+1}) \setminus C(t + 1, P_{t+1})$. All members of $\mathcal{S}$ must be of the form $F \cup \{a_{t+2}\}$ with $F \subset G \cup \{a_{t+1}\}$ and $|F \cap A| \leq t - 1$. Therefore, there is no set S of size $t + 1$ such that $\mathcal{S}$ is the collection of all $F \cup \{a_{t+2}\}$ with $F \in \binom{S}{t}$. Using this, one easily observes that $|\Delta(\mathcal{S}) \setminus \Delta(C(t + 1, P_{t+1}))| > \binom{t+1}{2}$ (which also follows from more general results due to Bezrukov [2], Mörs [16], and Füredi & Griggs [7] who, independently, characterized all parameters leading to unique solutions of the corresponding SMP in Boolean lattices). By this, we have

$$|\Delta(C(2t + 2, P_{t+1}))| > (t + 1)^2   .$$

On the other hand, consider the collection $\mathcal{F}$ of all $F \in P_{t+1}$ with $F \subset \{b_1, b_2, a_1, a_2, \ldots, a_t, a_{t+1}\}$. Using (2), a contradiction is implied by the trivial observations $|\mathcal{F}| = 2t + 2$ and $|\Delta(\mathcal{F})| = (t + 1)^2$.

2. Let $t \in \{1, k - 1\}$. We have to show that P is a Macaulay poset. This follows for $t = 1$ by Theorem 4. So let us assume that $t = k - 1$.

First we establish the Macaulay order on P. Suppose that

$$b_1 < a_1 < a_2 < \cdots < a_k < b_2 < b_3 < \cdots < b_m   .$$

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We are going to prove that the reverse-lexicographic order $\prec_{r\ell}$ as introduced for finite sets in Section 1 is a Macaulay order on P.

Clearly, (3) is satisfied for P and $\prec_{r\ell}$, it remains to show (2). Let $i \in \{1, 2, \ldots, k\}$ and $\emptyset \subset \mathcal{F} \subseteq P_i$. We have to prove $|\Delta(C(\mathcal{F}))| \leq |\Delta(\mathcal{F})|$. Since several steps of the proof are more or less routine, in order to keep it short, we omit detailed proofs of some statements which are (really) easy to verify.

First define a partition of P by

$$P = P(0) \cup P(1) \cup P(2) \cup \cdots \cup P(m) \quad,$$

where $P(0) := \{F \in P \mid F \cap B = \emptyset\}$ and $P(j) := \{F \in P \mid b_j \in F\}$ for $j = 1, 2, \ldots, m$. Together with the set inclusion, $P(0) \cup P(1)$ is a Boolean lattice of order $k + 1$ without its two last elements w.r.t. $\prec_{r\ell}$ (A and $A \cup \{b_1\}$), whereas each of the subposets $P(2), P(3), \ldots, P(m)$ is isomorphic to B_k^{k-1}. Moreover,

$$\bigl(P(0) \cup P(1)\bigr) \prec_{r\ell} P(2) \prec_{r\ell} \cdots \prec_{r\ell} P(m)$$

holds.

The corresponding partition of $\mathcal{F}$ is given by

$$\mathcal{F} = \mathcal{F}(0) \cup \mathcal{F}(1) \cup \cdots \cup \mathcal{F}(m) \quad,$$

where $\mathcal{F}(j) := \mathcal{F} \cap P(j)$ for $j = 0, 1, \ldots, m$. Without loss of generality, we assume

$$|\mathcal{F}(1)| \geq |\mathcal{F}(2)| \geq \cdots \geq |\mathcal{F}(m)| \quad. \tag{4}$$

By means of the Kruskal-Katona Theorem (i.e. the fact that $\prec_{r\ell}$ is a Macaulay order for B_n), $|\Delta(\mathcal{F})|$ does not increase when replacing $\mathcal{F}(j)$ by $C\bigl(|\mathcal{F}(j)|, N_i(P(j))\bigr)$ for all $j \in \{0, 1, \ldots, m\}$ (simultaneously). Hence, we assume

$$\mathcal{F}(j) = C\bigl(|\mathcal{F}(j)|, N_i(P(j))\bigr) \quad \text{for} \quad j = 0, 1, \ldots, m \quad. \tag{5}$$

If $N_i(P(0) \cup P(1)) \subseteq \mathcal{F}$, then the claim follows by $P(j) \cong B_k^{k-1}$ for $j = 2, 3, \ldots, m$ and the well-known fact (see [5] for example) that Boolean lattices are little-submodular, i.e. that for all $1 \leq m_1 \leq m_2 \leq \binom{k}{i-1}$ the inequality

$$|\Delta(C(m_1, N_{i-1}(B_k)))| + |\Delta(C(m_2, N_{i-1}(B_k)))|$$

$$\geq \begin{cases} |\Delta(C(m_1 + m_2, N_{i-1}(B_k)))| & \text{if } m_1 + m_2 \leq \binom{k}{i-1}, \\ \binom{k}{i-2} + |\Delta(C(m_1 + m_2 - \binom{k}{i-1}, N_{i-1}(B_k)))| & \text{else} \end{cases}$$

holds.

Hence, we assume $N_i(P(0) \cup P(1)) \not\subseteq \mathcal{F}$. Observe that for $F \in N_i(P(0))$ every element of $\Delta(F)$ is also contained in $\Delta(F')$ for some $F' \in N_i(P(1))$ with $F' \prec_{r\ell} F$. Therefore, we can suppose that the first element of $P_i \setminus \mathcal{F}$ w.r.t. $\prec_{r\ell}$ is in $P(1)$. Using this, (4), (5), and the Kruskal-Katona Theorem (applied to $P(0) \cup P(1)$), we can even assume that $\mathcal{F}(0) \cup \mathcal{F}(1)$ is a compressed subset of P_i.

We conclude by induction on m. If $m = 2$, then the claim is implied by Theorem 3. (Note that for $m = 2$ the Macaulay order from Theorem 3 coincides with $\prec_{r\ell}$ on P.) Assume that $m \geq 3$ and that the assertion holds for $m' < m$.

If $\mathcal{F}(m) = \emptyset$, then we are done by the induction hypothesis. Let $\mathcal{F}(m) \neq \emptyset$. By the fact that $\mathcal{F}(0) \cup \mathcal{F}(1)$ is already compressed together with (4), (5), and Theorem 3 (applied to $P(0) \cup P(1) \cup P(m) \cong B_k^{k-1} \times B_2^1$), $|\Delta(\mathcal{F})|$ does not increase when replacing $\mathcal{F}(0) \cup \mathcal{F}(1) \cup \mathcal{F}(m)$ by

$$C\Big(|\mathcal{F}(0) \cup \mathcal{F}(1) \cup \mathcal{F}(m)|, \, N_i(P(0) \cup P(1) \cup P(m))\Big) \quad.$$

After this replacement we have $N_i(P(0) \cup P(1)) \subseteq \mathcal{F}$ or $\mathcal{F}(m) = \emptyset$. This completes the proof since in both cases we are done by the preceding arguments. ∎

Lemma 5 yields a consequence for the case of two factors in general:

Corollary 6 *Let* $2 \leq t < k$ *and* $2 \leq s < m$ *be integers such that* $s \leq t$. *Furthermore, suppose that* $s = t$ *implies* $k \leq m$. *If* $P = B_k^t \times B_m^s$ *is a Macaulay poset, then* $t = k - 1$ *holds.*

Proof. Let A and B be disjoint sets with $|A| = k$ and $|B| = m$. We consider P as the collection of all subsets $F \subset A \cup B$ satisfying $|F \cap A| \leq t$ and $|F \cap B| \leq s$, ordered by inclusion.

Suppose that P is a Macaulay poset, and let $\prec$ denote the corresponding Macaulay order. If $F \in P_s$, then

$$|\nabla(F)| = \begin{cases} k & \text{if } F \subset B, \\ m & \text{if } s = t \text{ and } F \subset A, \\ k + m - s & \text{else} \end{cases}$$

holds. Let G be the last element of P_s w.r.t. $\prec$. By the above equality and Proposition 1, $G \subset B$ must hold if $s < t$ or $k < m$. If $s = t$ and $k = m$, then we have either $G \subset A$ or $G \subset B$. Without loss of generality, we assume $G \subset B$ in this case, too.

Let H denote the last element of P_{s-1} w.r.t. $\prec$. According to Proposition 1, $G \in \nabla(H)$ holds. Hence, we have $H \subset B$.

Finally, consider $\mathcal{F}(H)$, the filter generated by H. Clearly, $\mathcal{F}(H)$ is right-compressed and, consequently, is a Macaulay poset by Proposition 2. On the other hand, $\mathcal{F}(H)$ is the set of all $H \cup F$ with $F \subset A \cup (B \setminus H)$ and $|F \cap A| \leq t$, $|F \cap (B \setminus H)| \leq 1$, ordered by inclusion. Consequently, $\mathcal{F}(H)$ is isomorphic to $B_k^t \times B_{m-s+1}^1$, and the claim follows by Lemma 5. ∎

In particular, this means to the case of two isomorphic factors:

Corollary 7 *Let $1 \leq t < k$ be integers. $P = (B_k^t)^2$ is a Macaulay poset if and only if $t \in \{1, k-1\}$.*

Proof. By Theorems 4 and 3, P is a Macaulay poset if $t \in \{1, k-1\}$. On the other hand, Corollary 6 immediately implies that P is not a Macaulay poset if $t \notin \{1, k-1\}$. ∎

Finally, it is not hard to show the generalization to powers of truncated Boolean lattices.

Theorem 8 *Let $1 \leq t < k$ and $n \geq 2$ be integers. $P = (B_k^t)^n$ is a Macaulay poset if and only if $t \in \{1, k-1\}$.*

Proof. Let $A_1, A_2, \ldots, A_n$ be mutually disjoint sets of size k. Clearly, we can consider P to be the collection of all subsets $F \subset N := \bigcup_{i=1}^n A_i$ with $|F \cap A_i| \leq t$ for $i = 1, 2, \ldots, n$, ordered by inclusion.

By Theorems 3 and 4, P is a Macaulay poset if $t \in \{1, k-1\}$. Consequently, it suffices to show the following implication: If P is a Macaulay poset, then $t \in \{1, k-1\}$ holds.

The proof is by induction on n. If $n = 2$, then we are done by Corollary 7. Let $n \geq 3$, and for $2 \leq n' < n$ assume that $(B_k^t)^{n'}$ is a Macaulay poset if and only if $t \in \{1, k-1\}$. Furthermore, suppose that P is a Macaulay poset, and let $\prec$ denote the corresponding Macaulay order.

Consider an element $F \in P_t$. Trivially,

$$|\nabla(F)| = \begin{cases} (n-1)k & \text{if } F \subset A_i \text{ for some } i \in \{1, 2, \ldots, n\}, \\ nk - t & \text{else} \end{cases}$$

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holds. By this and by Proposition 1, if G denotes the last element of P_t w.r.t. $\prec$, then $G \subset A_i$ for some $i \in \{1, 2, \ldots, n\}$. Without loss of generality, we assume $G \subset A_n$.

Let $\mathcal{F}(G)$ denote the filter generated by G. By the choice of G, the filter $\mathcal{F}(G)$ is right-compressed. Hence, by Proposition 2, $\mathcal{F}(G)$ is a Macaulay poset. On the other hand, $\mathcal{F}(G)$ consists of all $G \cup F$ such that $F \subset N \setminus A_n$ with $|F \cap A_i| \le t$ for $i = 1, 2, \ldots, n-1$, ordered by inclusion. Consequently, $\mathcal{F}(G)$ is isomorphic to $(B_k^t)^{n-1}$, and $t \in \{1, k-1\}$ is implied by the induction hypothesis. ∎

Here we concentrated on powers of truncated Boolean lattices, and formulated just some initial observations for two non-isomorphic factors. In general, the following is still open:

Problem 9 *Characterize all parameters k_i, t_i $(i = 1, 2, \ldots, n)$ such that the direct product $B_{k_1}^{t_1} \times B_{k_2}^{t_2} \times \cdots \times B_{k_n}^{t_n}$ is a Macaulay poset.*

Let us conclude mentioning that there are indeed products of the above kind containing factors which are not stars or equal to B_k^{k-1} for some k. For instance, it is not hard to find a Macaulay order for $B_4^2 \times B_3^2$.

3 Products of truncated chain products

Another natural question suggested by Theorem 3 is: Is there a multiset version of the theorem, i.e. a similar statement for products each factor of which is a chain product that is truncated in some sense? First candidates are products of chain products whose maximal elements have been removed, but it is not complicated to figure out that, unfortunately, in general these are not Macaulay posets.

We suggest to generalize B_n^{n-1} in a different way: Let S_k^n denote the direct product of n chains of length k, i.e. S_k^n consists of all vectors $(x_1, x_2, \ldots, x_n)$ with integer entries $0 \le x_i \le k$ which are partially ordered by

$$(x_1, \ldots, x_n) \le (y_1, \ldots, y_n) \iff x_i \le y_i \text{ for } i = 1, 2, \ldots, n.$$

Furthermore, let $\mathbf{1}_n$ denote the n-ary vector $(1, 1, \ldots, 1)$, and let $F(\mathbf{1}_n)$ be the filter generated by $\mathbf{1}_n$ in S_k^n. As an example, S_2^3 is shown in Figure 1, where the dashed part is the filter $F(\mathbf{1}_3)$. In the Boolean case $k = 1$ the filter $F(\mathbf{1}_n)$ consists of just the maximal element.
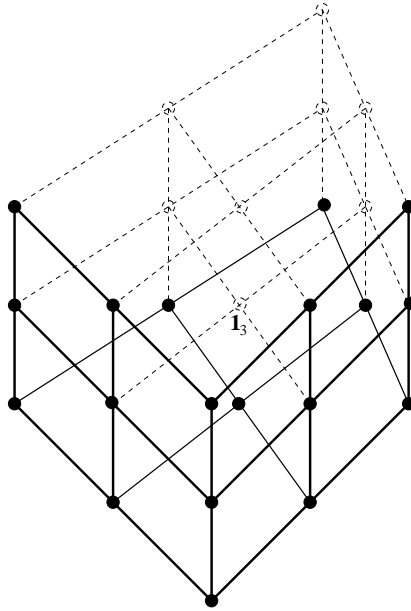
Figure 1: $S_2^3 \setminus F(\mathbf{1}_3)$ (solid) and $F(\mathbf{1}_3)$ (dashed).

Problem 10 *Let $P = S_k^n \setminus F(\mathbf{1}_n)$. Decide whether the cartesian power P^m is a Macaulay poset for all $m \geq 1$.*

Although we conjecture that Problem 10 gives the "right" extension of Theorem 3 to multisets, the problem is completely open, even the case $m = 1$ so far has not been studied seriously.

Obviously, Theorem 3 and the Colored Kruskal-Katona Theorem coincide for factors B_2^1, i.e. in Lindström's subcubes-of-a-cube case. Similarly, an affirmative answer to Problem 10 for $n = 2$ would coincide with the special case $\ell = 2$ of the following result due to Bezrukov and Elsässer [4]: Cartesian powers of the *spider poset* $Q_{k,\ell}$ $(k, \ell \geq 1)$ are Macaulay posets, where $Q_{k,\ell}$ denotes the poset whose Hasse diagram is obtained by identifying the minimal elements of ℓ chains of length k.

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